Topology

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These notes were prepared for a lecture course on topology at the Technical University of Munich in the Summer Semester of 2026. I have relied in part on the sources listed in the bibliography in preparing these notes; any errors are my own. The figures for these notes were made using TikZ and asymptote unless otherwise specified. All material not yet covered in lecture is subject to change without notice.

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Topologies and continuity

First lecture on 13. April

When we first encounter continuity — usually in an analysis class — we define continuous functions in terms of distances. The classic ϵ-δ definition fundamentally requires the structure of, for instance, a metric space to define continuity and associated notions.

The fundamental observation of topology is that continuity does not depend on the metric, but rather only on the open sets defined by the metric.[1] This opens the door to generalizing continuity-related arguments to a wide range of non-metric spaces: posets, polynomial rings over arbitrary fields, and spaces with indistinguishable points are all examples of topological spaces which do not admit a metric.

Preliminary to our exploration of topology, let us briefly recall the classic definition of continuity, and see how it only depends on open sets.

Definition 1.0.1. A *metric space* (X, d) is a pair consisting of a set X and a *metric* d on X, i.e., a function

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying the following properties.

1. *Symmetry*: For all $x, y \in X$,

$$d(x, y) = d(y, x).$$

2. *Identity of indiscernables*: For $x, y \in X$, $x = y$ if and only if $d(x, y) = 0$.

3. *Triangle inequality*: For $x, y, z \in X$,

$$d(x, y) + d(y, z) \geq d(x, z).$$

A function $f : (X, d_X) \to (Y, d_Y)$ is called *continuous at* $x \in X$ if, for every $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, there exists a $\delta \in \mathbb{R}$ with $\delta > 0$ such that, for all $z \in X$,

$$d_X(x, z) < \delta \Longrightarrow d_Y(f(x), f(y)) < \epsilon.$$

The function f is called *continuous* if it is continuous at all points $x \in X$.

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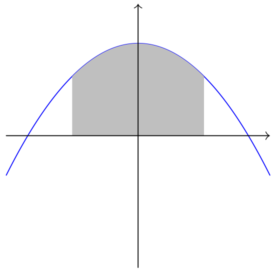
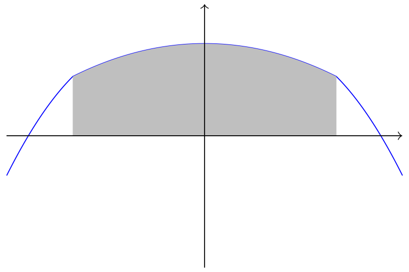
[1] As one way to illustrate that continuity doesn't depend on the specific metric, but rather on the open and closed sets, let's consider the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}$.



We've highlighted a section of the x-axis, and we now stretch that section out, while leaving the other distances unchanged.



The function remains continuous, even though we've clearly changed the metric we're using on $\mathbb{R}$.

Definition 1.0.2. If (X, d_X) is a metric space, $x \in X$ and $r \in \mathbb{R}_{>0}$, we define the *open ball of radius* r *around* x to be

$$B_r(x) := \{y \in X \mid d_X(x, y) < r\} \subset X.$$

We call a subset $U \subset X$ *open* if, for every $x \in U$, there exists a radius $r \in \mathbb{R}_{>0}$ such that $B_r(x) \subset U$.

We can then formalize our earlier observation about continuity.

Proposition 1.0.3. *Let (X, d) and (Y, s) be metric spaces. Then a function $s : X \to Y$ is continuous if and only if, for every open subset $U \subset Y$, the subset $f^{-1}(U) \subset X$ is open.*

Proof. First suppose that s is continuous and let $U \subset Y$ be an open set. Suppose $x \in f^{-1}(U)$. Since U is open, we can find an $\epsilon > 0$ such that the ball $B_\epsilon(f(x)) \subset U$. By continuity, there exists a $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$, and thus $B_\delta(x) \subset f^{-1}(U)$, proving that $f^{-1}(U)$ is open.

Now suppose that every preimage under f of an open set is open. In particular, given $x \in X$, and $\epsilon > 0$, the set $f^{-1}(B_\epsilon(f(x)))$ is open. Therefore, there is a $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$, and thus $f(B_\epsilon(x)) \subset B_\epsilon(f(x))$. Hence, f is continuous. $\square$

1.1 The category of topological spaces

Based on Proposition 1.0.3, we define a topological space to be an axiomatization of the key properties of the collection of open sets of a metric space.

Definition 1.1.1. Let X be a set. A *topology on* X is a subset $\tau_X \subset \mathbb{P}(X)$ of the power set such that

1. $X, \varnothing \in \tau_X$.[2]
2. Let $\{U_i\}_{i \in I}$ be a (possibly infinite) collection of sets in τ_X. Then
$$\bigcup_{i \in I} U_i \in \tau_X$$
3. Let $\{U_i\}_{i \in 1}^n$ be a finite collection of sets in τ_X. Then
$$\bigcap_{i=1}^n U_i \in \tau_X.$$

We call the elements $U \in \tau_X$ the *open sets* of the topology on X. We refer to a pair (X, τ_X), where τ_X is a topology on X, as a topological space.[3] We call a subset $C \subset X$ of a topological space *closed* if

$$C^c := X \setminus C$$

is an open set, i.e. is in τ_X.

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[2] Applying general set-theoretic conventions, this assertion actually follows from the following two. Since the empty union is empty, and the empty intersection of subsets of X is all of X, condition (2) implies that $X \in \tau_X$, and condition (3) implies that $\varnothing \in \tau_X$

[3] We will often abuse notation and write X for a topological space in cases where the choice of topology is clear from context.

As a first sanity check, we show that this does, indeed, generalize the situation of metric spaces.

Proposition 1.1.2. *Let (X, d) be a metric space. Denote by τ_d the collection of d-open subsets of X. Then τ_d is a topology on X.*[4]

Proof. It is immediate that X and $\emptyset$ are elements of τ_d. We now check the remaining properties.

Suppose that $\{U_i\}_{i\in I}$ is a collection of open sets of X, and let $x \in \bigcup_{i\in I} U_i$. Then, in particular, there exists a $j \in I$ such that $x \in U_j$. Since U_j is open, there is a radius r such that $B_r(x) \subset U_j$. However, this implies that $B_r(x) \subset \bigcup_{i\in I} U_i$, so the latter is open.

Now suppose that $\{U_i\}_{i=1}^n$ is a collection of open sets of X. Let $x \in \bigcap U_i$. For each $1 \leq i \leq n$, choose a radius $r_i > 0$ such that $B_{r_i}(x) \subset U_i$. Set $r = \min_i(r_i)$. Then $B_r(x) \subset U_i$ for any $q \leq i \leq n$. Consequently $B_r(x) \subset \bigcap U_i$, and such $\bigcap U_i$ is an open set. □

As alluded to earlier, there are also a wide variety of examples of topological spaces which do not arise as metric spaces.

Example 1.1.3. Suppose that $(P, \leq)$ is a partially ordered set (poset).[5] We can define a topology $\tau_\leq$ on P as follows. We define a set $U \subset P$ to be *downwards closed* if, for every $x \in U$ and $y \in P$, if $y \leq x$, then $y \in U$. We claim that the downwards closed sets form a topology τ_P on P. To see this, we check axioms (2) and (3) in the definition of topological spaces

2. Suppose that $\{U_i\}_{i\in I}$ is a collection of downwards-closed sets. Let $x \in \bigcup_{i\in I} U_i$ and $y \in P$ such that $y \leq x$. Then there is some $j \in I$ such that $x \in U_j$. Thus $y \in U_j$, and so $y \in \bigcup_{i\in I} U_i$. Thus, $\bigcup_{i\in I} U_i$ is downwards-closed.
3. Suppose $\{U_i\}_{i\in I}$ is a collection of downwards-closed sets. Let $x \in \bigcap_{i\in I} U_i$ and $y \in P$ such that $y \leq x$. Then for each $i \in I$, $x \in U_i$, and so $y \in U_i$. Thus $y \in \bigcap_{i\in I} U_i$, and so $\bigcap_{i\in I} U_i$ is a downwards-closed set.

Notice that this topology has a curious feature: an *arbitrary* intersection of open sets is still open. This is not true, for instance, in the metric topology on $\mathbb{R}^n$.

Example 1.1.4. Let k be a field, and consider the polynomial ring $k[x_1, \ldots, x_n]$ in n variables. We define the *Zariski topology* on k^n as follows. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. We define a corresponding subset of k^n, the *vanishing set* of I, to be

$$V(I) := \{a \in k^n \mid p(a) = 0 \ \ \forall p \in I\}.$$

Let us explore the behavior of the $V(I)$ under intersections and unions.

- Given I, J ideals, we can compute that

$$V(I) \cap V(J) = V(I + J)$$

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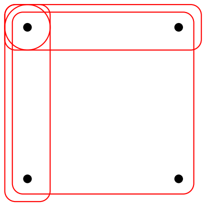
[4] This, together with Proposition 1.0.3 effectively tells us that we can study continuous functions between metric spaces in terms of the associated topological spaces.

[5] Let us consider a specific poset — the power set of $\{1, 2\}$ ordered by inclusion. We can draw the order relation as arrows:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] \arrow[dr] & \{1\} \arrow[d] \\
\{2\} \arrow[r] & \{1,2\}
\end{tikzcd}$$

Since this poset is finite, we can also draw all of the corresponding non-empty open sets.



This topology is not trivial in any way — it encodes information about the original order relation. Enough, in fact, that we can recover the poset structure from the topology.

where $I + J$ is the ideal consisting of elements of the form $p + q$ for $p \in I$ and $q \in J$. More generally, for an arbitrary collection $\{I_s\}_{s\in S}$ of ideals, we have

$$\bigcap_{s\in S} V(I_s) = V\left(\left\langle \bigcup_{s\in S} I_s \right\rangle\right)$$

where $\langle \bigcup_{s\in S} I_s \rangle$ denotes the ideal generated by all of the elements in the I_s.

- For a finite set S and a collection of ideals $\{I_s\}_{s\in S}$, we have[6]

$$\bigcup_{s\in S} V(I) = V\left(\prod_{s\in S} I_s\right).$$

This tells us that the collection of zero-sets of ideals is closed under arbitrary intersection and finite union — the opposite of what we want for a topology. However, this is a simple fix for this issue. We define the *Zariski topology* on k^n to be the topology whose open sets are of the form $k^n \setminus V(I)$ for some ideal $I \subset k[x_1, \ldots, x_n]$.

Definition 1.1.5. There are two key examples of topological spaces which often show up in computations. One is the *empty space* $\varnothing$, whose underlying set is the empty set and whose topology is $\{\varnothing\}$. The other is the *singleton space* — also called the *one-point space* or simply *the point* — denoted by $*$ whose underlying set is a singleton set $\{a\}$ and whose topology is $\mathbb{P}(\{a\}) = \{\varnothing, \{a\}\}$.

Definition 1.1.6. Let τ_1 and τ_2 be two topologies on a set X. If $\mathrm{id}_X : (X, \tau_2) \to (X, \tau_1)$ is continuous, we say that τ_1 is *coarser* than τ_2 or that τ_2 is *finer* than τ_1.

Examples. Some interesting examples of coarseness/fineness are the most extreme. Let X be a set

1. Define a topology τ_{dis} on X by declaring *every* subset of X to be an element of τ_{dis}. We call this the *discrete topology* on X. This is the finest possible topology on X, and it has a very interesting property. Let Y be any topological space, and let $f : X \to Y$ be any map of underlying sets. Then $f : (X, \tau_{dis}) \to (Y, \tau_Y)$ is continuous.
2. Define a topology τ_{ind} on X by $\tau_{ind} := \{\emptyset, X\}$. We call this the *indiscrete topology* on X – the coarsest possible topology on X. For any topological space Y and any map of sets $f : Y \to X$, the map $f : (Y, \tau_Y) \to (X, \tau_{ind})$ is continuous.

These two examples are *dual* to one another, in the same sense that the singleton and the empty space are dual to each other. We will explore this further in the next section.

Exercise 1.1.7. Show that the following statements are equivalent:

1. $\tau_1 \subset \tau_2$
2. τ_1 is coarser than τ_2.

Definition 1.1.8. A map $f : X \to Y$ between (the underlying sets of) topological spaces (X, τ_X) and (Y, τ_Y) is called a *continuous map of topological spaces* if, for every open subset $U \in \tau_Y$, the preimage $f^{-1}(U)$ is an element of τ_X. We say that f is a *homeomorphism* if it is bijective and both f and f^{-1} are continuous.[7] Further, we call f an *open map* if $f(U) \in \tau_Y$ for every $U \in \tau_X$, and a *closed map* if $f(C)$ is closed in Y for every closed set $C \subset X$.

Example 1.1.9. Define $B^n := \{x \in \mathbb{R}^n \mid |x| < 1\}$ to be the open unit ball in $\mathbb{R}^n$ (both spaces equipped with the topologies induced by the metrics). Define maps

$$\begin{aligned} f : B^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \tfrac{x}{1-|x|} \end{aligned}$$

and

$$\begin{aligned} g : \mathbb{R}^n &\longrightarrow B^n \\ x &\longmapsto \tfrac{x}{1+|x|} \end{aligned}$$

Then f is a homeomorphism with inverse g.

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[6] The interested reader may wish to attempt the following.

Exercise. Say what goes wrong here if we consider an infinite set of ideals.

[7] This is simply formalizing what we discovered about metric spaces into a definition. We can now talk about continuity of maps between general topological spaces and know that this subsumes the case of metric spaces.

Lemma 1.1.10. *Continuous maps have the following properties.*

1. *The composition of two continuous maps is continuous.*
2. *The identity map on any topological space is continuous.*

Proof. Left to the reader. ☐

We will not yet define categories, but we already have enough information to describe our key object of study: the category of topological spaces. Roughly speaking, a *category* is an axiomatization of a "place to do a certain kind of mathematics". It consists of a collection of "mathematical objects" — which in some simple cases can be thought of as sets equipped with additional structures like topologies or binary operations — and a collection of "morphisms" between any two objects — which in the aforementioned simple cases can be thought of as structure-preserving maps of sets.[8]

Definition 1.1.11. For topological spaces X and Y, we denote the set of continuous maps from X to Y by $\mathsf{Top}(X, Y)$.[9] The *category* Top *of topological spaces and continuous maps* consists of:

1. The collection[10] of all topological spaces. These will be referred to as the *objects* of Top. In a mild abuse of notation, we will often write $X \in \mathsf{Top}$ to mean "X is a topological space"
2. For every pair of spaces $X, Y \in \mathsf{Top}$, the set
$$\mathsf{Top}(X, Y)$$
of continuous maps from X to Y. These continuous maps are also called the *morphisms* of Top.
3. For every triple of spaces $X, Y, Z \in \mathsf{Top}$, the composition map
$$\mathsf{Top}(Y, Z) \times \mathsf{Top}(X, Y) \xrightarrow{-\circ-} \mathsf{Top}(X, Z)$$
which sends a pair of composable continuous maps to their composite.[11]

Notice that the composition map is associative and has units in the sense that, for $f : X \to Y$, $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$.

Exercise 1.1.12. Show that a continuous map $f : X \to Y$ is a homeomorphism if and only if it is both bijective and open.

1.2 Constructing new topologies

—— Second lecture on 14. April

Our next order of business is to understand how to build new topological spaces out of old ones. As one might expect, there are numerous different ways to approach all of these constructions, but we will focus on two: the *point-set* approach, where we explicitly write a set and characterize its open sets, and the *categorical*

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[8] While we will work with the category of topological spaces very heuristically here, we will return to delve more formally into category theory in Chapter 3. The reader desiring a more formal definition of category before continuing can consult Definition 3.1.1 and the follow pages for a more detailed exposition and exploration.

[9] This is also variously denoted by $C^0(X, Y)$ or $\mathrm{Hom}_{\mathsf{Top}}(X, Y)$.

[10] Not, sadly, a set, owing to Russel's Paradox.

[11] This is well-defined by Lemma 1.1.10.

approach, where we specify a space by specifying all of the continuous maps out of it (or into it). Thus, each definition — phrased entirely in terms of maps — will be accompanied by a construction, which explains how to specify a topological space which satisfies the given definition. This is in contrast to many texts, in which the construction is given as the definition of a space.

Warning 1.2.1. ⚠ Our definitions here in terms of continuous mappings are more usually called *universal properties*. Our terminology is chosen to emphasize that these properties characterize a topological space *uniquely up to unique homeomorphism*, and so can be considered as definitions of the space in question.

As a tool to ease the point-set approach, we first introduce *bases* for topologies.

Definition 1.2.2. Let X be a set, and $\mathcal{S} \subset \mathbb{P}(X)$ a subset of the power set. The *topology generated by* $\mathcal{S}$ is the coarsest topology containing $\mathcal{S}$. Equivalently, this is the intersection of all topologies on X which contain $\mathcal{S}$. If τ is the topology on X generated by $\mathcal{S}$, we call $\mathcal{S}$ a *subbasis* of τ.

Lemma 1.2.3. *Let X be a set and $\mathcal{S} \subset \mathbb{P}(X)$ a subset of the power set. The topology τ generated by $\mathcal{S}$ consists of the arbitrary unions of finite intersections of elements of $\mathcal{S}$.*

Proof. Since the topology generated by $\mathcal{S}$ clearly must contain the unions of finite intersections of elements of $\mathcal{S}$, it suffices to simply verify that these sets form a topology. This follows immediately from the fact that unions of unions are unions, and intersections distribute over unions. □

If we place some conditions on $\mathcal{S}$, however, the topology generated by $\mathcal{S}$ admits a simpler description.

Definition 1.2.4. We call $\mathcal{S} \subset \mathbb{P}(X)$ a *basis* if it satisfies the condition that, for any $U, V \in \mathcal{S}$ and any $x \in U \cap V$, there exists $W \in \mathcal{S}$ with $x \in W \subset U \cap V$.

Lemma 1.2.5. *If $\mathcal{S} \subset \mathbb{P}(X)$ is a basis, then the topology τ generated by U consists of the unions of elements of $\mathcal{S}$.*

Proof. By Lemma 1.2.3, it suffices to show that every finite intersection of elements of $\mathcal{S}$ can be written as a union of elements of $\mathcal{S}$. Given $x \in \bigcap_{i=1}^{n} U_i$, we first choose $V_2 \in \mathcal{S}$ such that $x \in V_1 \subset U_1 \cap U_2$. We then choose $V_3 \in \mathcal{S}$ such that $x \in V_2 \subset V_1 \cap U_3 \subset U_1 \cap U_2 \cap U_3$. Iterating, we find $V_x \in \mathcal{S}$ with $x \in V \subset \bigcap_{1=1}^{n} U_i$. By construction, $\bigcup_{x \in \bigcap U_i} V_x$ is $\bigcap_{i=1}^{n} U_i$, completing the proof. □

Lemma 1.2.6. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, let $f : X \to Y$ be a map of underlying sets, and let $\mathcal{B}$ be a basis for the topology τ_X. Then f is continuous if and only if, for every open $U \in \tau_Y$ and every $x \in f^{-1}(U)$, there is a basis element $V_{x,U} \in \mathcal{B}$ such that $x \in V_{x,U} \subset f^{-1}(U)$.*

Proof. For a fixed $U \in \tau_Y$, taking the union over all $x \in f^{-1}(U)$ of the corresponding $V_{x,U}$'s yields $f^{-1}(U)$, showing that $f^{-1}(U)$ is open. On the other hand, if f is continuous, and $U \in \tau_Y$, then $f^{-1}(U) \in \tau_X$. Since $\mathcal{B}$ is a basis, this means that $f^{-1}(U)$ is a union of elements of $\mathcal{B}$, and so every element of $f^{-1}(U)$ is contained in one such. □

Lemma 1.2.7. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, let $f : X \to Y$ be a map of underlying sets, and let $\mathcal{B}$ be a basis for the topology τ_Y. Then f is continuous if and only if $f^{-1}(U)$ is open for every $U \in \mathcal{B}$.*

Proof. One direction is immediate from the definitions. To demonstrate the other, suppose that preimages of basis elements are open, and let $U \in \tau_Y$. Then Choose basis elements $\{U_i\}_{i \in I}$ such that

$$\bigcup_{i \in I} U_i = U.$$

Then

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i)$$

which is a union of open sets, and thus open. □

To deal more easily with categorical definitions, we need *commutative diagrams*. The definition given here is preliminary, and will be revised substantially once we discuss category theory more formally.

Definition 1.2.8. A *diagram* in Top is a directed graph[12] whose vertices are labeled by objects of Top (topological spaces) and whose arrows are labeled by morphisms (continous maps) between the appropriate objects. A diagram is said to *commute* if, for any two directed paths between the same pair of vertices, the composites of the associated morphisms are equal.[13]

Examples 1.2.9.

1. There are several trivial examples of commutative diagrams: any diagram associated to the graph with a single vertex and no arrows trivially commutes, as does any diagram associated to the graph with two vertices and a single arrow between them.

2. The simplest non-trivial example of a commutative diagram consists of three topological spaces X, Y, and Z and three morphisms (continuous maps) f, g and h assigned to a triangle-shaped graph. We draw this commutative diagram as

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \arrow[ur, "g"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

   The condition that this diagram commutes mean that the composite $f \circ g$ (associated to the upper directed path) and the composite h associated to the lower, 1-edge, path, must be equal.

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[12] In principle, we can interpret the term *directed graph* broadly here, allowing parallel edges. In practice, the *commutative* diagrams we draw will almost always lack these. This is because the condition that a diagram commutes means that parallel edges must be assigned the same morphism.

[13] A better, and less long-winded, definition of a commutative diagram is given in Chapter 3 as Definition 3.2.1.

3. Another common shape of commutative diagram is the *commutative square*:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "h"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "k"'] & W \end{tikzcd}$$

   The condition that this diagram commute is that $f \circ g = k \circ h$.

Initially, we will give definitions both using equations of continuous maps and (in the sidebar) the corresponding commutative diagrams. Eventually, as we become more accustomed to categorical terminology and notation, we will switch to using commutative diagrams almost exclusively.

1.2.1 Products and coproducts

One of the themes we will encounter in this class is that categorical definitions come in pairs, related by reversing the directions of all of the morphisms involved. These definitions are said to be *dual* to one another, and the name of one is usually constructed by affixing a "co-" to the front of the name of the other. Our first example of this phenomenon is products and coproducts. We will return to products and coproducts more formally Example 3.6.6 (2) as examples of limits and colimits in a category.

The categorical definition of the product is phrased entirely in terms of the objects and morphisms of Top, but as we will see, uniquely specifies the same space.

Definition 1.2.10 (Universal property of the product topology)**.** Let I be a set, and $\{(X_i, \tau_i)\}_{i \in I}$ a collection of topological spaces indexed by I. A *product* of the X_i is a topological space Y together with a collection of continuous maps $\{\ p_i : Y \longrightarrow X_i\ \}_{i \in I}$, such that, for any other space Z and collection of continuous maps $\{\ q_i : Z \longrightarrow X_i\ \}_{i \in I}$ there is a unique continous map $f : Z \to Y$ such that, for every $i \in I$,[14]

$$p_i \circ f = q_i.$$

Definition 1.2.11 (Universal property of the coproduct topology)**.** Let I be a set, and $\{(X_i, \tau_i)\}_{i \in I}$ a collection of topological spaces indexed by I. A *coproduct* of the X_i is a topological space Y together with a collection of continuous maps $\{\ Y \longleftarrow X_i : \iota_i\ \}_{i \in I}$, such that, for any other space Z and collection of continuous maps $\{\ Z \longleftarrow X_i : h_i\ \}_{i \in I}$ there is a unique continous map $f : Y \to Z$ such that, for every $i \in I$,[15]

$$f \circ \iota_i = h_i.$$

There are three important points about these definitions to notice:

1. The continuous maps p_i are *part of the data of a product.*
2. The definition of a coproduct is obtained from the definition of a product by reversing the directions of all of the arrows, and keeping everything else the same.

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[14] Equivalently, this is the condition that all of the diagrams

$$\begin{tikzcd} & Y \arrow[dr, "p_i"] & \\ Z \arrow[ur, "f"] \arrow[rr, "q_i"'] & & X_i \end{tikzcd}$$

commute. The definition of the product is also sometimes written, schematically, as a commutative diagram

$$\begin{tikzcd} & Z \arrow[d, dashed, "\exists!"] \arrow[ddl, "q_i"'] \arrow[ddr, "q_j"] & \\ & Y \arrow[dl, "p_i"] \arrow[dr, "p_j"'] & \\ X_i & \cdots & X_j \end{tikzcd}$$

Here, the dashed arrow must exist uniquely for Y to be a product of the X_i.

[15] Equivalently, this is the condition that all of the diagrams

$$\begin{tikzcd} & Y \arrow[dl, "f"'] & \\ Z & & X_i \arrow[ul, "\iota_i"'] \arrow[ll, "h_i"] \end{tikzcd}$$

commute. The definition of the product is also sometimes written, schematically, as a commutative diagram

$$\begin{tikzcd} & Z & \\ & Y \arrow[u, dashed, "\exists!"'] & \\ X_i \arrow[uur, "h_i"] \arrow[ur, "\iota_i"'] & \cdots & X_j \arrow[uul, "h_j"'] \arrow[ul, "\iota_j"] \end{tikzcd}$$

Here, the dashed arrow must exist uniquely for Y to be a product of the X_i.

3. We have defined **a** product, rather than **the** product, and a priori, our definition may not specify a unique space (or any space at all). Our next order of business will be to show that products exist, and are unique up to unique homeomorphism. Since homeomorphic spaces are effectively the same, this means we may safely speak of **the** categorical product.

The introduction of the product topology introduces some ambiguity to the notation $\prod_{i=1}^n X_i$ when the topologies on the X_i are induced by metrics. It is, however, not hard to prove that if $\{(X_i, d_i)\}_{i=1}^n$ are metric spaces, and τ_i are the corresponding topologies, then the product metric

$$d(x,y) = \left(\sum_{i=1}^n d_i(x_i, y_i)^2 \right)^{\frac{1}{2}}$$

induces the product topology.

In particular, when we write $\mathbb{R}^n$, we can equivalently view the topology as being induced by the standard metric, or as being the product topology induced by viewing $\mathbb{R}^n$ as the product of n copies of $\mathbb{R}$.

Construction 1.2.12. Let I be a set, and $\{(X_i, \tau_i)\}_{i\in I}$ a collection of topological spaces indexed by I.

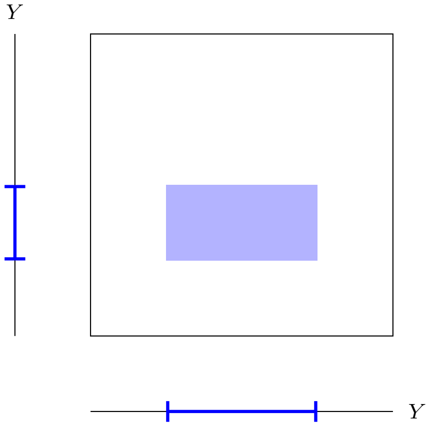
1. The *product* of the X_i, denoted by $\prod_{i\in I} X_i$, is the topological space whose underlying set is the Cartesian product of the sets X_i and whose topology is generated by the basis

$$\left\{ \prod_{i\in I} U_i \;\middle|\; U_i \in \tau_i \text{ and } U_i \neq X_i \text{ for finitely many } i \in I \right\}$$

2. The *coproduct* of the X_i, denote by $\coprod_{i\in I} X_i$, is the topological space whose underlying set is the disjoint union, and whose topology consists of those subsets $U \subset \coprod X_i$ such that $U \cap X_i$ is open for all $i \in I$.

We leave it to the reader to verify that the set in part (1) is, in fact, a basis.

The conventional picture of basis elements in the product topology on $X \times Y$ depicts X and Y as intervals and U and V as open intervals.



Proposition 1.2.13. *Let I be a set and $\{X_i\}_{i\in I}$ a collection of spaces indexed by I. Let $\prod_{i\in I} X_i$ be the product space as given in Construction 1.2.12, and denote by*

$$p_j : \prod_{i\in I} X_i \longrightarrow X_j$$

denote the projection map of underlying sets.

1. *The maps p_i are continuous.*
2. *The space $\prod_{i\in I} X_i$ together with the morphisms $\{p_i\}_{i\in I}$ satisfy Categorical Definition 1.2.10, i.e., form a product in* `Top`.

Proof. To prove part (1), we use Lemma 1.2.6. For an open subset $U \subset X_i$,

$$p_i^{-1}(U) = \prod_{j\in I} V_j$$

where $V_i = U$ and $V_j = X_j$ for $j \neq i$. As such, p_i is continuous.

To prove part (2), suppose we are given another space Z with continuous maps $q_i : Z \to X_i$. There is a unique map of sets

$$\begin{aligned} f : Z &\longrightarrow \prod_{i\in I} X_i \\ z &\longmapsto (q_i(z))_{i\in I} \end{aligned}$$

so it will suffice for us to check that it is continuous. Using Lemma 1.2.7, it suffices for us to check this on basis elements. Consider a basis element $U = \prod_{i\in I} U_i$, and let $i_1, \dots, i_k$ be the indices so that $U_{i_j} \neq X_{i_j}$. Then

$$f^{-1}(U) = \bigcap_{j=1}^{k} q_{i_j}^{-1}(U_{i_j})$$

which is a finite intersection of open sets, and thus open. □

Proposition 1.2.14. *Let $\{X_i\}_{i\in I}$ be an index collection of spaces, and let Y, $\{p_i : Y \to X_i\}_{i\in I}$ and Z, $\{q_i : Z \to X_i\}_{i\in I}$ be two products of the X_i in* Top. *Then there is a unique homeomorphism $f : Z \to Y$ such that $p_i \circ f = q_i$ for all $i \in I$.*

Proof. Since Y is a product of the X_i's in Top, there is a unique continuous map $f : Z \to Y$ satisfying the desired commutativity condition. Thus it suffices to see that f is a homeomorphism. Similarly, since Z is a product of the X_i's in Top there are unique maps $g : Y \to Z$ such that $q_i \circ g = p_i$ for all $i \in I$.

This means that $f \circ g$ is a continuous map satisfying the condition

$$p_i \circ (f \circ g) = q_i \circ g = p_i.$$

However, the identity id_Y on Y also satisfies this condition. Since Y is a product, there is a unique continuous map satisfying this condition, and so $f \circ g = \mathrm{id}_Y$. An identical argument shows that $g \circ f = \mathrm{id}_Z$, and so f is a homeomorphism. □

Roughly, diagrammatically, we can draw this argument as

$$\begin{tikzcd}
Y \arrow[r, "\exists! f"] \arrow[rr, bend left, "\exists! \mathrm{id}_Y"] \arrow[dr, "p_i"'] & Z \arrow[r, "\exists! g"] \arrow[d, "q_i"] & Y \arrow[dl, "p_i"] \\
& X_i &
\end{tikzcd}$$

Exercise 1.2.15. State and prove the analogues of Propositions 1.2.13 and 1.2.14 for the coproduct of topological spaces.

1.2.2 Subspaces and quotients

The next pair of constructions we consider are the so-called *subspace* and *quotient* topologies. Again, we will give categorical definitions and point-set definitions for each. Similarly, we will also write these definitions in such a way as to make it clear that they are *dual*: each definition is obtained from the other by reversing the directions of all of the arrows.[16]

To ease the categorical side of our definitions, we will introduce another category.

Definition 1.2.16. For sets X and Y, we denote the set of functions from X to Y by $\mathsf{Set}(X, Y)$.[17] The *category* Set *of sets and functions* consists of:

1. The collection[18] of all sets. These will be referred to as the *objects* of Set. In a mild abuse of notation, we will often write $X \in \mathsf{Set}$ to mean "X is a set"

2. For every pair of sets $X, Y \in \mathsf{Set}$, the set

$$\mathsf{Set}(X, Y)$$

of functions from X to Y. These functions are also called the *morphisms* of Set.

---

[16] Unlike with products and coproducts, we will not return directly to these subspaces and quotients when we discuss category theory. However, it turns out that they are examples of *equalizers* and *coequalizers*, see Example 3.6.6 (5).

[17] This is also sometimes denoted by $\mathrm{Hom}_{\mathsf{Set}}(X, Y)$.

[18] As was the case for Top, this is not a set.

3. For every triple of spaces $X, Y, Z \in \mathsf{Set}$, the composition map
$$\mathsf{Set}(Y, Z) \times \mathsf{Set}(X, Y) \xrightarrow{- \circ -} \mathsf{Set}(X, Z)$$
which sends a pair of composable functions to their composite.

Notice that the composition map is associative and has units in the sense that, for $f : X \to Y$, $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$.

We can also now define our first *functor*.[19]

Definition 1.2.17. The *forgetful functor from* Top *to* Set, written $U : \mathsf{Top} \to \mathsf{Set}$ consists of the following assignments.

1. U sends a topological space $X \in \mathsf{Top}$ to the underlying set $U(X) \in \mathsf{Set}$. That is, U forgets the topology on X.
2. U sends a continuous map $f : X \to Y$ in Top to the underlying function between sets $U(f) : U(X) \to U(Y)$ in Set.

Notice that U preserves composition and identities: $U(\mathrm{id}_X) = \mathrm{id}_{U(X)}$ and $U(f \circ g) = U(f) \circ U(g)$.

There are other useful functors between these two categories, but we will defer discussion of these to a later section.

Definition 1.2.18 (Universal property of the subspace topology)**.** Let $X \in \mathsf{Top}$ and let $\iota : Y \to U(X)$ be an injective morphism of Set. We call $(Y, \tau) \in \mathsf{Top}$ a *subspace topology on* Y if ι is continuous with respect to τ and, for every continuous map $f : Z \to X$ such that $U(f) : U(Z) \to U(X)$ factors through ι, the induced map $Z \to (Y, \tau)$ is continuous.[20]

Construction 1.2.19. Let X be a topological space, and let $\iota : Y \to U(X)$ be an injective map in Set — effectively a subset of the underlying set of X. The *subspace topology* on Y is the topology $\tau_{Y \subset X}$ defined by requiring that V is open in Y if and only if there is an open set U of X such that $\iota^{-1}(U) = V$.[21]

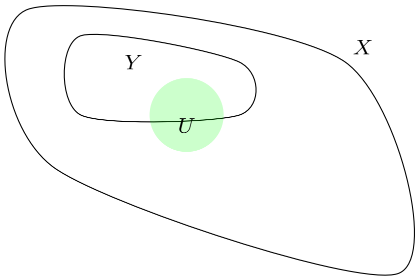
Proposition 1.2.20. *Let X be a topological space, and $\iota : Y \to U(X)$ an injective map of sets.*

1. *The subspace topology $(Y, \tau_{Y \subset X})$ of Construction 1.2.19 satisfies Definition 1.2.18.*
2. *If two topologies on Y satisfy Definition 1.2.18, then they are equal.*

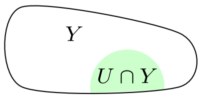
Proof. It is immediate from the definitions that $\iota : (Y, \tau_{Y \subset X}) \to X$ is continuous. Suppose given a continuous map $g : Z \to X$ such that $U(g)$ factors through ι as $\iota \circ h = g$. We claim that $h : Z \to (Y, \tau_{Y \subset X})$ is continuous. To see this, let $U \in \tau_{Y \subset X}$. Then there is a $V \in \tau_X$ with $\iota^{-1}(V) = U$. Since $g = \iota \circ h$, we have that $g^{-1}(V) = h^{-1}(\iota^{-1}(V)) = h^{-1}(U)$, which is open by the continuity of g. Thus, h is continuous.

—— Third lecture on 20. April

A schematic depiction of the open sets in the supspace topology is as follows. In the first drawing we have a space X, a subset Y, and an open set U of X.



In the second, we have the corresponding open subset $Y \cap U$ of Y in the subspace topology



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[19] Note that, as with categories, we have not given a formal definition of functors yet.

[20] Notice that this is secretly saying that there exists a unique continuous map satisfying a condition: there is a unique continuous map $g : Z \to (Y, \tau)$ such that $\iota \circ U(g) = f$.

[21] Note that when Y is truly a subset, this reads $Y \cap U = V$.

To prove the second claim, suppose that τ_1 and τ_2 are topologies on Y satisfying Categorical Definition 1.2.18. Since $\iota : Y \to U(X)$ clearly factors through Y (as id_Y), it follows that $\mathrm{id}_Y : (Y, \tau_1) \to (Y, \tau_2)$ and $\mathrm{id}_Y : (Y, \tau_2) \to (Y, \tau_1)$ are both continuous. Thus $\tau_1 \subset \tau_2$ and $\tau_2 \subset \tau_1$, so the topologies are equal. □

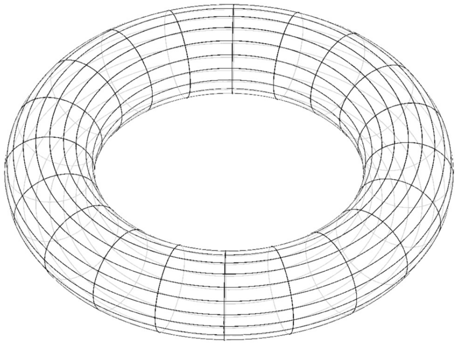
As with the product topology, there is an potential ambiguity when talking about subspaces of metric spaces. However, it is once again easy to show that the restriction of a metric to a subset induces the same topology as the subspace topology.

Example 1.2.21. For $n \geq 0$, the *n-sphere* S^n is the subset

$$S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} \subset \mathbb{R}^{n+1}$$

equipped with the subspace topology inherited from the standard topology on $\mathbb{R}^n$. Equivalently, we will often view the *circle* S^1 as a subspace of $\mathbb{C}$ (with the metric topology).

When drawn, the torus is a surface which looks a bit like a doughnut:



Example 1.2.22. The *torus* $T^2 = S^1 \times S^1$ is another fundamental example of a topological space. We can view the torus either as the product of two copies of the circle (with the product topology) or as a subspace of $\mathbb{R}^2$. It is a useful exercise to show that these definitions coincide.

Definition 1.2.23. A continuous map $f : X \to Y$ between topological spaces is called an *embedding* if f induces a homeomorphism between X and the image of X, where the latter is equipped with the subspace topology.

In effect, an embedding $f : X \to Y$ identifies X with a subspace of Y.

Proposition 1.2.24. *Let $f : X \to Y$ be a continuous injective map of topological spaces and suppose that f is either closed or open. Then f is an embedding.*

Proof. We give the proof for an open map, the proof for a closed map is similar. Suppose that f is open. Then for any open set $U \subset X$, $f(U) = f(U) \cap f(X)$, and so the continuous map onto the image $f(X)$

$$\widetilde{f} : X \longrightarrow f(X)$$

is open as well. Since $\widetilde{f}$ is continuous, open, and bijective by assumption, it follows that $\widetilde{f}$ is a homeomorphism, completing the proof. □

The dual definition to the subspace topology is the *quotient* topology. Notice that this definition really does arise by simply reversing the arrows in Definition 1.2.18.

Definition 1.2.25 (Universal property of the quotient topology)**.** Let $X \in \mathsf{Top}$, and let $\pi : U(X) \to Y$ be a surjective[22] morphism in Set. We call (Y, τ) a *quotient topology* if π is continuous with respect to τ, and, for every morphism $f : X \to Z$ in Top such that $U(f)$ factors through π, the induced map $(Y, \tau) \to Z$ is continuous.

Construction 1.2.26. Let X be a topological space, let Y be a set, and let $\pi : X \to Y$ be a surjective map of sets. The *quotient topology on Y* is defined by declaring $U \subset Y$ to be open if and only if $f^{-1}(U)$ is open.

---

[22] In Set, surjectivity is formally dual to injectivity in a way we will not discuss in this class. The interested student can read about *epimorphisms* and *monomorphisms* for further details.

Proposition 1.2.27. *Let $X \in \mathsf{Top}$, and let $\pi : U(X) \to Y$ be a surjective map of sets.*

1. *The quotient topology of Construction 1.2.26 satisfies Definition 1.2.25.*
2. *If τ_1 and τ_2 are two topologies satisfying Definition 1.2.25, then $\tau_1 = \tau_2$.*[23]

Proof. Exercise. □

Notation 1.2.28.

1. The most common source of quotient maps comes from equivalence relations. If X is a topological space, and $\sim$ is an equivalence relation on X, we will typically write $X_{/\sim}$ for the quotient set equipped with the quotient topology inherited via the quotient map $X \to X_{/\sim}$.
2. As a special case of the previous example, if G is a group which acts on a topological space, we obtain an equivalence relation by setting $x \sim g \cdot x$ for $g \in G$. In this case, we write the quotient space as $X_{/G}$.

It will be useful for us later to quickly recognize quotient maps.

Lemma 1.2.29. *Let $\pi : X \to Y$ be a continuous, surjective map of topological spaces. If π is either an open map or a closed map, then π is a quotient map.*

Proof. First suppose π is an open map, and let $U \subset Y$ such that $\pi^{-1}(U)$ is open. Then $\pi(\pi^{-1}(U)) = U$, and since π is an open map, U is open. Thus, the topology on Y is the quotient topology.

Now suppose π is a closed map, and let $U \subset Y$ such that $\pi^{-1}(U)$ is open. Then $\pi^{-1}(U)^c$ is closed. Since π is surjective, $\pi(\pi^{-1}(U)^c) = U^c$, and since π is a closed map, this implies U^c is closed. Thus U is open, and the topology on Y is the quotient topology. □

Lemma 1.2.30. *A product of continuous open surjections is open.*

Proof. Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be collections of spaces indexed by I, and let $f_i : X_i \to Y_i$ be continuous open surjections. Write $f : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ for the product map which sends $(x_i)_{i\in I} \mapsto (f(x_i))_{i\in I}$. If $\prod_{i\in I} U_i$ is a basic open in $\prod_{i\in I} X_i$, then

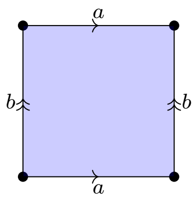
$$f\left(\prod_{i\in I} U_i\right) = \prod_{i\in I} f(U_i)$$

is a basic open (the requirement that at most finitely many factors are not equal to the whole space is satisfied by surjectivity). Taking unions implies that f maps open sets to open sets. □

Example 1.2.31 (The torus as a quotient in two ways)**.** Consider the surjective, continuous map

$$\begin{aligned} p : \quad \mathbb{R}^2 &\longrightarrow S^1 \times S^1 \\ (x, y) &\longmapsto (e^{2\pi i x}, e^{2\pi i y}). \end{aligned}$$

We will often draw pictures meant to convey quotients in terms of equivalence relations. For example, the quotient of $[0,1] \times [0,1]$ from Example 1.2.31 might be drawn as



The labels tell us which intervals should be identified homeomorphically, and the arrows tell us whether to reverse the direction when we glue or not. The corresponding equivalence relation is generated by $(s,0) \sim (s,1)$ and $(0,t) \sim (1,t)$.

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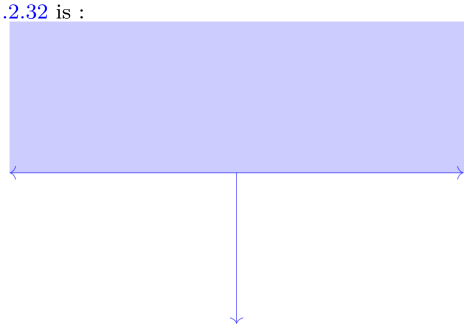
[23] Previously, we said that our categorical definitions specify a topological up to unique homeomorphism. Here, however, we see that we obtain *equal* topologies on the same space. This is because we wrote Definitions 1.2.18 and 1.2.25 in such a way that the underlying set and the underlying map of sets is fixed.

It is sufficient for us to show that the exponential map is continuous, surjective, and open. We know the former facts from analysis. We turn to seeing that the exponential map is open. Let $x \in S^1$ be a point in the circle, and let $t_0 \in \mathbb{R}$ be such that $e^{2\pi i t} = x$. Then for $|y - x|$ sufficiently small, we have that the function $y \mapsto \frac{1}{2\pi}(t_0 + \arccos(\mathrm{Re}(y)))$ is a continuous section to $t \mapsto e^{2\pi i t}$. A brief calculation shows that

$$|e^{2\pi_i t} - e^{2\pi i t_0}| = \sqrt{2}|\sin(2\pi|x-y|)|$$

Thus, for sufficiently small $r \geq 0$, $B_r(t)$ is mapped homeomorphically to $B_{\sqrt{2}|\sin(2\pi r)|}(x)$. Since the open balls of radius below any fixed value greater than 0 form a basis for the topology on $\mathbb{R}$, it follows that p is an open map. As such, the torus is a quotient space of $\mathbb{R}^2$ with quotient map p.

A schematic depiction of the space from Example 1.2.32 is :



Consider the composite

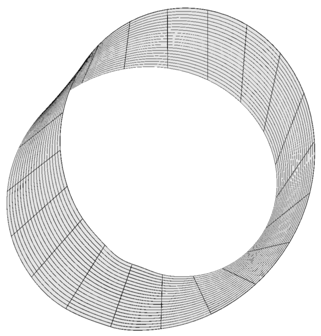
$$[0,1] \times [0,1] \hookrightarrow \mathbb{R}^2 \xrightarrow{p} S^1 \times S^1$$

of p with the inclusion, which we will temporarily term $\overline{p}$. A similar argument to the above can be used to show that this is a closed map, so that $\overline{p}$ is a quotient map.

⚠ The map p is not closed, and the map $\overline{p}$ is not open.

Example 1.2.32. ⚠ A quotient map can be neither open nor closed. Consider the subspace $\mathbb{R} \times \{0\} \cup [0,\infty) \times \mathbb{R} \subset \mathbb{R}^2$. Then the projection onto the first coordinate is a quotient map that is clearly neither open nor closed.

The Möbius band is a rare example of a *non-orientable* surface which is easy to visualize:



Indeed, one can easily construct a Möbius band from paper or fabric.

Examples 1.2.33.

1. Consider $\mathbb{R}$ equipped with the metric topology, and define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if and only if there exists $n \in \mathbb{Z}$ with $x = y + n$. Then the quotient space $\mathbb{R}_{/\sim}$ is homeomorphic to S^1. This is effectively the same as the quotient map explored in Example 1.2.31.

2. Let $X = [0,1] \times [-1,1] \subset \mathbb{R}^2$, equipped with the subspace topology. Define an equivalence relation on X by setting $(0,x) \sim (1,-x)$. The quotient space $(X_{/\sim}, \tau_\sim)$ is called the *Möbius band.*

Example 1.2.34. Consider $\mathbb{R}^{n+1} \setminus \{0\}$ with the subspace topology coming from $\mathbb{R}^{n+1}$. Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by $x \sim \lambda x$ for every $\lambda \in \mathbb{R} \setminus \{0\}$. We define the *$n$-dimensional real projective space* $\mathbb{R}P^n$ to be the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})_{/\sim}$.

Note that we can also consider $\mathbb{R}P^n$ as the quotient of the unit sphere S^n by the equivalence relation $x = -x$.

—— Fourth lecture on 21. April

1.2.3 Pullbacks and pushouts

Our final set of constructions building new spaces from old (at least for the time being) are pullbacks and pushouts. These are again dual constructions — each categorical definition obtained from the other by reversing all of the arrows. In this case, however, we have a new an interesting feature: each of our new constructions can be obtained by performing two of our previous constructions in sequence.[24]

Our definition of pullback will be the first in which we give the definition using commutative diagrams, rather than equations of continuous maps.

Categorical Definition 1.2.35 (Universal property of the pullback)**.** Let $X, Y, Z \in$ Top, and let $f : X \to Z$ and $g : Y \to Z$ be morphisms of Top. A *pullback* of the diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is an object $P \in \mathsf{Top}$ equipped with morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes, and satisfying the property that, for any $W \in \mathsf{Top}$ equipped with morphisms $q_1 : W \to X$ and $q_2 : W \to Y$ such that the diagram

$$\begin{tikzcd} W \arrow[r, "q_2"] \arrow[d, "q_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes, there is a unique morphism $u : W \to P$ such that the diagram

$$\begin{tikzcd} W \arrow[dr, "u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes.

We can compactly summarize the universal property with the single commutative diagram

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

We immediately give the dual definition — *pushouts.*

Categorical Definition 1.2.36 (Universal property of the pushout)**.** Let $X, Y, Z \in$ Top and let $i : Z \to X$ and $j : Z \to Y$ be morphisms of Top. A *pushout* of X and Y over Z is an object $P \in \mathsf{Top}$ equipped with morphism $k_1 : X \to P$ and $k_2 : Y \to P$ satisfying the universal property expressed by the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "j"'] & X \arrow[d, "k_1"] \arrow[ddr, bend left] & \\ Y \arrow[r, "k_2"'] \arrow[drr, bend right] & P \arrow[dr, dashed, "\exists!"] & \\ & & W \end{tikzcd}$$

---

[24] As before, we will return to pullbacks and pushouts when we discuss categories. See Example 3.6.6 3.

Construction 1.2.37. Let X, Y, and Z be topological spaces, and let $f : X \to Z$ and $g : Y \to Z$ be continuous maps. The *pullback*[25] $X \times_Z Y$ of X and Y over Z[26] is the subspace of the product

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

Construction 1.2.38. Let X, Y, and Z be topological spaces, and let $i : Z \to X$ and $j : Z \to Y$ be continuous maps. The *pushout* $X \amalg_Z Y$ of X and Y over Z[27] is the quotient of the coproduct

$$(X \amalg Y)_{/\sim}$$

by the equivalence relation generated by $i(z) \sim j(z)$ for $z \in Z$.

Proposition 1.2.39. *Let X, Y, and Z be topological spaces, and let $f : X \to Z$ and $g : Y \to Z$ be continuous maps.*

1. *For any two pullbacks P, $p_1 : P \to X$, $p_2 : P \to Y$, and Q, $q_1 : Q \to X$ and $q_2 : Q \to Y$, there is a unique homeomorphism from P to Q which commutes with the p_i and q_i.*
2. *Construction 1.2.37 satisfies Definition 1.2.35.*

Proof. Part (1) is the usual routine. By Categorical Definition 1.2.35, there are unique morphisms $h : P \to Q$ and $k : Q \to P$ which commute with the p_i's and q_i's. The composite $k \circ h$ commutes with the p_i's, and so by the uniqueness guaranteed by Categorical Definition 1.2.35 must equal id_P. Similarly, $h \circ k = \mathrm{id}_Q$, and so h and k are mutually inverse homeomorphisms.

Part (2) follows by considering the corresponding statements for products and subspaces. Let $W \in \mathsf{Top}$, and let $q_1 : W \to X$ and $q_1 : W \to Y$ be two maps. By Categorical Definition 1.2.10, there is a unique map $u : W \to X \times Y$ that commutes with the p_i's and q_i's. Moreover, the underlying map of sets of u factors through the subset $X \times_Z Y$ if and only if $f \circ q_1 = g \circ q_2$, and so by Categorical Definition 1.2.18, there is a unique continuous map $\overline{u} : W \to X \times_Z Y$ making the desired diagram commute. □

Exercise 1.2.40. Formulate and prove the dual statement (i.e., the analogue for pushouts) of Proposition 1.2.39.

The pushout has a particularly nice visual interpretation as a *gluing*: The idea is that we first take $X \amalg Y$ — a space which has two unrelated pieces, and then glue them together according to the maps provided.

Examples 1.2.41.

1. Consider the diagram

$$\begin{tikzcd}
\{0,1\} \arrow[r, hook] \arrow[d] & {[0,1]} \\
*
\end{tikzcd}$$

---

[25] Also sometimes called the *fiber product* or *fibered product*.

[26] ⚠ This is misleading terminology and notation, though extremely common, since the pullback depends on f and g, rather than only depending on the spaces X, Y, and Z.

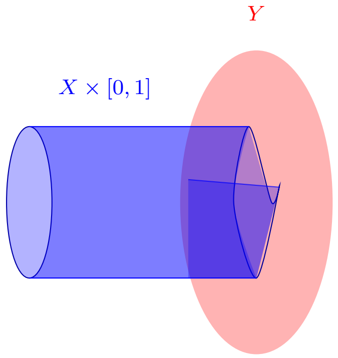
[27] ⚠ The previous warning about pullbacks applies here as well. The notation is misleading, and the pushout fundamentally depends on the maps i and j.

where $*$ represents the one-point space, and $\{0,1\}$ is equipped, equivalently, with either the discrete topology, or the subspace topology inherited from $[0,1]$. Then the pushout is (homeomorphic to) $[0,1]$ quotiented by the relation $0 \sim 1$, and thus is homeomorphic to S^1.

2. Let $f : X \to Y$ be a continuous map of topological spaces. The *mapping cylinder* M_f of f is the pushout of the diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, hook] \arrow[d, "f"'] & X \times [0,1] \\ Y & \end{tikzcd}$$

We can picture the mapping cylinder as follows



Remark 1.2.42. Effectively, the universal property of pushouts (Definition 1.2.36) is the reason that we can define functions piecewise. If, for example, we note that for any $b \in \mathbb{R}$

$$\mathbb{R} \cong (-\infty, b] \amalg_{\{b\}} [b, \infty)$$

then Definition 1.2.36 tells us that we can define a continuous function f on all of $\mathbb{R}$ uniquely by specifying continuous functions on $(-\infty, b]$ and $[b, \infty)$, and checking that they agree on b.

—— fifth lecture on 27. April

1.2.4 Directed (co)limits (direct and inverse limits)

Our final dual pair of constructions are fundamentally infinite in nature. These two constructions, and particularly directed colimits, are the key tool by which infinite, iterative constructions of topological spaces are achieved, as we will see in the next section.[28]

Definition 1.2.43. A *directed system* of topological spaces consists of

1. An object $F_i \in \mathsf{Top}$ for each $i \in \mathbb{N}$.
2. Whenever $i \leq j$ a morphism $f_{i,j} : F_i \to F_j$ in Top.

subject to the conditions

1. The morphisms $f_{i,i}$ are identity morphisms.
2. For $i \leq j \leq k$, we have
$$f_{i,k} = f_{j,k} \circ f_{i,j}.$$

Note that these conditions imply that the directed system is completely and uniquely determined by the spaces F_i and the morphisms $f_{i,i+1} : F_i \to F_{i+1}$. We will abbreviate the latter by f_i, and write $F := (\{F_i\}, \{f_i\})$ for the directed system.

An *inverse system* of topological spaces is the dual notion, given by reversing the directions of all of the $f_{i,j}$'s.

---

[28] See Example 3.6.6 (4) for the categorical approach to this construction.

Our notation for directed and inverse systems is meant to suggest something that we will see later: that these *functors* from $\mathbb{N}$ to Top. Compare with Definition 1.2.17.

Definition 1.2.44. A *cone under* a directed system F is an object $L \in \mathsf{Top}$ equipped with morphisms $\ell_i : F_i \to L$ for all $i \in \mathbb{N}$, and such that, for all $i \leq j$ in $\mathbb{N}$, $\ell_j \circ f_{i,j} = \ell_i$. Equivalently, this can be visualized as a commutative diagram

$$\begin{tikzcd}
F_1 \arrow[r, "f_1"] \arrow[rrrd, "\ell_1"'] & F_2 \arrow[r, "f_2"] \arrow[rrd, "\ell_2"] & F_3 \arrow[r] \arrow[rd, "\ell_3"] & \cdots \\
 & & & L
\end{tikzcd}$$

A *colimit*[29] of the directed system F is a cone $(L, \{\ell_i\})$ such that, for every other cone $(M, \{m_i\})$ under F, there is a unique morphism $g : L \to M$ such that, for every $i \in \mathbb{N}$, $g \circ \ell_i = m_i$.

A *cone over* an inverse system F is the dual notion to a cone under a directed system, and a *limit* of an inverse system is the dual notion of a colimit of a directed system.

We can now give an explicit characterization of direct and inverse limits.

Construction 1.2.45. Given a directed system F, we define a space $\operatorname{colim}_{\mathbb{N}} F$ to be the quotient space

$$\operatorname*{colim}_{\mathbb{N}} F := \left(\coprod_{i \in \mathbb{N}} F_i \right)_{\sim}$$

by the equivalence relation generated by $x \sim f_{i,j}(x)$ for $i, j \in \mathbb{N}$ and $x \in F_i$. There is a canonical cone $(\operatorname{colim}_{\mathbb{N}} F, \{\iota_i\})$ under F whose components are given by the composites

$$F_i \longrightarrow \textstyle\coprod_{i \in \mathbb{N}} F_i \longrightarrow \operatorname{colim}_{\mathbb{N}} F$$

of the quotient map with the summand inclusions.

Proposition 1.2.46. *Given a directed system F, the cone $(\operatorname{colim}_{\mathbb{N}} F, \{\iota_i\})$ is a colimit of F.*

Proof. Given another cone $(L, \{\ell_i\})$ under F, we need to construct a unique morphisms $g : \operatorname{colim}_{\mathbb{N}} \to L$ which commutes with the ℓ_i's and ι_i's. We first note that, by Definition 1.2.11 (the universal property of the coproduct), there is a unique morphism

$$\widetilde{g} : \textstyle\coprod_{i \in \mathbb{N}} F_i \longrightarrow L$$

which commutes with the product. It therefore suffices for us to show that this descends to a unique morphism on the quotient. However, by Definition 1.2.25 (the universal property of the quotient), it suffices to show that $\widetilde{g}$ descends uniquely on the level of sets, i.e., that $\widetilde{g}$ respects the equivalence relation. However, by construction, we have that, for $x \in F_i$ and $i < j$,

$$\widetilde{g}(\iota_j(f_{i,j}(x))) = \ell_j(f_{i,j}(x)) = \ell_i(x) = \widetilde{g}(\iota_i(x))$$

and so $\widetilde{g}$ respects the equivalence relation, completing the proof. ☐

[29] In older references, this is sometimes called a *direct limit*, but this terminology should be avoided if at all possible.

Construction 1.2.47. Given an inverse system F we define a space $\lim_{\mathbb{N}} F$ to be the subspace

$$\lim_{\mathbb{N}} F := \left\{ (x_i) \in \prod_{\mathbb{N}} F_i \;\middle|\; f_{i,j}(x_j) = x_i \right\} \subset \prod_{i \in \mathbb{N}} F_i.$$

There is a canonical cone $(\lim_{\mathbb{N}} F, \{\pi_i\})$ over F whose components are given by the composites

$$\lim_{\mathbb{N}} F \longrightarrow \textstyle\prod_{i \in \mathbb{N}} F_i \longrightarrow F_j$$

of the subspace inclusion with the product projections.

Proposition 1.2.48. *Given an inverse system F, the cone $(\lim_{\mathbb{N}} F, \{\pi_i\})$ is a limit of F.*

Proof. Dual to the proof of Proposition 1.2.46. □

Exercise 1.2.49. Show that colimits of directed systems and limits of inverse systems are unique up to unique homeomorphism which commutes with the defining cones.

Remark 1.2.50. In the case where we are given a directed system F consisting of subspace inclusions

$$F_1 \subset F_2 \subset F_3 \subset \cdots$$

then the underlying set of $\operatorname{colim}_{\mathbb{N}} F$ can be identified with $\bigcup_{i \in \mathbb{N}} F_i$, and a set U is open in the topology of $\operatorname{colim}_{\mathbb{N}} F$ if and only if $U \cap F_i$ is open for every i. It is common to write $\bigcup_{i \in \mathbb{N}} F_i$ for the colimit in this instance.

Warning 1.2.51. ⚠ Given a fixed space X and a directed system of subspace inclusions

$$F_1 \subset F_2 \subset F_3 \subset \cdots \subset X,$$

the topology on $\operatorname{colim}_{\mathbb{N}} F$ may not be the same as the subspace topology on $\bigcup_{i \in \mathbb{N}} F_i \subset X$.

Example 1.2.52 (The Cantor Set)**.** We define an inverse system of subspaces $K_i \subset [0,1] \subset \mathbb{R}$ recursively.

- We set $K_0 = [0,1]$.
- We set
$$K_{i+1} = \frac{1}{3}K_i \cup \left(\frac{2}{3} + \frac{1}{3}K_i\right)$$

This construction of K_i amounts to taking the (closed) upper and lower thirds of every closed interval in K_i. The inclusions form an inverse system of topological spaces, and we define the Cantor set C to be the limit of the K_i. This can be identified with the subspace of $\mathbb{R}$ whose underlying set is the intersection of the K_i. One can show that the Cantor set has the cardinality of $\mathbb{R}$.

1.2.5 CW complexes

We now leverage the constructions we have provided to give a description of a particularly useful class of spaces: *CW complexes*. These, loosely speaking are the spaces that can be build by gluing together n-dimensional disks.

Notation 1.2.53. We denote by D^n the closed unit disk in $\mathbb{R}^n$:

$$D^n := \{x \in \mathbb{R}^n \quad | \quad |x| \leq 1\}$$

There is a canonical inclusion $S^{n-1} \hookrightarrow D^n$ which views S^{n-1} as the 'boundary'[30] of D^n.

Notice that $D^0 = *$ is the one-point space. We fix the convention that $S^{-1} = \varnothing$, as the "boundary" of D^0.[31]

Definition 1.2.54. A *CW complex* K consists of the following data

- For every $n \in \mathbb{N}$, a set $\mathrm{Cell}_n(K)$, called the *set of n-cells* of K.
- A sequence of topological spaces spaces K_n for $n \geq -1$ with $K_{-1} = \varnothing$.
- For every $n \in \mathbb{N}$ and every $i \in \mathrm{Cell}_n(K)$, a continuous map

  $$\iota_i : S^{n-1} \longrightarrow K_{n-1}$$

  called the *attaching map* of the cell i.
- For each $n \in \mathbb{N}$, pushout diagrams

  $$\begin{tikzcd} \coprod_{i\in \mathrm{Cell}_n(K)} S^{n-1} \arrow[r, "\amalg \iota_i"] \arrow[d, hook] & K_{n-1} \arrow[d, "j_n"] \\ \coprod_{i\in \mathrm{Cell}_n(K)} D^{n} \arrow[r] & K_n \end{tikzcd}$$

We call the space

$$|K| = \operatorname*{colim}_{\mathbb{N}} K_n = \bigcup_{i \in \mathbb{N}} K_n$$

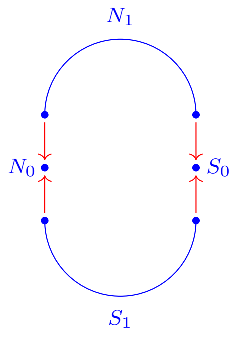
the *realization* of the complex K. We call a CW complex *n-dimensional* if $\mathrm{Cell}_m(K) = \varnothing$ for $m > n$.

In a sense, a CW complex is a "recipe" for building a space. We start, *tabula rasa*, with an empty space K_{-1}. We then add a (possibly infinite) collection of points (0-disks), connect these points with a (possibly infinite) collection of line segments (1-disks), and so on. If there are an infinite number of steps, we combine them all into one space by taking a "union" (directed colimit).

Example 1.2.55. There is a standard (finite-dimensional) CW complex structure on the n-sphere S^n given as follows. In each dimension i less than or equal to n, there are two cells, which I will call N_i and S_i. In the case where $n = 0$, this gives the 0-sphere, which is simply two points, without any further data. We then inductively suppose that the cells up to dimension n give us the n-sphere, and define the attaching maps for N_{n+1} and S_{n+1} to be the identity maps $\mathrm{id} : S^n \to S^n$.

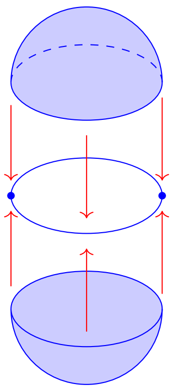
We can illustrate the CW-structure of the n-sphere from Example 1.2.55 as follows. We first add the cells N_0 and S_0, to get the 0-sphere.

We then attach the two 1-cells (intervals) as follows



to get the circle S^1.

Next, we attach a pair of 2-cells (2-dimensional disks)



to obtain the sphere S^2.

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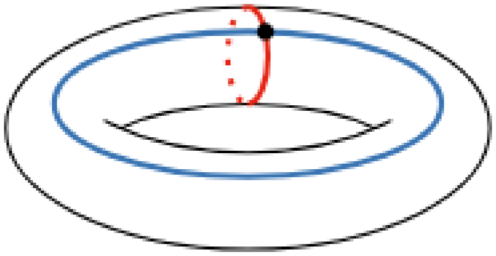
[30] We will shortly give a topological definition of the boundary, but it is not necessary for our purposes here

[31] Note that this convention accords with our definition of S^n. Since there are no points of norm 1 in $\mathbb{R}^0$, $S^{-1} = \varnothing$.

Example 1.2.56. There is a standard CW structure on the torus $S^1 \times S^1$ consisting of one 0-cell, two 1-cells, and one 2-cell. We can picture this as



(Image created by Ulrich Bauer)

—— Sixth lecture on 28. April

1.3 First properties of topological spaces

Now that we have built up a small bestiary of topological spaces, we begin to explore the properties these spaces can have and the construction which can take place internal to a given space.

1.3.1 Interior, closure, and boundary

Already intuitively familiar from analysis are the interrelated notions of interior, closure, and boundary. Our task now is simply to understand how best to formulate them in terms of open and closed sets.

Definition 1.3.1. Let X be a topological space, and $A \subset X$ a subset.

1. The *interior of A in X*, denoted by $\mathring{A}$, is the largest open set of X contained in A, or, equivalently, the union of all open sets of X contained in A.
2. The *closure of A in X*, denoted by $\overline{A}$, is the smallest closed set containing A, or, equivalently, the intersection of all closed sets of X containing A.
3. The *boundary of A in X*, denoted by ∂A, is the set-theoretic difference $\partial A = \overline{A} \setminus \mathring{A}$.

Lemma 1.3.2. *Let X be a topological space and $A \subset X$. Then*

1. $\partial A = \overline{A} \cap \overline{X \setminus A}$.
2. *$x \in \overline{A}$ if and only if, for every open set U containing X, $U \cap A \neq \varnothing$.*
3. *$x \in \mathring{A}$ if and only if there is an open set U with $x \in U \subset A$.*

Proof. Exercise. □

Definition 1.3.3. A subset $A \subset X$ of a topological space X is called

1. *dense* if $\overline{A} = X$, and

For any topological space, there is a way of constructing a preorder from that space which has a close connection to the construction of topologies from posets.

Example. Let X be a topological space. The *specialization preorder on X* is a preorder on the underlying set of X defined by declaring $x \leq y$ if $x \in \overline{y}$.

This construction, in fact, defines an *adjoint functor* (a topic we will not explore here, but about which there is copious literature) to a functor from preorders to topological spaces defined analogously to Example 1.1.3. In fact, we can consider preorders to be a special case of topological spaces.

2. *nowhere dense* if $\overline{\mathring{A}} = \varnothing$.

Remark 1.3.4. Let (X, d) be a metric space, and $A \subset X$ be a subset. Then the metric definitions of closure, interior, dense, and nowhere dense agree with the topological definitions.

1.3.2 Connectedness and path-connectedness

Above, we constructed the *disjoint union* of topological spaces, $X \amalg Y$, which can be viewed as consisting of two separate 'parts': X and Y. However, given a topological space (X, τ_X), we do not yet have any way of testing whether it has been built in this way. Such a criterion is provided by notions of *connectedness.*

Definition 1.3.5. Let X be a topological space. If, for every pair of non-empty open sets $U, V \subset X$ such that $U \cup V = X$ the intersection $U \cap V \neq \emptyset$, we say that X is connected. A *connected component* of X is a maximal connected subspace $Y \subset X$.

Lemma 1.3.6. *A topological space X is connected if and only if it has a single connected component: X.*

Proof. Left as an exercise for the reader. ☐

Remark 1.3.7. Notice that if X is not connected, then there are non-empty open sets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V = \varnothing$. As such, U and V are both open and closed. Indeed, this is an equivalent characterization of connectedness.

Lemma 1.3.8. *A topological space X is not connected if and only if it is homeomorphic to a disjoint union $X_0 \amalg X_1$ of spaces.*

Proof. If $X \cong X_0 \amalg X_1$, then it is clear that the open subsets X_0 and X_1 display X as disconnected.

If, on the other hand, X is not connected, let $Y \subsetneq X$ be a proper subset which is both open and closed. Suppose $U \subset X$ is a subset such that $U \cap Y$ and $U \cap (X \setminus Y)$ are open in the subspace topologies. Then, since Y and $(X \setminus Y)$ are open, $U \cap Y$ and $U \cap (X \setminus Y)$ are open in X, and so their union, U must be open as well. Thus U is open if and only if $U \cap Y$ and $U \cap (X \setminus Y)$ are open in the subspace topologies, and so $X \cong Y \amalg (X \setminus Y)$ as desired. ☐

Examples 1.3.9.

1. A subspace of $\mathbb{R}$ is connected if and only if it is an open, closed, or half-open interval.

2. Any open or closed ball in $\mathbb{R}^n$ is connected.

In a sense made precise by the following proposition, connectedness measures 'discreteness of maps out of X'.

Proposition 1.3.10. *Let X be a topological space. The following are equivalent*

1. *X is connected.*
2. *Every continuous map $f : X \to Y$ to a discrete space is constant.*

Proof. We first show 2. $\Rightarrow$ 1. Suppose that X is not connected. Then there are two non-empty open sets $U, V \subset X$ with $U \cup V = X$ and $U \cap V = \emptyset$. Define a map to $f : X \to \{0,1\}$ by sending every element of U to 0, and every element of V to 1. It is immediate from the definition that f is continuous with respect to the discrete topology on $\{0,1\}$ and non-constant.

We now show 1. $\Rightarrow$ 2. Suppose that there is a continuous, non-constant map $f : X \to Y$, where Y is equipped with the discrete topology. In particular, choose two distinct elements y_0 and y_1 in Y such that both are in the image of f. Choose any map of sets $p : Y \to \{0,1\}$ such that $p(y_0) = 0$ and $p(y_1) = 1$. Since this is continuous with respect to the discrete topologies, we get a non-constant continuous map $p \circ f : X \to \{0,1\}$. Since this is continuous, the sets $U := (p \circ f)^{-1}(0)$ and $V := (p \circ f)^{-1}(1)$ are open. Since $p \circ f$ is non-constant, both U and V are non-empty. By definition $U \cup V = X$ and $U \cap V \neq X$. $\square$

Definition 1.3.11. Let (X, τ_X) be a topological space, and let $A \subset X$. We define the *closure* of A to be the subset $\overline{A} \subset X$ which is the intersection of all closed subsets of X which contain A.

Lemma 1.3.12. *Let $f : X \to Y$ be a continuous map, and let X be connected. Then $f(X) \subset Y$ is connected.*

Proof. Let $U, V \subset Y$ be open subsets with $f(X) \subset U \cup V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of X whose union equals X. As such, there is an $x \in f^{-1}(U) \cap f^{-1}(V)$ since X is connected, and thus, $f(x) \in U \cap V$, so $f(X)$ is connected. $\square$

Proposition 1.3.13. *Let X be a topological space, and A a connected subset. If $B \subset X$ such that $A \subset B \subset \overline{A}$, then B is connected.*

Proof. Suppose that there were two open sets $U, V \subset X$ such that $U \cup V = B$ and $U \cap V \cap B = \emptyset$. Since A is connected, we must then have that $A \subset U$ or $A \subset V$. WLOG, assume $A \subset U$. But then, for $b \in B$, we have that $b \in \overline{A}$ and V is an open subset containing b. Therefore, $V \cap A \neq \emptyset$, and thus, $V \cap U \cap B \neq \emptyset$, which is a contradiction. $\square$

Definition 1.3.14. A topological space X is called *totally disconnected* if every connected component is a singleton subset.

Example 1.3.15. The Cantor set of Example 1.2.52 is totally disconnected.

So if connectedness measures the discreteness of maps *out* of X, can we also measure the discreteness of maps *into* X?

Definition 1.3.16. A *path* in a topological space X is a continuous map $\alpha : [0,1] \to X$, where $[0,1]$ is equipped with the subspace topology inherited from $\mathbb{R}$. We say that α is a *path from x to y* if $\alpha(0) = x$ and $\alpha(1) = y$.

We define an equivalence relation on X by $x \sim y$ if and only if there exists a path in X from x to y. A *path component of* x is an equivalence class $[x] \in X_{/\sim}$, viewed as a subspace $[x] \subset X$. We denote by $\pi_0(X)$ the set of path components of X. We say that X is *path connected* if $\pi_0(X)$ is a singleton.

Proposition 1.3.17. *The relation $\sim$ of Definition 1.3.16 is indeed an equivalence relation on X.*

Proof. Reflexivity follows immediately from the existence of constant paths. Given $x \in X$, there is a path

$$c_x : [0,1] \longrightarrow X$$

from x to x which sends every point of $[0,1]$ to x.[32]

Given a path α from x to y and a path β from y to z, we can define a path $\beta * \alpha$ as a piecewise function

$$\beta * \alpha(t) = \begin{cases} \alpha\,(2) & t \in \left[0, \frac{1}{2}\right] \\ \beta(2(t-1)) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

which goes from x to z, showing transitivity.[33]

Finally, for symmetry, notice that the map

$$\begin{aligned} \tau : [0,1] &\longrightarrow [0,1] \\ t &\longmapsto 1-t \end{aligned}$$

is continuous, and if α is a path from x to y, then $\alpha \circ \tau$ is a path from y to x. ☐

Proposition 1.3.18. *Let X be a path-connected topological space. Then X is connected.*

Proof. Suppose X is not connected. Then there exists a continous map $f : X \to \{0,1\}$ (where $\{0,1\}$ is equipped with the discrete topology) such that f is non-constant. Let $x \in f^{-1}(0)$ and $y \in f^{-1}(1)$. A path $\alpha : [0,1] \to X$ from x to y would then yield a continuous, non-constant map $f \circ \alpha : [0,1] \to \{0,1\}$. Since $[0,1]$ is connected, this cannot occur, and thus, X is not path connected. ☐

Warning 1.3.19. ⚠ The converse of Proposition 1.3.18 is *not* true. There are connected spaces which are not path-connected. We need to make additional assumptions about our space X if we want connectedness and path-connectedness to be equivalent.

Definition 1.3.20. We call a topological space X *locally path-connected* if, for every $x \in X$ and every open U containing x, there is an open V with $x \in V \subset U$ such that V is path connected.

Proposition 1.3.21. *If X is a connected, locally path-connected topological space, then X is path-connected.*

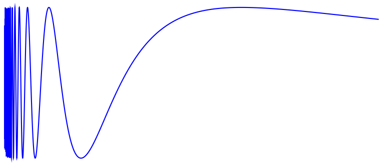
A standard counterexample in topology is the *topologist's sine curve*. Let $X \subset \mathbb{R}^2$ be the collection of all points $(x, \sin(\frac{1}{x}))$ for $x > 0$, together with the segment of the y-axis from -1 to 1. This inherits a topology from $\mathbb{R}^2$.

Lemma. *The topologist's sine curve is connected.*

Proof. We simply need note that the subspace $A := \{(x, \sin(\frac{1}{x})) \mid x > 0\}$ is path-connected, and thus connected. Moreover $A \subset X \subset \overline{A}$. Therefore, by Proposition 1.3.13, X is connected. ☐

Lemma. *The topologist's sine curve is not path connected.*

Proof. Suppose we have a path $p : [0,1] \to X$ going from $(1, \sin(1))$ to $(0,0)$. Consider the component functions $p_x, p_y : [0,1] \to \mathbb{R}$. Since p_x is continuous, its image is connected, and therefore is the interval $[0,1]$. But then, p is the map $t \mapsto (t, \sin(\frac{1}{t}))$. But for every $\delta > 0$, there is a $0 < t < \delta$ such that $\sin(\frac{1}{t}) = 1$, i.e $|p(t) - (0,0)| > 1$. Therefore, p cannot be continuous. ☐



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[32] This is clearly continuous, since the preimage of any set U are either $[0,1]$ itself or $\varnothing$, depending on whether U contains x.

[33] For continuity, see Remark 1.2.42.

Proof. Let $x \in X$, and denote by $[x] \subset X$ the path component of x in X. We claim that $[x]$ is both open and closed, so since $[x]$ is non-empty and X is connected, $[x]$ must equal X.

To see that $[x]$ is closed, let $y \in \overline{[x]}$, and use the locally path-connected property of X to choose an open path-connected subset U containing y. Then $Y \cap [x]$ is non-empty, and so by the transitivity of the path equivalence relation $y \in [x]$. Thus, $\overline{[x]} = [x]$, and so $[x]$ is closed.

To see that $[x]$ is open, let $y \in [x]$, and choose an open path-connected subset U containing Y. Then by the transitivity of the path equivalence relation $U \subset [x]$, and so every point of $[x]$ has an open neighborhood contained in $[x]$. Thus $[x]$ is open. ☐

Proposition 1.3.22. *The assignment* Top $\to$ Set *which sends every space X to its set of path components is* functorial. *That is,*

1. *For every morphism $f : X \to Y$ in* Top*, there is a morphism $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ in* Set.
2. *For the identity morphism $\mathrm{id}_X : X \to X$ in* Top*, $\pi_0(\mathrm{id}_X)$ is the identity morphism on $\pi_0(X)$.*
3. *For composable morphisms*
$$X \xrightarrow{g} Y \xrightarrow{f} Z$$
in Top*,*
$$\pi_0(f \circ g) = \pi_0(f) \circ \pi_0(g).$$

Proof. Since the composition of a path with a continuous map preserves the equivalence relation $\sim$ of Definition 1.3.16, there is a unique map $\pi_0(f)$ induced by each continuous f which sends $[x] \mapsto [f(x)]$, showing part (1). To see part (2), note that

$$\pi_0(\mathrm{id}_X)([x]) = [\mathrm{id}_X(x)] = [x].$$

Finally, to see part (3), notice that

$$\pi_0(f \circ g)([x]) = [f(g(x))] = \pi_0(f)([g(x)]) = \pi_0(f)(\pi_0(g)([x])).$$ ☐

2

Classical topology

—— Seventh lecture on 4. May

We now move from first constructions to the exploration of key theorems in point-set topology. Most of the material we will cover here is very classical in flavor, and, as such, the category-theoretic considerations introduced in the first chapter will fall briefly by the wayside. They will, however, return when we begin our exploration of homotopy theory.

2.1 Separation axioms

Our first topic in this section will be conditions on topological spaces which help to ensure that they better match our intuition. To get a sense of why we might want this, let us first consider a counter-example.

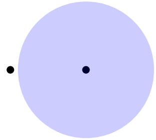
Example 2.1.1. Consider any set X equipped with the *indiscrete topology*: the topology in which on $\varnothing$ and X are open. In this topology, the open sets tell us nothing about the points of X — every point of X is contained in exactly the same open sets.

Definition 2.1.2. Given a topological space X, we call a subset $U \subset X$ containing a point $x \in X$ and such that there is an open V in X with $x \in V \subset U$ a *neighborhood of* x. We call U an *open neighborhood* or a *closed neighborhood* of x when U is open or closed, respectively. We write N_x for the set of neighborhoods of x in X.

Two points $x, y \in X$ are called *topologically distinquishable* if $N_x \neq N_y$.

Our first separation axiom will be the mildest

Schematically, we can draw the T_0 property as



Definition 2.1.3. A topological space X is called T_0 or a *Kolmogorov space* if, for every two points $x_1, x_2 \in X$, there exists $i \in \{1, 2\}$ and an open neighborhood U of x_i which does not contain the other point.

Lemma 2.1.4. *For a topological space X, the following are equivalent.*

1. *The space X is T_0.*
2. *Any two points of X are topologically distinguishable.*

Proof. It is immediate that in a T_0 space, every two points are topologically distinguishable. Suppose on the other hand that every two points of X are topologically distinguishable. For $x, y \in X$, there is then (without loss of generality) a neighborhood C of X which is not a neighborhood of y. In particular, there is an open set V with $x \in V \subset C$. Since C is not a neighborhood of y, $y \notin V$, and so X is T_0. ☐

Lemma 2.1.5. *Let $(P, \leq)$ be a poset, and let $\tau_{\leq}$ be the poset topology on P.*[1] *Then $(P, \tau_{\leq})$ is T_0.*

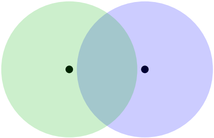
Proof. Suppose that $x, y \in P$ such that every open set containing one contains the other. Since the sets

$$\{z \in P \mid z \leq x\} \quad \text{and} \quad \{z \in \mid z \leq y\}$$

are open, it follows that $x \leq y$ and $y \leq x$, and so $x = y$. ☐

Definition 2.1.6. A topological space is called T_1, or *Fréchet*[2] if, for every $x, y \in X$, there exists open neighborhoods U_x of x and U_y of y such that $x \notin U_y$ and $y \notin U_x$.

Schematically, we can draw the T_2 property as



Lemma 2.1.7. *For a topological space X, the following are equivalent.*

1. *The space X is T_1.*
2. *Every singleton $\{x\} \subset X$ is closed.*
3. *Every subset $A \subset X$ is the intersection of all open sets containing A.*

Proof. To see that (1)⇒(2), let $x \in X$, and for every other point $y \in X \setminus \{x\}$, choose an open neighborhood V_y of y such that $x \notin V_y$. Then

$$X \setminus \{x\} = \cup_{y \in X \setminus \{x\}} V_y$$

is open, and so $\{x\}$ is closed.

To see that (2)⇒(3), let $A \subset X$. Then for $x \in X \setminus A$, $X \setminus \{x\}$ is open. Since

$$A = \bigcap_{x \in X \setminus A} X \setminus \{x\}$$

we have in particular that A is the intersection of the open sets containing A.

Finally, to see that (3)⇒(1), let $x \neq y$. Then since

$$\{x\} = \bigcap_{x \in U \overset{\circ}{\subset} X} U$$

there must be at least one open neighborhood V of x which does not contain y. ☐

Example 2.1.8. There are poset topologies which are not T_1. If, for example $(P, \leq)$ has an element x which is greater than every other element, then every open set containing x will contain every other point.

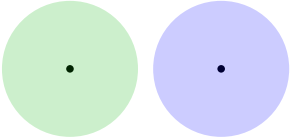
---

[1] See Example 1.1.3.

[2] Note that this has nothing to do with the functional-analytic notion of a Fréchet space.

Example 2.1.9. Let X be a set. The *cofinite topology* on X is the topology given by declaring a set U to be open if and only if $X \setminus U$ is finite. The cofinite topology on an infinite set is always T_1.

Schematically, we can draw the Hausdorff property as



Definition 2.1.10. A topological space X is called T_2 or *a Hausdorff space* if, for every $x \neq y$ in X, there are open neighborhoods U of x and V of y such that $U \cap V = \varnothing$.

Lemma 2.1.11. *For a topological space X, the following are equivalent.*

1. *The space X is T_2.*
2. *Every point is the intersection of all its closed neighborhoods.*

Proof. To show that (1)⇒(2), it suffices to show that for every x and every $y \neq x$, there is a closed neighborhood of x not containing y. Let U and V be the open neighborhoods of x and y guaranteed by the Hausdorff property. Then $x \in U \subset X \setminus V$, so $X \setminus V$ is a closed neighborhood of x not containing y.

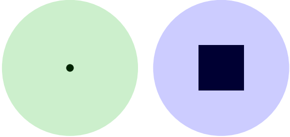
To show that (2)⇒(1), let $x, y \in X$ with $x \neq y$ and choose a closed neighborhood C of x not containing y. Then there is an open U with $x \in U \subset C$, and so U and $X \setminus C$ are the desired open neighborhoods in the Hausdorff property. □

Remark 2.1.12. One key takeaway from the preceding lemma is that points are closed in Hausdorff spaces. In particular, every Hausdorff space is T_1.

Example 2.1.13. Every metric space (X, d) is a Hausdorff space. Every CW complex is a Hausdorff space.

Warning 2.1.14. ⚠ Our next few separation axioms create problems, because the terminology is not standardized. I am opting to use the same terminology as the previous version of this course, but bear in mind that in many sources (including Wikipedia), the meaning of the terms, e.g., T_3 and *regular* are reversed.

Schematically, we can draw the T_3 property as

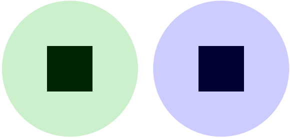


Definition 2.1.15. A topological space X is called T_3 if, for every closed set $C \subset X$ and every point $x \in X \setminus C$, there exists open sets $U, V \subset X$ with $x \in U$, $C \in V$, and $U \cap V \neq \varnothing$.

The space X is called *regular* if it is T_3 and T_1 (or, equivalently, if it is T_3 and Hausdorff).

Remark 2.1.16. Note that T_3 does not imply Hausdorff, precisely because it may not guarantee that singletons are closed.

Schematically, we can draw the T_4 property as



Definition 2.1.17. A topological space X is called T_4 if, for any closed sets $C, K \subset X$ such that $C \cap K = \varnothing$, then there are open sets U and V, which respectively contain C and K and such that $U \cap V = \varnothing$.

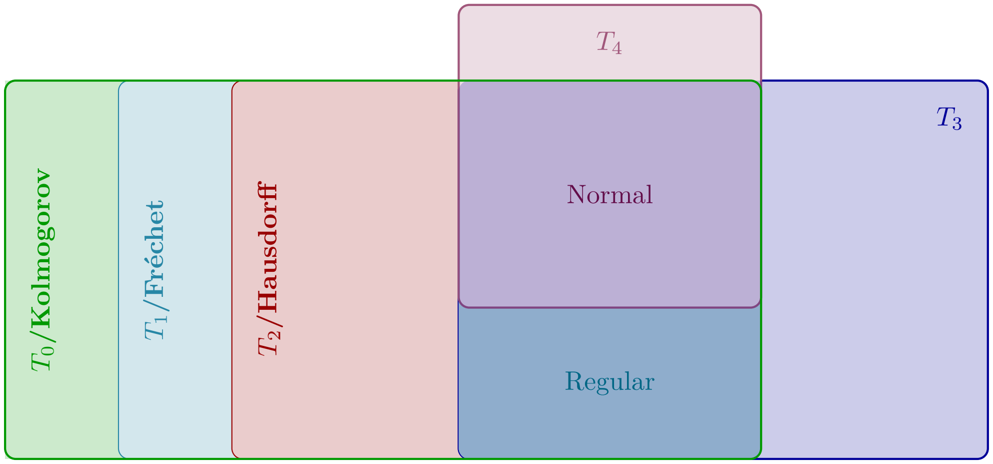
The space X is called *normal* if it is T_4 and T_1 (or, equivalently, T_4 and Hausdorff, or, equivalently, T_4 and regular).

Example 2.1.18. Every CW-complex is normal. We will see later that every metric space is also normal.

Lemma 2.1.19. *A topological space X is T_4 if and only if, for every pair of disjoint closed sets A and B in X, there is an open set U containing A such that $\overline{U}$ is disjoint from B.*

Proof. This is effectively just a reformulation. If X satisfies the property specified in the lemma, we can set $V = X \setminus \overline{U}$ to obtain the T_4 property. On the other hand, if U and V are the sets guaranteed by the T_4 property, $X \setminus V$ is closed, and so $\overline{U} \subset X \setminus V \subset X \setminus B$. ☐

Before we get to our first major theorem of the course, let us briefly provide a graphical summary of the implications among the various separations axioms.



2.1.1 *The Tietze Extension Theorem*

We now come to the core of this section: the theorems of Urysohn and Tietze.

Definition 2.1.20. Let X be a topological space, and let $A, B \subset X$ be disjoint closed subsets of X. An *Urysohn function* is a continuous function

$$f : X \longrightarrow [0,1]$$

such that $f(A) = 0$ and $f(B) = 1$

Our first technical result which will pave the way to proving Urysohn's Metrization Theorem is the following proposition.[3]

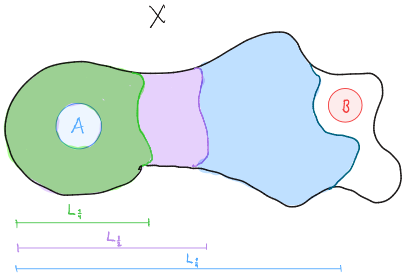
Proposition 2.1.21 (Urysohn's Lemma)**.** *A topological space is T_4 if and only if for any closed, disjoint subsets $A, B \subset X$, there is an Urysohn function for A and B.*

[3] The grounds for calling this proposition "Urysohn's Lemma" are somewhat obscure — likely because it is now seen as a simple helpmeet in the proof of the Tietze Extension Theorem and Urysohn's Metrization Theorem. However, the proof is somewhat technical, and in Urysohn's original paper proving it (*Über die Mächtigkeit der zusammenhängenden Mengen*, 1925) the statement appears as a "Satz" (Theorem, or Proposition) rather than as a "Hilfssatz" (Lemma).

Proof. If, for any such $A, B \subset X$ there exists an Urysohn function $f : X \to [0,1]$, then $f^{-1}([0,\frac{1}{2}))$ and $f^{-1}((\frac{1}{2},1])$ are the disjoint open sets containing A and B required for X to be T_4.

If, on the other hand, X is T_4, the proof is more involved. The strategy will be to define a countable sequence of open subsets of X which contain A and expand out to "almost fill up" $X \setminus B$. These subspaces will be indexed by binary fractions in the interval $[0,1]$. From such a sequence, we then construct a function by using the density of binary fractions in the unit interval.

We can picture the first few steps of this construction as follows.



Constructing the spaces: We first construct a space $L_{\frac{1}{2}}$. Using Lemma 2.1.19, there is an open set $L_{\frac{1}{2}}$ such that

$$A \subset L_{\frac{1}{2}} \subset \overline{L_{\frac{1}{2}}} \subset X \setminus B.$$

We then apply Lemma 2.1.19 twice, (to the pairs $(A, X \setminus L_{\frac{1}{2}})$ and $(\overline{L_{\frac{1}{2}}}, B)$) to obtain a sequence

$$A \subset L_{\frac{1}{4}} \subset \overline{L_{\frac{1}{4}}} \subset L_{\frac{1}{2}} \subset \overline{L_{\frac{1}{2}}} \subset L_{\frac{3}{4}} \subset \overline{L_{\frac{3}{4}}} \subset X \setminus B$$

Iterating, we get open subspaces

$$\{L_{\frac{m}{2^n}} \mid m, n \in \mathbb{N} \text{ and } 0 < m < 2^n\}$$

such that for binary fractions $p < q$, we have $\overline{L_p} \subset L_q$.

Defining the function: Write $D \subset (0,1)$ for the set of binary fractions. We define

$$f(x) := \inf(\{p \in D \mid x \in L_p\} \cup \{1\}).$$

We notice that, for $b \in B$, $f(b) = 1$, since b cannot be contained in any of the L_p, and for $a \in A$, $f(a) = 0$ since a is contained in every L_p.

Proving continuity: As the final step, we show that f is continuous. Let $U \subset [0,1]$ be an open set, and let $x \in f^{-1}(U)$. We have three cases:

1. If $f(x) = 0$, then choose $p \in D$ with $[0,p) \subset U$. Then L_p is an open subset containing x which maps into U.
2. If $f(x) = 1$, then choose $p \in D$ with $(p,1] \subset U$. Then $X \setminus \overline{L_p}$ is an open subset containing x which maps into U.
3. If $0 < f(x) < 1$, choose $p, q \in D$ with $p < f(x) < q$ and such that $(p,q) \subset U$. Then $L_q \setminus \overline{L_p}$ is an open subset containing x which maps into U.

thus f is continuous, completing the proof. ☐

Definition 2.1.22. Let X be a topological space, and let $\{f_n : X \to \mathbb{R}\}_{n \in \mathbb{N}}$ be a sequence of continuous real-valued functions on X. We say that the f_n's *converge uniformly* to a function $f : X \to \mathbb{R}$ if, for every $\epsilon > 0$ there is an $m_\epsilon \in \mathbb{N}$ such that, for all $x \in X$ and all $n \geq m$,

$$|f_n(x) - f(x)| < \epsilon.$$

Lemma 2.1.23. *Let X be a topological space, and suppose that $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ is a sequence of functions which converge uniformly to $f : X \to \mathbb{R}$. If the f_n are continuous, then f is continuous.*

Proof. We check continuity on the basis of open ϵ-balls. Consider $y \in \mathbb{R}$ and an ϵ-ball $B_\epsilon(y)$. Let $x \in f^{-1}(B_\epsilon(y))$ and let $r = |f(x) - y| < \epsilon$. For any fixed $\gamma > 0$ we can choose $n \in \mathbb{N}$ and an open subset $U \subset X$ containing x such that (1) for $z \in X$ we have

$$|f_n(z) - f(z)| < \gamma$$

and (2) for $z \in U$ we have $|f_n(z) - f_n(x)| < \gamma$. Thus, for $z \in Z$

$$\begin{aligned} |f(z) - y| &= |f(z) - f_n(z) + f_n(z) - f_n(x) + f_n(x) - f(x) + f(x) - y| \\ &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y| \\ &< 3\gamma + r. \end{aligned}$$

taking γ to be less than $\frac{1}{3}(\epsilon - r)$ thus shows that U is an open set containing x and contained in $f^{-1}(B_\epsilon(y))$, completing the proof. □

Theorem 2.1.24. *Let X be a T_4 topological space, let $A \subset X$ be closed, and let $f : A \to [-1, 1]$ be a continuous function. Then there is a continuous extension of f to X, i.e., a continuous function*

$$F : X \longrightarrow [-1, 1]$$

such that $F|_A = f$.

Proof. We will use Urysohn's lemma to construct a sequence of functions which converge to the desired F. The central notion is an *approximate ϵ-extension* of f: A continuous function g such that, for all $a \in A$,

$$|f(a) - g(a)| \leq \epsilon.$$

The key step of the proof will be to construct a better approximation from a worse one.

IMPROVING EXTENSIONS: Let $g : X \to [-1, 1]$ be an approximate ϵ-extension of f. Consider the closed subsets

$$C = (f - g)^{-1}([-\epsilon, -\frac{1}{3}\epsilon]), \quad D = (f - g)^{-1}([\frac{1}{3}\epsilon, \epsilon])$$

These are precisely the sets such that the error in the approximation is greater than $\frac{\epsilon}{3}$. By Proposition 2.1.21 (The Urysohn Lemma), there is a function

$$\gamma : X \longrightarrow [-\tfrac{\epsilon}{3}, \tfrac{\epsilon}{3}]$$

which takes value $-\frac{\epsilon}{3}$ on C and $\frac{\epsilon}{3}$ on D. Define $h = g + \gamma$. Then for, e.g., $a \in A \cap C$, we have

$$\begin{aligned} f(x) - h(x) &= f(x) - g(x) - \gamma(x) \\ &> -\epsilon + \frac{\epsilon}{3} = -\frac{2\epsilon}{3} \end{aligned}$$

and

$$\begin{aligned} f(x) - h(x) &= f(x) - g(x) - \gamma(x) \\ &< -\frac{\epsilon}{3} + \frac{\epsilon}{3} = 0 \end{aligned}$$

Similar arguments on $A \cap D$ and $A \setminus (C \cup D)$ show that h is an approximate $\frac{2\epsilon}{3}$-extension of f.

The remainder of the proof is easy. We note that the constant zero function $g = 0$ is an approximate 1-extension of f. We then use our step above to improve it to an approximate $\frac{2}{3}$-extension g_1 of f. This is improved to an approximate $(\frac{2}{3})^2$-extension of f, and so on. This yields a sequence $\{g_n\}$ of approximate $(\frac{2}{3})^n$-extensions which converge uniformly to some function F which agrees with f on A. By Lemma 2.1.23, this is continuous, completing the proof. □

2.1.2 Metrization

We now can give a partial answer to the question of when a topology is induced by a metric.

Definition 2.1.25. A topological space X is called *metrizable* if there exists a metric d on X which induces the topology on X.

Lemma 2.1.26. *Every metric space* (X, d) *is normal.*

Proof. It is easy to see that any metric space is Hausdorff. We thus prove only the T_4 axiom. Let $A, B \subset X$ be closed disjoint subsets. We define

$$\begin{aligned} d_A : X &\longrightarrow [0, \infty) \\ x &\longmapsto \inf\{d(a, x) \mid a \in A\} \end{aligned}$$

and note that for $x, y \in X$

$$|d_A(x) - d_A(y)| \leq \inf\{|d(a, x) - d(a, y)| \mid a \in A\} \leq d(x, y).$$

If $d_A(x) = 0$, we can chose a sequence $\{a_n\}_{n\in\mathbb{N}}$ which converges to x. Since A is closed, this implies $x \in A$. As such, since A and B are disjoint, it follows that

$$d_A(x) + d_B(x) > 0$$

for all $x \in X$.

We then define a continuous function $f : X \to [0, 1]$ by

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}$$

which is clearly an Urysohn function for A and B. Thus by Proposition 2.1.21, X is T_4. □

Urysohn's Metrization theorem provides a partial converse, giving us a broad class of examples in which topological spaces admit metrics.

Definition 2.1.27. A topological space X is called *second countable*[4] if there is a countable basis $\mathcal{B}$ for the topology on X.

Example 2.1.28. For any $n \in \mathbb{N}$, the set of balls around rational points with rational radii provides a countable basis for $\mathbb{R}^n$. Similarly, S^n, $\mathbb{R}P^n$, T^2, etc. are second countable. Subspaces of second countable spaces are second countable, as are countable products of second countable spaces.

Theorem 2.1.29 (Urysohn's Metrization Theorem)**.** *If X is a second countable, normal topological space, then X is metrizable.*

Proof. We will prove the theorem as a sequence of claims.

Claim 1: There is a countable collection of continuous functions

$$\{\ f_\alpha : X \to [0,1]\ \}_{\alpha \in A}$$

which *separates points from closed sets.* That is, for every $x \in X$ and $C \subset X$ closed such that $x \notin C$, there is an $\alpha \in A$ such that $f_\alpha(x) \notin \overline{f_\alpha(C)}$.

> *Proof of Claim 1.* Let A be the set of pairs (U, V) of basis elements such that $\overline{U} \subset V$ and for $(U, V) \in A$, let $f_{U,V}$ be an Urysohn function for $\overline{U}$ and $X \setminus V$. We claim that this (countable) set of continuous maps separates points from closed sets. To see this, let $C \subset X$ be closed and let $x \notin C$. Choose a basis element V with $x \in V$ and $V \cap C = \varnothing$. Then $X \setminus V$ is closed, and so we can find a further basis element U with $x \in U$ and $\overline{U} \subset V$. Thus, $f_{(U,V)}$ separates x and C by construction. ■

Claim 2: Let $\{f_\alpha\}_{\alpha \in A}$ be the functions from Claim 1. The product $F = \prod_{\alpha \in A} f_\alpha$ defines an embedding $F : X \to [0,1]^A$.

> *Proof of Claim 2:* Note that by construction, F separates points from closed subsets. Since X is T_1, singletons are closed, and so F separates distinct points, i.e., is injective.
>
> We then show that F is a closed map onto its image. Let $C \subset X$ be closed, and suppose $x \in X$ such that $f(x) \in \overline{F(C)}$. This implies $x \in C$, since F separates points from closed sets. Thus, $F(X) \cap \overline{F(C)} = F(C)$, and so $F(C)$ is closed in $F(X)$. ■

We now can complete the proof. Since $[0,1]^A$ is a countable product of metric spaces, it is a metric space. Thus, X is (homeomorphic to) a subspace of a metric space, and so is itself metrizable. □

2.2 Compactness

We now briefly digress into one of the most important (and, at times, mysterious) properties in topology and analysis: compactness. The definition is familiar from analysis courses, but it takes on new importance in topology.

---

[4] While there is a notion of first countability, we will not discuss it in this course.

Definition 2.2.1. Let X be a topological space. An *(open) cover* of X is a set $\mathcal{U}$ of open subsets of X such that

$$X = \bigcup_{U \in \mathcal{U}} U.$$

A *subcover* of $\mathcal{U}$ is a subset $\mathcal{V} \subset \mathcal{U}$ such that $\mathcal{V}$ is still a cover of X.

The space X is said to be *compact* if every open cover of X admits a finite subcover.

Examples 2.2.2.

1. Any finite space X is compact.
2. A subset of $\mathbb{R}^n$ is compact if and only it is closed and bounded. This is the Heine-Borel Theorem, which we will prove in the next section.
3. The spheres S^n are all compact, as we will see shortly.

Remark 2.2.3. If we consider a subspace $A \subset X$, we can equivalently define compactness by considering covers of A *by open subsets of* X, using the definition of the subspace topology.

Example 2.2.4 (non-example)**.** Let $p, q \in \mathbb{Q}$ with $p < q$. Then the subset $[p, q] \subset \mathbb{Q}$ is not compact. To see this, let $r \in \mathbb{Q} \setminus \mathbb{R}$ with $p < r < q$. Let $m = \min\{q - r, r - p\}$, and consider the cover $\{U_\epsilon\}_{0<\epsilon<m}$ of $[p, q]$ by the opens

$$U_\epsilon = \{x \in \mathbb{Q} \mid |x - r| > \epsilon\}.$$

If there were a finite subcover $\{U_{\epsilon_1}, \ldots, U_{\epsilon_n}\}$, then there would be an i such that $\epsilon_i = \min\{\epsilon_j \mid 1 \leq j \leq n\}$. This would imply that $[p, q] \subset U_{\epsilon_i}$. However, by the density of the rationals in $\mathbb{R}$, there is a rational between r and $r + \epsilon_i$, a contradiction.

Lemma 2.2.5. *Let $f : X \to Y$ be continuous, and let X be compact. Then $f(X)$ is compact.*

Proof. Given a cover $\mathcal{U}$ of $f(X)$ (by open subsets of X), the preimages of the sets in $\mathcal{U}$ are open in X by continuity. As such,

$$\mathcal{V} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$$

is an open cover of X. Let $\{V_1, \ldots, V_n\}$ be a finite subcover. Then $\{f(V_1), \ldots, f(V_n)\} \subset \mathcal{U}$, and covers $f(X)$. Thus $f(X)$ is compact. $\square$

Lemma 2.2.6. *Let X be compact and $A \subset X$ be closed. Then A is compact.*

Proof. Given a cover $\mathcal{U}$ of A by open subsets of X, then $\mathcal{U} \cup \{X \setminus A\}$ is an open cover of X. The compactness of X gives us a finite subcover $\mathcal{V}$, and thus $\mathcal{V} \setminus \{X \setminus A\}$ is a finite subcover of $\mathcal{U}$. $\square$

Lemma 2.2.7. *If X is a Hausdorff space and $A \subset X$ is compact, then A is closed.*

Proof. For any fixed point $x \in X \setminus A$ and every point $a \in A$, we can use the Hausdorff property to find disjoint open subsets with $a \in U_{x,a}$ and $x \in V_{x,a}$. Then $\{U_{x,a} \mid a \in A\}$ is an open cover of A, so admits a finite subcover $U_{x,a_1}, \dots, U_{x,a_k}$ by compactness. Then

$$V_x := \bigcap_{i=1}^{k} V_{x,a_i}$$

is an open neighborhood of x which is disjoint from A. Taking the union over all $x \in X \setminus A$, we find that $X \setminus A$ is open, as desired. □

Corollary 2.2.8. *Let $f : X \to Y$ be a continuous map. If X is compact and Y is a Hausdorff space, then f is a closed map.*

Proof. Let $A \subset X$ be closed. Then since X is compact, A is compact by Lemma 2.2.6, and so $f(A)$ is compact by Lemma 2.2.5. Thus by Lemma 2.2.7, $f(A)$ is closed. □

Remark 2.2.9. In practice, this makes it particularly easy to display compact Hausdorff spaces as quotients of compact spaces. For instance, the surjective map

$$[0,1] \longrightarrow S^1$$

is a surjective map from a compact space to a Hausdorff space, and so is closed by Corollary 2.2.8. As such, by Lemma 1.2.29, it follows that it is a quotient map.

Lemma 2.2.10. *For a topological space X, any finite union of compact subspaces of X is itself compact.*

2.3 The Subbase Theorem, Tychonoff, and Heine-Borel

We now come to the last major theorem of the chapter, and two of its corollaries.[5]

Theorem 2.3.1 (Alexander's subbase theorem)**.** *Let X be a topological space, and let $\mathcal{B}$ be a subbasis of X. Then X is compact if and only if X fulfills the following condition*

> For any cover $\mathcal{U}$ of X by elements of $\mathcal{B}$, there is a finite subcover of $\mathcal{U}$.

Proof. Since every element of $\mathcal{B}$ is open, it is clear that compact spaces satisfy the given condition.

On the other hand, suppose X satisfies the given condition, and let $\mathbf{N}$ be the set of open covers of X not admitting a subcover. Suppose, by way of contradiction, that $\mathbf{N}$ is non-empty.

By Zorn's Lemma[6], there is a maximal cover $\mathcal{U}$ (under subset inclusion, rather than refinement of covers). Then $\mathcal{B} \cap \mathcal{U}$ is not a cover of X, for if it were, it would admit a finite subcover, contradicting our definition of $\mathcal{U}$. Thus, we can choose a point x in the complement of the points covered by $\mathcal{B} \cap \mathcal{U}$, and an open $V \in \mathcal{U}$ containing x. Since $\mathcal{B}$ is a subbase, we can choose $B_1, \dots, B_n \in \mathcal{B}$ whose intersection contains x and lies in V.

---

[5] There are many proofs of the two corollaries we give here, and they are often proven without reference to Alexander's subbase theorem. We here opt for a unified treatment, but the interested reader can find a variety of other proofs in standard texts on the subject.

[6] To apply Zorn's Lemma, we must show that every chain has an upper bound. However, a finite subcover of a union of covers must, in particular be a subset of one of the original covers, and so the union of a chain of covers in $\mathbb{N}$ must also not admit a finite subcover.

Since the B_i are not in $\mathcal{B} \cap \mathcal{U}$[7] each of the covers $\mathcal{U}_i = \{B_i\} \cup \mathcal{U}$ strictly contains $\mathcal{U}$. By the maximality of $\mathcal{U}$, it follows that each of these covers admits a finite subcover $\mathcal{F}_i$. It follows that $\mathcal{F}_i \setminus \{B_i\}$ must cover $X \setminus B_i$. However, this means that

$$\mathcal{F} := \bigcup_{i=1}^{n} (\mathcal{F}_i \setminus \{B_i\})$$

covers

$$\bigcup_{i=1}^{n} (X \setminus B_i) = X \setminus \left(\bigcap_{i=1}^{n} B_i \right)$$

and so, since the latter intersection is contained in V, $\mathcal{F} \cup \{V\}$ covers X. This is a finite subcover of $\mathcal{U}$ — a contradiction. ☐

Corollary 2.3.2 (Tychonoff's Theorem)**.** *Let $\{X_i\}_{i\in I}$ be an collection of topological spaces indexed by a set I. Then $\prod_{i\in I} X_i$ is compact if and only if each X_i is.*

Proof. On the one hand, the continuous image of a compact subset is compact by Lemma 2.2.5. Since the projections $p_j : \prod_{i\in I} X_i \to X_j$ are continuous and surjective, it follows that if the product is compact, each factor is compact.

On the other hand, suppose that each X_i is compact. Let $p_j : \prod_i X_i \to X_j$ denote the projections. By Theorem 2.3.1, it suffices to consider covers coming from the subbase

$$\{p_i^{-1}(U_i) \mid i \in I \text{ and } U_i \in \tau_i\}.$$

Let $\mathcal{U}$ be such a cover. If there is a $j \in I$ such that

$$\mathcal{U}_j := \{p_j(V) \mid V \in \mathcal{U}\} \setminus \{X_j\}$$

is an open cover of X_j, then we are done by the compactness of X_j. If we assume this is not the case, then consider the set

$$\prod_{i\in I} \left(X_i \setminus \bigcup_{U \in \mathcal{U}_i} U \right).$$

By the Axiom of Choice, this is non-empty, and so contains a point $x = (x_i)_{i\in I}$. However, for every $V \in \mathcal{U}$ with $p_j(V) \neq X_j$, $x_j \notin p_j(V)$, and so $x \notin V$. This contradicts the assumption that $\mathcal{U}$ was a cover, completing the proof. ☐

Corollary 2.3.3 (The Heine-Borel Theorem)**.** *Let $C \subset \mathbb{R}^n$. Then C is compact if and C is closed and bounded.*

Proof. On the one hand, if C is bounded, then it is contained within a product $\prod_{i=1}^n [a_i, b_i]$, which is closed by Corollary 2.3.2. If C is a additionally closed, then as closed subset of a compact space, C is compact by lem 2.2.6.

On the other hand, if C is compact, then since $\mathbb{R}^n$ is a Hausdorff space, C is closed by Lemma 2.2.7. Since $\{B_r(0)\}_{r>0}$ is an open cover, there is a finite subcover of C. Thus, in particular, there is a maximal radius R to this subcover, and so $C \subset B_R(0)$, meaning C is bounded. ☐

[7] Since otherwise x would be covered by $\mathcal{B} \cap \mathcal{U}$.

2.4 Locally compact Hausdorff spaces

To be added later

3

Interlude: some category theory

In this chapter, we digress into the basics of category theory and make rigorous some of the concepts we have begun using in the previous chapters.

3.1 Categories, functors, and natural transformations

In some sense, a category is a *setting in which to do mathematics*. It tells us what kind of mathematical objects we are considering, and how we relate them. More formally

Definition 3.1.1. A *category* $\mathcal{C}$ consists of

- A set[1] $\mathrm{Ob}(\mathcal{C})$, called the *objects* of $\mathcal{C}$.
- For each pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, a set[2] $\mathcal{C}(x, y)$ called the *morphisms* from x to y. We will write $f : x \to y$ for a morphism $f \in \mathcal{C}(x, y)$.
- For every triple of objects $x, y, z \in \mathrm{Ob}(\mathcal{C})$, a map of sets

  $$\circ : \mathcal{C}(y, z) \times \mathcal{C}(x, y) \longrightarrow \mathcal{C}(x, z)$$

  $$(f, g) \longmapsto f \circ g$$

  called *composition*.
- For every $x \in \mathrm{Ob}(\mathcal{C})$, a distinguished element $\mathrm{id}_x \in \mathcal{C}(x, x)$ called *the identity on x.*

These data are then required to satisfy the following conditions.

- (Associativity) For every triple of morphisms $x \xrightarrow{h} y$, $y \xrightarrow{g} z$, and $z \xrightarrow{f} w$, the composites

  $$f \circ (g \circ h) = (f \circ g) \circ h$$

  are equal.
- (Unitality) For $f : x \to y$ and $g : y \to x$, we have

  $$f \circ \mathrm{id}_x = f \quad \text{and} \quad \mathrm{id}_x \circ g = g.$$

---

[1] What we mean here should be made a little more precise, as the definition here could lead to set-theoretic paradoxes. I will, for the most part, sweep such technicalities under the rug, and ask the reader to trust that they can be resolved. The more naturally suspicious reader may satisfy their curiousity in, for example, [6].

[2] *Vide supra.*

This seems like a lot to unpack, but really it amounts to an axiomatization of the things which we do with functions in mathematics. We can compose functions, there are identity functions, and these behave in precisely the way we described above. So how does a category represent a 'place to do mathematics'?

Examples 3.1.2.

1. There is a category Set whose objects are sets[3], and such that, for every two sets X and Y, we have
$$\mathsf{Set}(X,Y) := \{\text{functions } f : X \to Y\}.$$
The composition is the usual composition of functions, and the identies are the identity maps.
2. There are categories Grp and Ab whose objects are groups and abelian groups, respectively, and whose morphisms are group homomorphisms.
3. There is a category Top whose objects are topological spaces, and whose morphisms are continuous maps of topological spaces.
4. There is a category Top_* whose objects are *pointed topological spaces* — pairs (X,x), where X is a topological space and $x \in X$ is a basepoint. Morphisms in Top_* from $(X,x) \to (Y,y)$ are continuous maps $f : X \to Y$ such that $f(x) = y$.
5. Let $(P, \geq)$ be a partially ordered set. We can define a category $\mathcal{P}$ with $\mathrm{Ob}(\mathcal{P}) := P$, and a unique morphism $p \to p'$ if and only if $p \leq p'$.
6. Let G be a group. Then there is a category BG with a single object $*$, and $BG(*,*) = G$. The composition is given by $g \circ f = g \cdot f$, and the identity is the neutral element of G.
7. We define *the simplex category* Δ to be the category whose objects are the totally ordered sets $[n] := \{0,1,\dots,n\}$ for $n \geq 0$, and whose morphisms $[n] \to [m]$ are maps of sets $f : [n] \to [m]$ such that, if $a \leq b$, $f(a) \leq f(b)$.
8. Let $\mathcal{C}$ be a category. We can define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ to have $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, and, for every $x, y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$
$$\mathcal{C}^{\mathrm{op}}(x,y) = \mathcal{C}(y,x).$$
The composition and identities are the same maps as in $\mathcal{C}$. One thinks of the opposite category as 'reversing the arrows' of $\mathcal{C}$, i.e. reversing the direction of morphisms.
9. There is a category **Metric** whose objects are metric spaces, and whose morphisms are continuous maps.

We will often have cause to sketch categories arising from posets. Let us consider, for instance, the category associated to the power set $\mathbb{P}(\{0,1\})$, ordered by inclusion. We will draw an arrow for every non-identity morphism:

$$\begin{tikzcd}
\{0\} \arrow[r] & \{0,1\} \\
\emptyset \arrow[u] \arrow[r] \arrow[ur] & \{1\} \arrow[u]
\end{tikzcd}$$

In the picture, we assume that the category is a poset, e.g. that the arrow from $\emptyset \to \{0,1\}$ is the composite of the arrows $\emptyset \to \{1\} \to \{0,1\}$ and the composite of the arrows $\emptyset \to \{0\} \to \{0,1\}$. For ease of drawing, we will sometimes omit composite arrows:

$$\begin{tikzcd}
\{0\} \arrow[r] & \{0,1\} \\
\emptyset \arrow[u] \arrow[r] & \{1\} \arrow[u]
\end{tikzcd}$$

---

[3] Here's where the set-theoretic technicalities come into play: This definition, on its face, would require us to have a 'set of all sets' which ends up leading us down the dark path to Russell's paradox. The usual way of resolving this is to define 'little sets' and 'big sets' such that the set of all little sets is a big set, and then defining Set to be the category of all little sets. More technically, the frameworks of *inaccessible cardinals* and *Grothendieck universes* are typical ways to address this issue, as is the Von Neumann-Bernays-Godel axiomatization of sets and classes. A good summary of approaches to the foundations of category theory can be found in [3].

10. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can form a new category $\mathcal{C} \times \mathcal{D}$, called the *product category*. The objects are $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$, and the morphisms are given by

$$\mathcal{C} \times \mathcal{D}((x_1, x_2), (y_1, y_2)) := \mathcal{C}(x_1, y_1) \times \mathcal{D}(x_2, y_2).$$

The identities are $\mathrm{id}_{(x,y)} = (\mathrm{id}_x, \mathrm{id}_y)$.

Definition 3.1.3. Let $\mathcal{C}$ be a category, and let $f : x \to y$ be a morphism in $\mathcal{C}$. We call $g : y \to x$ a *right inverse*[4] of f if $f \circ g = \mathrm{id}_y$. We say that g is a *left inverse* of f if $g \circ f = \mathrm{id}_x$. If g is both a left and right inverse to f, then we call g the *inverse* of f, and say that f is an isomorphism in $\mathcal{C}$.

Example 3.1.4. In a category $\mathcal{C}$, every identity morphism id_x is an isomorphism. In the category BG associated to a group G, every morphism is an isomorphism.

Exercise 3.1.5. Justify calling g *the* inverse of f by showing that any two inverses of f are equal. Show that every identity is an isomorphism.[5]

Exercise 3.1.6. Show that the isomorphisms in Set, Top and Grp are, respectively, the bijections, homeomorphisms, and group isomorphisms. Show that the only isomorphisms in the simplex category Δ are the identities.

Lemma 3.1.7. *Let $\mathcal{C}$ be a category. The isomorphisms of $\mathcal{C}$ satisfy the* 2-out-of-3 *property: given morphisms $g : x \to y$ and $f : y \to z$ in $\mathcal{C}$, if any two of the morphisms f, g, and $f \circ g$ are isomorphisms, then the third must be as well.*[6]

Proof. Since the inverse of an isomorphism is itself an isomorphism, it suffices for us to write the proof in the case where f and g are isomorphisms. Let $f^{-1} : z \to y$ and $g^{-1} : y \to x$ be their respective inverses. Then we can simply compute

$$(f \circ g)(g^{-1} \circ f^{-1}) = f \circ (g \circ g^{-1}) \circ f^{-1} = f \circ \mathrm{id}_y \circ f^{-1} = f \circ f^{-1} = \mathrm{id}_z$$

and similarly

$$(g^{-1} \circ f^{-1}) \circ (f \circ g) = \mathrm{id}_x.$$

Thus $g^{-1} \circ f^{-1}$ is inverse to $f \circ g$ and the lemma is proved. □

Lemma 3.1.8. *Functors preserve isomorphism. That is, given a functor $F : \mathcal{C} \to \mathcal{D}$, and an isomorphism $f : x \to y$ in $\mathcal{C}$, then $F(f) : F(x) \to F(y)$ is also an isomorphism.*

Proof. Follows by unwinding the definitions. □

From categories we can also extract more familiar algebraic structures: monoids and groups.

Definition 3.1.9. Let $\mathcal{C}$ be a category and $x \in \mathrm{Ob}(\mathcal{C})$ an object. An *endomorphism* of x in $\mathcal{C}$ is a morphism $f : x \to x$ in $\mathcal{C}$. An *automorphism* of $x \in \mathcal{C}$ is an <u>iso</u>morphism $f : x \to x$ in $\mathcal{C}$. We write

$$\mathrm{End}_{\mathcal{C}}(x) := \mathcal{C}(x, x)$$

---

[4] This is also sometimes called a *section* of f.

[5] This is analogous to showing that inverses are unique in a group G, and showing that the neutral element $e \in G$ has an inverse.

[6] In particular, the composition of isomorphisms is an isomorphism.

for the set of endomorphisms of x, and $\mathrm{Aut}_{\mathcal{C}}(x) \subset \mathrm{End}_{\mathcal{C}}(x)$ for the subset consisting of automorphisms. Notice that the composition of morphisms of $\mathcal{C}$ provides a binary operation

$$\mathrm{End}_{\mathcal{C}}(x) \times \mathrm{End}_{\mathcal{C}}(x) \longrightarrow \mathrm{End}_{\mathcal{C}}(x)$$

and, by Lemma 3.1.7, this induces a binary operation

$$\mathrm{Aut}_{\mathcal{C}}(x) \times \mathrm{Aut}_{\mathcal{C}}(x) \longrightarrow \mathrm{Aut}_{\mathcal{C}}(x).$$

Lemma 3.1.10. *Let $\mathcal{C}$ be a category and $x \in \mathrm{Ob}(\mathcal{C})$ an object. The composition of morphisms makes $\mathrm{End}_{\mathcal{C}}(x)$ into a monoid with neutral element id_x and makes $\mathrm{Aut}_{\mathcal{C}}(x)$ into a group with neutral element id_x.*

Proof. Associativity and unitality follow directly from associativity and unitality for the composition of morphisms in a category. In the case of $\mathrm{Aut}_{\mathcal{C}}(x)$, the inverse morphism defines an inverse element in the usual group-theoretic sense. □

So how do we relate different categories to one another? We need to define a 'map of categories' which tells us not only how objects relate, but also how morphisms relate.

Definition 3.1.11. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of a map of sets $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, and, for each $x, y \in \mathrm{Ob}(\mathcal{C})$ a map of sets $F : \mathcal{C}(x,y) \to \mathcal{D}(F(x), F(y))$ such that:

- For every $x \in \mathrm{Ob}(\mathcal{C})$, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.
- For every pair of composable morphisms $x \xrightarrow{g} y \xrightarrow{f} z$ in $\mathcal{C}$, we have that $F(f \circ g) = F(f) \circ F(g)$.

One can compose functors in the obvious way — simply compose the defining maps of sets. We will leave it as an exercise to the reader to check that this is actually a functor:

Exercise. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. Give a definition of the composite functor $F \circ G : \mathcal{B} \to \mathcal{D}$. Show that your definition is, in fact, a functor.

Examples 3.1.12.

1. For every category $\mathcal{C}$, there is an 'identity' functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which sends $x \mapsto x$ and $f \mapsto f$.
2. There is a functor $U : \mathsf{Top} \to \mathsf{Set}$ which sends each topological space to its underlying set, and each continuous map to the underlying map of sets. (See 1.2.17)
3. Our results in Propositions 1.0.3 and 1.1.2 can be used to show that there is a functor $F : \mathbf{Metric} \to \mathsf{Top}$, which sends each metric space to it's underlying topological space.

Lemma 3.1.13. *For a group G, a functor $F : BG \to \mathsf{Set}$ consists of a set X and a group action $G \curvearrowright X$.*

Proof. Let as an exercise. □

Construction 3.1.14. Let $\mathcal{C}$ be a category, and $S \subset \mathrm{Ob}(\mathcal{C})$ be a set of objects. We define the *full subcategory of $\mathcal{C}$ on S* to be the category $\mathcal{D}$ with $\mathrm{Ob}(\mathcal{D}) = S$ and, for each $x, y \in S$

$$\mathcal{D}(x,y) = \mathcal{C}(x,y).$$

Definition 3.1.15. There is a category Cat whose objects are categories, and whose morphisms are functors.[7]

Lemma 3.1.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $x \in \mathrm{Ob}(\mathcal{C})$. A functor $F : \mathcal{C} \to \mathcal{D}$ induces a homomorphism*

$$\mathrm{End}_{\mathcal{C}}(x) \longrightarrow \mathrm{End}_{\mathcal{D}}(F(x))$$

of monoids and a homomorphism

$$\mathrm{Aut}_{\mathcal{C}}(x) \longrightarrow \mathrm{Aut}_{\mathcal{C}}(x)$$

of groups.

Proof. In both cases the binary operation is composition. Since F preserves composition, the lemma follows. ☐

Exercise 3.1.17. The totally ordered sets $[n]$ from the definition of the simplex category are, in particular, posets, and thus give rise to categories $\overline{n}$. Show that the simplex category Δ can be identified with the full subcategory of Cat on the objects $\overline{n}$, $n \geq 0$.

3.2 A note on commuting diagrams

Before we continue, let us briefly revise Definition 1.2.8, and talk about commutative diagrams.

Definition 3.2.1. A *commutative diagram* in a category $\mathcal{C}$ is a functor $F : P \to \mathcal{C}$ out of a poset category P.

This definition is very important, so let us unpack some basic facts about it.

Firstly, if the partial order $\leq$ on P is generated under transitivity and reflexivity (i.e., by adding composites and identities) by some set $\{x_i \leq y_i\}_{i \in I}$ of relations, then the functor F is uniquely determined by its value on the objects of P and its value on the (unique) morphisms $x_i \to y_i$ in P. For example, if we let P be the poset of subsets of $\{0,1\}$ ordered by inclusion, then we can simply remember the values of F on the morphisms $\varnothing \to \{0\}$, $\varnothing \to \{1\}$,$\{0\} \to \{0,1\}$, and $\{1\} \to \{0,1\}$. We very often draw these data, as we have already been doing with commutative diagrams, as follows.

$$\begin{tikzcd}
F(\varnothing) \arrow[r] \arrow[d] & F(\{0\}) \arrow[d] \\
F(\{1\}) \arrow[r] & F(\{0,1\})
\end{tikzcd}$$

Secondly, the condition that P is a poset means that, in our picture above, any two composite morphisms which can be equal will be equal. This is, indeed, a common definition of commutative diagrams. If, for instance, I draw a square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "k"'] & Y \arrow[d, "g"] \\
Z \arrow[r, "h"'] & W
\end{tikzcd}$$

---

[7] This runs into the same set-theoretic difficulties alluded to earlier. While we won't go into these here, it is important to note that Cat, though it is a category, is not an object of Cat. That is, the category of categories does not contain itself.

and say that it *commutes*, this means that the data pictured define a functor out of the (unique up to isomorphism) poset generated by the arrows, in this case, the power set of $\{0,1\}$.

3.3 A note on duality

One key construction above was the opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$. This is like $\mathcal{C}$ in all ways, except that we view the morphisms as going in the opposite direction. As we have already seen, definitions in category theory come in pairs, with each definition in the pair given by reversing the direction of all the arrows in the definition. The opposite category gives us a nice way to formalize it.

Definition 3.3.1. If a *foo in* $\mathcal{C}$ is a categorical definition[8] then a *cofoo in* $\mathcal{C}$ is defined to be a foo in $\mathcal{C}^{\mathrm{op}}$.

3.4 Naturality and equivalence

Functors play the same role in relating categories to one another as homomorphisms play in relating groups to one another. However, there is a bit of an issue. The isomorphisms in Cat (called, with the blithe directness typical of mathematicians, *isomorphisms of categories*) are a little useless.

Example 3.4.1. Let's define a category $\mathsf{Set}^{\mathrm{un}}$ to have precisely one object of each cardinality, and morphisms the maps of sets. It shouldn't really matter whether we work in Set or $\mathsf{Set}^{\mathrm{un}}$, but the two categories aren't isomorphic. To see this, we need only note that the isomorphism class of singletons in $\mathsf{Set}^{\mathrm{un}}$ consists of a single object, and the isomorphism class of singletons in Set is infinite.

How do we find a way around this problem? Well, we can effectively do what we did when talking about homotopy: try to define a relation between functors, and then define a weaker notion of equivalence.

Definition 3.4.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation from F to G* $\alpha : F \Rightarrow G$ consists of a morphism $\alpha_x : F(x) \to G(x)$ in $\mathcal{D}$ for every $x \in \mathrm{Ob}(\mathcal{C})$ such that, for every morphism $f : x \to y$ in $\mathcal{C}$ the following diagram commutes:[9]

$$\begin{tikzcd}
F(x) \arrow[r, "\alpha_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\
F(y) \arrow[r, "\alpha_y"] & G(y).
\end{tikzcd}$$

We call a natural transformation a *natural isomorphism* if each of its components α_x is an isomorphism in $\mathcal{D}$.

Examples 3.4.3.

1. There is an 'identity' natural isomorphism $\mathrm{Id}_F : F \to F$ for every $F : \mathcal{C} \to \mathcal{D}$. The components of Id_F are $(\mathrm{Id}_F)_x := \mathrm{id}_{F(x)}$

---

[8] By this, we loosely mean a definition made in terms of objects, morphisms, composition, and identities.

[9] When we say a diagram commutes, we mean that it follows the convention described above for drawing posets: if two composites can be equal, they are equal. In this case, this would mean in equations that $G(f) \circ \alpha_x = \alpha_y \circ F(f)$.

2. Consider the two functors $\mathrm{id}, C : \mathsf{Grp} \to \mathsf{Grp}$, where $C(G) = G/[G,G]$ is the abelianization. There is a natural transformation $\alpha : \mathrm{id} \Rightarrow C$ with component α_G given by the quotient map $G \to G/[G,G]$.

Construction 3.4.4. Let $\beta : F \Rightarrow G$ and $\alpha : G \Rightarrow H$ be natural transformations among functors $F, G, H : \mathcal{C} \to \mathcal{D}$. We define the composite $\alpha \circ \beta$ via the formula, for all $x \in \mathcal{C}$, $(\alpha \circ \beta)_x = \alpha_x \circ \beta_x$. To check that this is natural, note that for each morphism $f : x \to y$ in $\mathcal{C}$ we have a pair of squares:

$$\begin{tikzcd}
F(x) \arrow[r, "\beta_x"] \arrow[d, "F(f)"'] & G(x) \arrow[r, "\alpha_x"] \arrow[d, "G(f)"] & H(x) \arrow[d, "H(f)"] \\
F(y) \arrow[r, "\beta_y"'] & G(y) \arrow[r, "\alpha_y"'] & H(y).
\end{tikzcd}$$

where the left and right hand squares both commute. It is easy to verify from the diagram that this means the external square commutes as well[10]

Exercise 3.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Show that there is a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with objects given by functors from $\mathcal{C}$ to $\mathcal{D}$, and morphisms given by natural transformations.

This notion of natural transformation is our categorical analogue of homotopy. This can even be made more precise, as the following exercise shows:

Exercise 3.4.6. Let $\overline{1}$ be the category from Exercise 3.1.17 — that is, the category with two objects 0 and 1, and a unique non-identity morphism from 0 to 1. Check that, for any category $\mathcal{C}$ There are two functors

$$\begin{aligned} \iota_i : \mathcal{C} &\to \overline{1} \times \mathcal{C} \\ x &\mapsto (i, x) \end{aligned}$$

where $i = 0$ or $i = 1$. Given functors $F, G : \mathcal{C} \to \mathcal{D}$, define

$$\mathrm{Nat}(F, G) := \left\{ H \in \mathrm{Fun}(\overline{1} \times \mathcal{C}, \mathcal{D}) \mid H \circ \iota_0 = F \text{ and } H \circ \iota_1 = G \right\}$$

and show that $\mathrm{Nat}(F, G)$ is in bijection with the set of natural transformations from F to G.

Definition 3.4.7. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there is a functor $G : \mathcal{D} \to \mathcal{C}$, and natural isomorphisms $\alpha : G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $\beta : F \circ G \cong \mathrm{Id}_{\mathcal{D}}$. We then say that the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, and call G a *weak inverse* to F.

Definition 3.4.8. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We call F

1. *full* if, for every $x, y \in \mathcal{C}$, the map $F : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ is surjective.

2. *faithful* if , for every $x, y \in \mathcal{C}$, the map $F : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ is injective.

---

[10] This is a kind of *pasting law*, and we will often make use of such arguments without comment.

3. *fully faithful* if it is both full and faithful.
4. *essentially surjective* if, for every $y \in \mathcal{D}$, there exists an $x \in \mathcal{C}$ such that $F(x)$ is isomorphic to y.

Examples 3.4.9.

1. Consider the functor $F : \mathsf{Top} \to \mathsf{Set}$ which sends each topological space (X, τ_X) to the underlying set X, and each continuous map to the underlying map of sets. This functor is faithful, since a continuous map $X \to Y$ is uniquely determined by the underlying map of sets, and it is essentially surjective, since for any set X, we can define a topology on X. However, F is *not* full, since, given two spaces (X, τ_X) and (Y, τ_Y), not every map of sets $X \to Y$ defines a continuous function.
2. Consider a homomorphism of groups $\phi : G \to H$. The induced functor $BG \to BH$ is always essentially surjective, is full if and only if ϕ is surjective, and is faithful if and only if ϕ is injective.
3. Consider the 'inclusion' $\iota : \Delta \to \mathsf{Cat}$, which sends each poset $[n]$ to the associated poset category, and each monotone map to the associated functor. The functor ι is fully faithful, however, it is not essentially surjective, since not every category is isomorphic to a category with finitely many objects.
4. Consider a category Bun whose objects are triples (M, V, p) where M is a smooth manifold, and $p : V \to M$ is a vector bundle on M. The morphisms of Bun are commutative diagrams

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "p"'] & W \arrow[d, "q"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

where ϕ is a morphism of vector bundles and f is a smooth map. We can define a functor $F : \mathrm{Bun} \to \mathsf{Top}$ which sends each diagram as above to the underlying continuous map between topological spaces

$$f : M \to N.$$

The functor F is not full, not faithful, and not essentially surjective. It is not full because not every continuous map between smooth manifolds is smooth; it is not faithful because there are multiple bundle maps which lie over the same smooth map; and it is not essentially surjective because not every topological space admits the structure of a smooth manifold.

The term *faithful* in category theory has a precise analogue in representation theory. A representation ρ of a group G on a vector space V is said to be *faithful* if every group element $g \in G$ is represented by a *distinct* linear map $\rho(g) : V \to V$. Rephrasing this in categorical terms, a representation of G on a vector space V is the same thing as a functor

$$\rho : BG \longrightarrow \mathsf{Vect}_k$$

which sends the unique object $*$ of BG to the vector space V. A representation is faithful if and only if the associated functor is faithful.

Proposition 3.4.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then the following are equivalent:*

1. *F is fully faithful and essentially surjective.*

2. *F is an equivalence of categories.*

Proof. We first show 2. ⇒ 1. Let G be a weak inverse to F, and α and β the natural isomorphisms displaying the weak invertibility of F. Given y in $\mathcal{D}$, the y-component of β is an isomorphism $\beta_y : F(G(y)) \overset{\cong}{\to} y$, showing that F is essentially surjective.

Let $f : x \to y$ be a morphism in $\mathcal{C}$. Then, by the naturality of α, the diagram

$$\begin{tikzcd} G(F(x)) \arrow[r, "\alpha_x"] \arrow[d, "G(F(f))"'] & x \arrow[d, "f"] \\ G(F(y)) \arrow[r, "\alpha_y"'] & y \end{tikzcd}$$

commutes. Since α_x is invertible, this means that

$$f = \alpha_y \circ G(F(f)) \circ \alpha_x^{-1},$$

i.e., f is uniquely determined by $G(F(f))$. Thus $G \circ F : \mathcal{C}(x, y) \to \mathcal{C}(G(F(x)), G(F(y)))$ is injective, and hence, $F : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ is injective, and F is faithful. Note that we can make the same argument with β to find that G is faithful.

Now let $h : F(x) \to F(y)$ in $\mathcal{D}$ be a morphism, and define $f := \alpha_y \circ G(h) \circ \alpha_x^{-1}$. Again we find a commutative square:

$$\begin{tikzcd} G(F(x)) \arrow[r, "\alpha_x"] \arrow[d, "G(h)"'] & x \arrow[d, "f"] \\ G(F(y)) \arrow[r, "\alpha_y"'] & y \end{tikzcd}$$

Which implies that $G(F(f)) = G(h)$. However, since G is faithful, this implies $F(f) = h$, and thus, $F : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ is surjective and F is full.

We now show 1. ⇒ 2. Suppose F is essentially surjective and fully faithful. For every $y \in \mathrm{Ob}(\mathcal{D})$, we choose[11] an object which we call $G(y) \in \mathrm{Ob}(\mathcal{C})$ together with an isomorphism $\beta_y : F(G(y)) \overset{\cong}{\to} y$ (by essential surjectivity). We then define, for every $h : x \to y$ in $\mathcal{D}$, the morphism $G(h) := F^{-1}(\beta_y^{-1} \circ h \circ \beta_x) \in \mathcal{C}(G(x), G(y))$.[12]

Since

$$G(\mathrm{id}_x) = F^{-1}(\beta_x^{-1} \circ \mathrm{id}_x \circ \beta_x) = F^{-1}(\beta_x^{-1} \circ \beta_x) = F^{-1}(\mathrm{id}_x),$$

we see that $G(\mathrm{id}_x) = \mathrm{id}_{G(x)}$. Moreover, since

$$G(f \circ g) = F^{-1}(\beta_z^{-1} \circ f \circ g \circ \beta_x) = F^{-1}(\beta_z^{-1} \circ f \circ \beta_y \circ \beta_y^{-1} \circ g \circ \beta_x)$$

we have $G(f \circ g) = G(f) \circ G(g)$. Thus, G defines a functor.[13] Tracing the definitions, it is immediate that the squares

$$\begin{tikzcd} F(G(x)) \arrow[r, "\beta_x"] \arrow[d, "F(G(f))"'] & x \arrow[d, "f"] \\ F(G(y)) \arrow[r, "\beta_y"'] & y \end{tikzcd}$$

---

[11] Note that this requires the axiom of choice. As a rule, we will assume the axiom of choice in our categorical arguments.

[12] We can do this precisely because F is fully faithful, so the map of sets

$$F : \mathcal{C}(G(x), G(y)) \to \mathcal{D}(F(G(x)), F(G(y)))$$

is a bijection.

[13] We have used the fully faithfulness of F for both computations here, as well as the functoriality of F.

commute for any $f : x \to y$ in $\mathcal{D}$, so that $\beta : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ is a natural isomorphism.

We now seek to obtain the natural isomorphism $\alpha : G \circ F \Rightarrow \mathrm{Id}_{\mathcal{C}}$. Note that $\beta_{(F(x))} : F(G(F(x))) \to F(x)$ is an isomorphism. By the fully faithfulness of F, there is a unique isomorphism[14] $\alpha_x := F^{-1}(\beta_{F(x)}) : G(F(x)) \to x$. Since β is a natural transformation, for any $f : x \to y$ in $\mathcal{C}$, we get a commutative square

$$\begin{tikzcd}
F(G(F(x))) \arrow[r, "\beta_{F(x)}"] \arrow[d, "F(G(F(f)))"'] & F(x) \arrow[d, "F(f)"] \\
F(G(F(y))) \arrow[r, "\beta_{F(y)}"'] & F(y)
\end{tikzcd}$$

i.e. $F(f \circ \alpha_x) = F(\alpha_y \circ G(F(f)))$. But, since F is fully faithful, this implies

$$f \circ \alpha_x = \alpha_y \circ G(F(f))$$

and thus the square

$$\begin{tikzcd}
G(F(x)) \arrow[r, "\alpha_x"] \arrow[d, "G(F(f))"'] & x \arrow[d, "f"] \\
G(F(y)) \arrow[r, "\alpha_y"'] & y
\end{tikzcd}$$

commutes. This means that α is natural, and thus provides the desired natural isomorphism $\alpha : G \circ F \Rightarrow \mathrm{Id}_{\mathcal{C}}$. □

Exercise 3.4.11. Show that the two categories from Example 3.4.1 are equivalent.

3.5 *Universality*

One of the most important uses of category theory is to give 'universal' constructions, which determine objects in a category by conditions on their morphisms. Before getting into the general definition of a universal property, let's look at a familiar example.

Example 3.5.1. Let k be a field, and define Vect_k be the category whose objects are k-vector spaces, and whose morphisms are k-linear maps. Given $V, W \in \mathrm{Ob}(\mathsf{Vect}_k)$, a *tensor product* of V and W in Vect_k is an object $V \otimes W$ together with a bilinear map $l : V \times W \to V \otimes W$ satisfying the following

UNIVERSAL PROPERTY: Any bilinear map $f : V \times W \to U$ of k-vector spaces factors uniquely though l. That is, given a bilinear map $f : V \times W \to U$, there is a unique k-linear map $\tilde{f} : V \otimes W \to U$ such that the diagram

$$\begin{tikzcd}
V \times W \arrow[r, "f"] \arrow[d, "l"'] & U \\
V \otimes W \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

commutes. This establishes a bijection (for any $U \in \mathrm{Ob}(\mathsf{Vect}_k)$) between bilinear maps $V \times W \to U$ and linear maps $V \otimes W \to U$.

There is an explicit construction of $V \otimes W$ as a quotient of $V \times W$ by relations imposing bilinearity. The result of this construction is often called *the* tensor product of V and W. From the perspective of universal properties, this is an abuse of terminology. It would be better to call it *a* tensor product of V and W. We can work equally well with any other tensor product of V and W.

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[14] We leave it as an exercise to the interested reader to check that this is actually an isomorphism using the fully-faithfulness of F.

This universal property determines $V \otimes W$ and l up to unique isomorphism as follows. Let $V \boxtimes W$ be another tensor product of V and W, with defining bilinear map $r : V \times W \to V \boxtimes W$. Then, by universal property, r factors uniquely through f as $\tilde{r} : V \otimes W \to V \boxtimes W$, and similarly, l factors uniquely through r as $\tilde{l} : V \boxtimes W \to V \otimes W$. However, this implies that the diagram

$$\begin{tikzcd}
V \times W \arrow[r, "l"] \arrow[d, "l"'] & V \otimes W \\
V \otimes W \arrow[ur, "\tilde{r}\circ\tilde{l}"'] &
\end{tikzcd}$$

commutes. However, again by universal property, the morphism making this diagram commute is *unique*. Since $\mathrm{id}_{V\otimes W}$ also makes this diagram commute, this means that $\tilde{r} \circ \tilde{l} = \mathrm{id}_{V\otimes W}$. Similarly, by universal property $\tilde{l} \circ \tilde{r} = \mathrm{id}_{V\boxtimes W}$, so $\tilde{r}$ and $\tilde{l}$ are isomorphism. Since $\tilde{r}$ and $\tilde{l}$ are uniquely determined, we say the universal property determines the tensor product *up to unique isomorphism*.

So how do we formulate universal properties categorically? The answer lies in the *Yoneda lemma*.[15] This simple lemma might reasonably be called the fundamental theorem of category theory, and encapsulates an important philosophy of category theory:

SLOGAN: By their morphisms shall ye know them.

In practice, this means that, if we want to study mathematical objects or structures, we should study the structure-preserving maps between them. It doesn't make much sense to study groups by considering only arbitrary maps of underlying sets. Instead we consider group homomorphisms. Similarly, as we have already seen, the study of topological spaces is, in some sense, the study of abstract continuous maps. The Yoneda lemma makes this precise by showing that, given a category $\mathcal{C}$, an object c in $\mathcal{C}$ is, in a sense, uniquely determined up to unique isomorphism by the collection of all morphisms coming out of c.

Definition 3.5.2. Let $\mathcal{C}$ be a category[16], and $c \in \mathrm{Ob}(\mathcal{C})$. We define a functor

$$\begin{aligned} h^c : \mathcal{C} &\longrightarrow \mathsf{Set} \\ d &\longmapsto \mathcal{C}(c, d) \end{aligned}$$

called the *representable functor* associated to c. For a morphism $f : x \to y$ in $\mathcal{C}$, we define $h^c(f)$ to send $g : c \to x$ to $f \circ g : c \to y$.

Similarly, we obtain a representable functor

$$\begin{aligned} h_c : \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ d &\longmapsto \mathcal{C}(d, c) \end{aligned}$$

This is sometimes called the *contravariant*[17] *representable functor* associated to c.

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[15] Nobuo Yoneda, a computer scientist from Japan, formulated and proved the Yoneda lemma, but never published it. He communicated it to Saunders MacLane, one of the originators of category theory. The name 'the Yoneda Lemma' was given to the lemma by MacLane.

[16] Once again, set-theoretic technicalities come into play. Technically, what we need for this definition is a *locally small category*, that is, a category such that all of the Hom-sets are 'small sets.'

[17] This is an old terminological convention. A functor $F : \mathcal{C} \to \mathcal{D}$ is sometimes called a *covariant* functor from $\mathcal{C}$ to $\mathcal{D}$, and a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is called a *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$. Covariant functors preserve the direct of morphisms, whereas contravariant functors reverse the direction of morphisms.

Exercise 3.5.3. Show that

$$\begin{aligned} \mathcal{Y} : \mathcal{C} &\longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \\ c &\longmapsto h_c \end{aligned}$$

defines a functor. This functor is called the *Yoneda embedding.*

Notation 3.5.4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. We denote the set of natural transformations from F to G by $\mathrm{Nat}(F, G)$. Note that $\mathrm{Nat}(-, G) : \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \to \mathsf{Set}$ is the representable functor h_G on the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proposition 3.5.5 (The Yoneda Lemma)**.** *Let* $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ *be a functor and* $c \in \mathrm{Ob}(\mathcal{C})$. *There is an isomorphism*

$$\mathrm{Nat}(h_c, F) \overset{\cong}{\Rrightarrow} F(c)$$

Moreover, this isomorphism is natural in both c *and* F.

Before beginning the proof, we first comment on what we mean by "natural in c and F". We view $\mathrm{Nat}(h_{(-)}, -)$ as a functor $\mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathsf{Set}$, which sends (x, F) to $\mathrm{Nat}(h_x, F)$, and sends a pair of morphisms $f : x \to y$ in $\mathcal{C}$ and $\alpha : F \Rightarrow G$ to the morphism

$$\begin{aligned} \mathrm{Nat}(h_y, F) &\longrightarrow \mathrm{Nat}(h_x, G) \\ \beta &\longmapsto (\alpha \circ \beta \circ \mathcal{Y}(f)) \end{aligned}$$

Similarly, we define a functor $(c, F) \mapsto F(c)$ with the obvious functoriality.

Proof. We first prove that there is an isomorphism for any c and F. We define a map

$$\begin{aligned} \chi_{(c,F)} : \mathrm{Nat}(h_c, F) &\longrightarrow F(c) \\ \alpha &\longmapsto \alpha_c(\mathrm{id}_c). \end{aligned}$$

We will show that this is a bijection (i.e. an isomorphism in Set.)

First, let $\alpha : h_c \Rightarrow F$ be an arbitrary natural transformation, and let $f : d \to c$ be a morphism in $\mathcal{C}$ (i.e., a morphism $c \to d$ in $\mathcal{C}^{\mathrm{op}}$). Then naturality yields a commutative square

$$\begin{tikzcd} \mathcal{C}(c,c) \arrow[r, "h_c(f)"] \arrow[d, "\alpha_c"'] & \mathcal{C}(d,c) \arrow[d, "\alpha_d"] \\ F(c) \arrow[r, "F(f)"'] & F(d) \end{tikzcd}$$

Since $h_c(f)(\mathrm{id}_c) = f$, this, in particular implies that $\alpha_d(f) = F(f)(\alpha_c(\mathrm{id}_c))$. Therefore, α is uniquely determined by $\alpha_c(\mathrm{id}_c)$, and thus, $\chi_{(c,F)}$ is injective.

Now, given $x \in F(c)$ we define a natural transformation β with by setting $\beta_d(f) := F(f)(c)$. We now check that β is natural. Let $g : e \to d$ be a morphism in $\mathcal{C}$, and let $f \in \mathcal{C}(d, c)$. We then have that

$$\alpha_e(h_c(g)(f)) = F(h_c(g)(f))(\mathrm{id}_c) = F(g \circ f)(\mathrm{id}_c) = F(g)(F(f)(\mathrm{id}_c))$$

and by definition

$$F(g)(\alpha_d(f)) = F(g)(F(f)(\mathrm{id}_c)).$$

So the diagram

$$\begin{tikzcd}
\mathcal{C}(d,c) \arrow[r, "h_c(g)"] \arrow[d, "\alpha_c"'] & \mathcal{C}(e,c) \arrow[d, "\alpha_d"] \\
F(d) \arrow[r, "F(g)"'] & F(e)
\end{tikzcd}$$

commutes. Hence, β is a natural transformation with $\chi_{(c,F)}(\beta) = x$, and $\chi_{(c,F)}$ is surjective.

We now show naturality in c. Let $f : c \to d$ be a morphism in $\mathcal{C}$. We wish to show that, for any $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ the diagram

$$\begin{tikzcd}
\mathrm{Nat}(h_d, F) \arrow[r, "- \circ \mathcal{Y}(f)"] \arrow[d, "\chi_{(d,F)}"'] & \mathrm{Nat}(h_c, F) \arrow[d, "\chi_{(c,F)}"] \\
F(d) \arrow[r, "F(f)"] & F(c)
\end{tikzcd}$$

commutes. We therefore compute, for an arbitrary natural transformation $\alpha : h_d \Rightarrow F$,

$$\chi_{(c,F)}(\alpha \circ \mathcal{Y}(f)) = (\alpha \circ \mathcal{Y}(f))_c(\mathrm{id}_c) = \alpha_c(\mathcal{Y}(f)_c(\mathrm{id}_c)) = \alpha_c(f \circ \mathrm{id}_c) = \alpha_c(f)$$

Similarly, we compute

$$F(f)(\chi_{(d,F)}(\alpha)) = F(f) \circ \alpha_d(\mathrm{id}_d).$$

However, since α is, itself a natural transformation, our work above shows that

$$F(f)(\alpha_d(\mathrm{id}_d)) = \alpha_c(f).$$

so the diagram commutes, and χ is natural in c.

Finally, we show naturality in F. Let $\beta : F \to G$ be a natural transformation, and c an arbitrary object of $\mathcal{C}$. We want to show that the diagram

$$\begin{tikzcd}
\mathrm{Nat}(h_c, F) \arrow[r, "\beta \circ -"] \arrow[d, "\chi_{(c,F)}"'] & \mathrm{Nat}(h_c, G) \arrow[d, "\chi_{(c,G)}"] \\
F(c) \arrow[r, "\beta_c"] & G(c)
\end{tikzcd}$$

commutes. We again compute for an arbitrary natural transformation $\alpha : h_c \Rightarrow F$,

$$\chi_{(c,G)}(\beta \circ \alpha) = (\beta \circ \alpha)_c(\mathrm{id}_c)$$

and

$$\beta_c(\chi_{(c,F)}(\alpha)) = \beta_c(\alpha_c(\mathrm{id}_c))$$

This shows that the diagram commutes, and so χ is natural in F. $\square$

Exercise 3.5.6. Show that the Yoneda embedding

$$\mathcal{Y} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$$

is fully faithful.[18] In particular, if there is a natural isomorphism $\alpha : h_c \cong h_d$, then there is a unique isomorphism $c \cong d$ in $\mathcal{C}$ corresponding to α under the Yoneda embedding.

The exercise above is a key consequence of the Yoneda lemma. It tells us that the representable functor associated to an object allows us to retrieve that object up to isomorphism. This now allows us to define universal properties in great generality.

Definition 3.5.7. A *universal property* on a category $\mathcal{C}$ is a functor

$$U : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}.$$

A pair (c, α), where $c \in \mathrm{Ob}(\mathcal{C})$ and $\alpha : h_c \Rightarrow U$ is a natural isomorphism is called a *representation of* U. We also say that (c, α) *satisfies the universal property* U. We often abuse notation by saying that the object c satisfies U, and leaving the natural isomorphism implicit.

Remark 3.5.8. We will also consider the dual case, that of a functor $U : \mathcal{C} \to \mathsf{Set}$, as a universal property on $\mathcal{C}$. Here an pair satisfying U is an object c and a natural isomorphism $h^c \cong U$.

Example 3.5.9. We now can rephrase Example 3.5.1. Define a functor

$$U_{V,W} : \mathsf{Vect}_k \to \mathsf{Set}$$

by mapping $Z \in \mathsf{Vect}_k$ to the set $\mathrm{Bilin}(V, W; Z)$ of bilinear maps $V \times W \to Z$ and mapping a linear map $f : X \to Z$ to the map

$$\begin{aligned} \mathrm{Bilin}(V, W; X) &\longrightarrow \mathrm{Bilin}(V, W; Z) \\ g &\longmapsto f \circ g \end{aligned}$$

A tensor product of V and W over k is then a representation of $U_{V,W}$.

3.6 Limits and colimits

We now turn to a key application of universal properties: limits and colimits. These generalize notions like: cartesian products, disjoint unions, quotients, subobjects, etc.

Definition 3.6.1. Let $\mathcal{C}$ be a category, and define a functor

$$* : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$$

which sends every object of $\mathcal{C}$ to the singleton set $\{*\}$. A *terminal object* of $\mathcal{C}$ is a representation (c, α) for $*$.

Let's unpack what the definition means for a terminal object. Suppose $c \in \mathcal{C}$ is terminal. This means that α provides a natural isomorphism

$$\mathcal{C}(-, c) = h_c(-) \cong \{*\}$$

That is, $\mathcal{C}(d, c)$ is a singleton set for every $d \in c$. Since there is a unique bijection between any two singleton sets, naturality is automatic. The definition therefore amounts to saying that an object c is terminal if and only if there is a unique morphism $d \to c$ for every $d \in \mathcal{C}$.

An object c is then initial if and only if there is a unique morphism $f : c \to d$ for every object d in $\mathcal{C}$.

---

[18] This is, in fact, what we mean by an *embedding* of categories: a fully faithful functor.

An *initial object* is a terminal object in $\mathcal{C}^{\text{op}}$, that is, a representation of the functor

$$* : \mathcal{C} \to \mathsf{Set}$$

which sends every object to the singleton set.[19]

Examples 3.6.2.

1. A terminal object in Set is a singleton set. The initial object in Set is the empty set.
2. A terminal object in Top is the unique topological space with underlying set the singleton set. An initial object in Top is the empty topological space.
3. The trivial group $\{e\}$ is both initial and terminal in Grp. When an object is both initial and terminal, we call it a *zero object.*

Warning 3.6.3. Not every category has an initial or terminal object. For instance, viewing a poset P as a category, an object $p \in P$ is terminal if and only if, for all $q \in P$, $p \geq q$.

We now come to a vitally important example of universal constructions in categories: limits.

Definition 3.6.4. Let $\mathcal{C}$ and I be categories, and let $F : I \to \mathcal{C}$ be a functor. We define the *overcategory* of F, $\mathcal{C}_{/F}$ as follows. The objects of $\mathcal{C}_{/F}$ are *cones over* F, and consist of the following data:

1. An object $c \in \mathcal{C}$, called the *cone point*
2. For every $i \in \mathrm{Ob}(I)$, a morphism $\alpha_i : c \to F(i)$ such that, for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} & c \arrow[dl, "\alpha_i"'] \arrow[dr, "\alpha_j"] & \\ F(i) \arrow[rr, "F(f)"] & & F(j) \end{tikzcd}$$

   commutes.

A morphism in $\mathcal{C}_{/F}$ is a *morphism of cones* $(c, \{\alpha_i\}_{i \in \mathrm{Ob}(I)}) \to (d, \{\beta_i\}_{i \in \mathrm{Ob}(I)})$. Such a morphism consists of a morphism $g : c \to d$ in $\mathcal{C}$ such that, for every $i \in \mathrm{Ob}(I)$, the diagram

$$\begin{tikzcd} c \arrow[rr, "g"] \arrow[dr, "\alpha_i"'] & & d \arrow[dl, "\beta_i"] \\ & F(i) & \end{tikzcd}$$

commutes.

Definition 3.6.5. Let $F : I \to \mathcal{C}$ be a functor. A *limit cone of F in* $\mathcal{C}$ is a terminal object in $\mathcal{C}_{/F}$. We will refer to the cone point c of a limit cone $(c, \{\alpha_i\})$ for F as a *limit* of F. We often write $\lim_I F$ for a limit of F.

In category theory, nearly every concept has it's *dual.* The concept that corresponds to swapping the directions of the arrows. As we discuss limits and colimits, I will include dual notions and examples in the margin.

Definition (Dual definition)**.** Let $F : I \to \mathcal{C}$ be a functor. We define the *undercategory* of F, $\mathcal{C}_{F/}$ as follows. The objects of $\mathcal{C}_{F/}$ are *cones under* F (also called *cocones*), and consist of the following data:

1. An object $c \in \mathcal{C}$, called
2. For every $i \in \mathrm{Ob}(I)$, a morphism $\alpha_i : F(i) \to c$ such that, for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} & c & \\ F(i) \arrow[ur, "\alpha_i"] \arrow[rr, "F(f)"] & & F(j) \arrow[ul, "\alpha_j"'] \end{tikzcd}$$

   commutes.

A morphism in $\mathcal{C}_{/F}$ is a *morphism of cocones* $(c, \{\alpha_i\}_{i \in \mathrm{Ob}(I)}) \to (d, \{\beta_i\}_{i \in \mathrm{Ob}(I)})$. Such a morphism consists of a morphism $g : c \to d$ in $\mathcal{C}$ such that, for every $i \in \mathrm{Ob}(I)$, the diagram

$$\begin{tikzcd} c \arrow[rr, "g"] & & d \\ & F(i) \arrow[ul, "\alpha_i"] \arrow[ur, "\beta_i"'] & \end{tikzcd}$$

commutes.

Definition. Let $F : I \to \mathcal{C}$ be a functor. A *colimit cone of F in* $\mathcal{C}$ is an initial object in $\mathcal{C}_{F/}$. We will refer to the cone point c of a colimit cone $(c, \{\alpha_i\})$ for F as a *colimit* of F. We often write $\mathrm{colim}_I F$ for a colimit of F.

Remark. A colimit of $F : I \to \mathcal{C}$ is exactly the same as a limit of $F^{\text{op}} : I^{\text{op}} \to \mathcal{C}^{\text{op}}$. This is what we mean when we say the notions are dual.

Examples.

1. The dual notion of a product in $\mathcal{C}$ is a *coproduct* in $\mathcal{C}$
   - (a) In Set or Top, the coproduct is given by the disjoint union, together with the canonical inclusions $X \to X \coprod Y$ and $Y \to X \coprod Y$.
   - (b) In Grp, the coproduct of G and H is the free product of groups $G * H$.
   - (c) In Ab, the coproduct of G and H is the direct sum $G \oplus H$.

---

[19] An initial object is the *dual* concept to a terminal object — it is obtained by simply reversing the definitions of the arrows.

Examples 3.6.6.

1. Let I be the category with two objects and no non identity morphisms. A functor $F : I \to \mathcal{C}$ is uniquely specified by a pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$. A cone over F consists of an object c and morphisms

   $$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x & & y \end{tikzcd}$$

   If the cone $(c, \{f, g\})$ is a limit cone, this means that it is terminal in the overcategory of F, i.e, for any other cone $(d, \{h, k\})$ there is a unique morphism of cones $u : (d, \{h, k\}) \to (c, \{f, g\})$.

   Diagramatically, this means that, for any $(d, \{h, k\})$, as in the diagram

   $$\begin{tikzcd} & d \arrow[d, dashed] \arrow[ddl, bend right, "h"'] \arrow[ddr, bend left, "k"] & \\ & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x & & y \end{tikzcd}$$

   There exists a dashed arrow u making the diagram commute. The limit over such a diagram is called a *product* in $\mathcal{C}$. Some examples of products:

   (a) In Set, the product of X and Y is simply the Cartesian product $X \times Y$, together with the projections $X \times Y \to X$ and $X \times Y \to Y$.

   (b) In Top, the product of spaces X and Y is again the product $X \times Y$, equipped with the product topology and the canonical projections.

   (c) In Top_*, the product of (X, x) with (Y, y) is the pointed space $(X \times Y, (x, y))$.

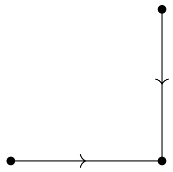
   (d) In Grp the product of G and H is the direct product of groups $G \times H$, equipped with the canonical projections.

   (e) In Ab the product of G and H is the same as in Grp.

   Even in an arbitrary category $\mathcal{C}$, a product of x and y is usually denoted $x \times y$.

2. More generally, let I be a category with only identity morphisms (which we can think of as just being a set). Given a diagram $F : I \to \mathcal{C}$, a cone over F consists of an object X and morphisms $p_i : X \to F(i)$ for $i \in \mathrm{Ob}(I)$. A limit of such a diagram is called a *product* of the objects $F(i)$, and denoted by $\prod_{i \in I} F(i)$.

3. Let Pb be the category



A limit over a functor $F : \mathrm{Pb} \to \mathcal{C}$ is called a *pullback*. In Top, the pullback of a diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & A \end{tikzcd}$$

is the subspace $X \times_A Y$ of $X \times Y$ consisting of those pairs $(x, y) \in X \times Y$ such that $f(x) = g(y)$. In Set the pullback is the subset $X \times_A Y \subset X \times Y$ defined in the same way.

4. Consider the poset $\mathbb{N}$ as a category. A limit over a diagram $F : \mathbb{N}^{\mathrm{op}} \to \mathsf{Ab}$ is an *inverse limit* in the classical algebraic terminology. In Set, an inverse limit of a functor F is the subset of the product

$$\prod_{i \in \mathbb{N}} F(i)$$

which consists of elements $(a_i)_{i \in \mathbb{N}}$ such that $f_i(a_{i+1}) = a_i$ for all $i \in \mathbb{N}$.

5. We can consider the category Eq with two objects 0 and 1, and two parallel, non-equal, non-identity morphisms f and g from 0 to 1. A functor $F : \mathrm{Eq} \to \mathcal{C}$ consists of a diagram

$$\begin{tikzcd} F(0) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] & F(1) \end{tikzcd}$$

in $\mathcal{C}$. A limit of such a diagram is called an *equalizer*. In Set, an equalizer is given by the subset $A \subset F(0)$ consisting of those $a \in F(0)$ such that $F(f)(a) = F(g)(a)$. In Top, an equalizer is constructed in the same way, and equipped with the subspace topology.

Lemma 3.6.7. *Let $\mathcal{C}$ be a category, and*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & A \end{tikzcd}$$

a diagram in $\mathcal{C}$ such that f is an isomorphism in $\mathcal{C}$. A commutative square

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & A \end{tikzcd}$$

is a pullback square if and only if p is an isomorphism.

2. Let $\mathrm{Po} := \mathrm{Pb}^{\mathrm{op}}$. A colimit over Po is called an *pushout*. In Top, let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

be a diagram over Po. The pushout in Top is the topological space $(X \coprod Y)_{/\sim}$, where we define the relation $x \sim y$ if and only if $f(x) = g(y)$. In other words, the pushout is obtained by 'gluing X and Y together along A'

3. A colimit of a diagram $F : \mathbb{N} \to \mathsf{Ab}$ is a *direct limit* in the classical algebraic terminology.

4. Note that $\mathrm{Eq}^{\mathrm{op}} \cong \mathrm{Eq}$. The colimit of a diagram

$$F : \mathrm{Eq} \longrightarrow \mathcal{C}$$

is called a *coequalizer*. In Set, a coequalizer is constructed as the quotient of $F(1)$ by the relation $F(f)(a) \sim F(g)(a)$ for all $a \in F(0)$. In Top, the coequalizer is constructed in the same fashion, and is equipped with the quotient topology.

Proof. It will suffice to show that the diagram

$$\begin{tikzcd}
Y \arrow[r, "f^{-1}\circ g"] \arrow[d, equal] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & A
\end{tikzcd}$$

is pullback.[20] Suppose we are given an object $Z \in \mathcal{C}$, and a commutative diagram

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "k"] \arrow[rdd, bend right, "\ell"'] & & \\
& & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & A
\end{tikzcd}$$

We want to show that there is a unique completion to a commutative diagram

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "k"] \arrow[rd, dashed, "q"] \arrow[rdd, bend right, "\ell"'] & & \\
& Y \arrow[r, "f^{-1}\circ g"] \arrow[d, equal] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & A
\end{tikzcd}$$

However, the requirement that the left-hand triangle commute forces $q = \ell$ (and hence if there is a completion, it is unique.), and since $f \circ f^{-1} \circ g \circ \ell = g \circ \ell = f \circ k$, this choice defines a commutative diagram. ☐

Exercise 3.6.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories such that every diagram in $\mathcal{D}$ has a limit. Consider a diagram

$$F : I \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Show that F has a limit, and that there is a canonical isomorphism $(\lim_I F)(c) \cong \lim_I (F(c))$ in $\mathcal{D}$.

Proposition 3.6.9. *Let $F : I \to \mathcal{D}$ be a functor, and suppose that $\mathcal{D}$ has all coproducts and coequalizers. Then there is a colimit of F in $\mathcal{D}$, and it is equivalent to the coequalizer*

$$\textstyle\coprod_{f\in \mathrm{Mor}(I)} F(s(f)) \overset{A}{\underset{B}{\rightrightarrows}} \coprod_{i\in \mathrm{Ob}(I)} F(i)$$

where $\mathrm{Mor}(I)$ *denotes the set of all morphisms of I, and $s(f)$ denotes the source of f. The morphisms A and B in the coequalizer diagram are induced by the canonical inclusion map*

$$F(s(f)) \longrightarrow \textstyle\coprod_{i\in \mathrm{Ob}(I)} F(i)$$

and the composite

$$F(s(f)) \xrightarrow{F(f)} F(t(f)) \longrightarrow \textstyle\coprod_{i\in \mathrm{Ob}(I)} F(i)$$

respectively.

---

[20] The fact that this suffices may be taken as an exercise.

Proof. We will build a correspondence between cones over F and cones over the coequalizer diagram above. Denote by

$$G : \mathrm{Eq} \longrightarrow \mathcal{D}$$

the coequalizer diagram from the lemma.

We define a functor

$$\phi : \mathcal{D}_{F/} \longrightarrow \mathcal{D}_{G/}$$

as follows. Given a cone (d, η) under F in $\mathcal{D}$, we assign the cone $\phi(d, \eta)$ with tip d over G, by defining two morphisms:

$$\textstyle\coprod_{i \in \mathrm{Ob}(I)} \eta_i : \coprod_{i \in \mathrm{Ob}(I)} F(i) \longrightarrow d$$

and

$$\textstyle\coprod_{f \in \mathrm{Mor}(I)} \eta_{s(f)} : \coprod_{f \in \mathrm{Mor}(I)} F(s(f)) \longrightarrow d$$

To see that the appropriate triangles commute, we note that (1) the triangles

$$\begin{tikzcd}
F(s(f)) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta_{s(f)}"'] & & F(s(f)) \arrow[dl, "\eta_{s(f)}"] \\
& d &
\end{tikzcd}$$

always commute, and (2) the naturality of η shows that the triangles

$$\begin{tikzcd}
F(s(f)) \arrow[rr, "F(f)"] \arrow[dr, "\eta_{s(f)}"'] & & F(t(f)) \arrow[dl, "\eta_{t(f)}"] \\
& d &
\end{tikzcd}$$

commute. Thus, $\phi(d, \eta)$ is, in fact, a cone over G. Given a morphism $g : (c, \mu) \to (d, \eta)$, the morphism $\phi(g) : \phi(c, \mu) \to \phi(d, \eta)$ is still given by $g : c \to d$. It is an easy check to show that this still defines a morphism of cones in $\mathcal{D}_{G/}$.

On the other hand, we define a functor

$$\psi : \mathcal{D}_{G/} \longrightarrow \mathcal{D}_{F/}$$

as follows. Given a cone (d, ρ) under G in $\mathcal{D}$, we define the i^{th} component $\psi(d, \rho)_i$ of $\psi(d, \rho)$ to be the composite

$$F(i) \longrightarrow \textstyle\coprod_{i \in \mathrm{Ob}(I)} \xrightarrow{\rho} d$$

with the cone defining the coproduct. Naturality follows from the commutative diagrams

$$\begin{tikzcd}
F(s(f)) \arrow[r] \arrow[d, "F(f)"'] & \coprod_{f \in \mathrm{Mor}(I)} F(s(f)) \arrow[d, "B"] \arrow[dr, "\rho"] & \\
& & d \\
F(t(f)) \arrow[r] & \coprod_{i \in \mathrm{Ob}(I)} F(i) \arrow[ur, "\rho"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(s(f)) \arrow[r] \arrow[dd, "F(f)"'] & \coprod_{f\in \mathrm{Mor}(I)} F(s(f)) \arrow[dd, "A"] \arrow[dr, "\rho"] & \\
 & & d \\
F(s(f)) \arrow[r] & \coprod_{i\in \mathrm{Ob}(I)} F(i) \arrow[ur, "\rho"'] &
\end{tikzcd}$$

On morphisms, ψ does not change the morphism of cone tips.

We leave it to the reader to check that ϕ and ψ are weakly inverse functors, which completes the proof. □

Corollary 3.6.10. *The categories* Set, Top, Ab, Grp, *and* Vect_k *admit all (small) limits and colimits.*

Proof. We can apply Proposition 3.6.9 and its dual, and note that the categories listed have all products, coproducts, equalizers, and coequalizers. □

Remark 3.6.11. Proposition 3.6.9 makes more explicit a general description of limits and colimits. Colimits are something like 'quotients of coproducts' and limits are something like 'subobjects of products'. What precisely this means, of course, varies based on the category in question, and can only be a heuristic.[21]

3.7 Groupoids and groups

In the next section, we will need to make heavy use of groupoids. As we will see, groupoids function as an analog of groups, but their formal properties are often better behaved, making them a useful tool to simplify the proofs of powerful theorems. Similarly, the theory of groupoids mirrors the theory of topological spaces in many useful ways, making them an ideal bridge between topology and algebra. Balanced against this, however, is the fact that groups are often computationally simpler than groupoids, and so we will need a dictionary which translates easily between them.

Definition 3.7.1. A *groupoid* is a category $\mathcal{G}$ in which every morphism is an isomorphism.[22] We denote by Grpd the category whose objects are groupoids and whose morphisms are functors.

Remark 3.7.2. Notice that for a groupoid $\mathcal{G}$, $\mathrm{End}_{\mathcal{G}}(x) = \mathrm{Aut}_{\mathcal{G}}(x)$. By Lemma 3.1.10 this is a group under the composition of automorphisms.

We have already encountered the key example of groupoids in Example 3.1.2 (6). Namely, for every group G, there is a groupoid BG with one object, and automorphisms given by the group G. To explore how important this construction is, we borrow a definition from topology.

Definition 3.7.3. A groupoid $\mathcal{G}$ is called *path-connected* if, for every two objects $x, y \in \mathrm{Ob}(\mathcal{G})$, there is a morphism from x to y in $\mathcal{G}$.[23] A *path component* of G is

[21] One extreme example of just how heuristic this is, is that limits in $\mathcal{C}$ are colimits in $\mathcal{C}^{\mathrm{op}}$. This means that a description like 'limits are subobjects of Cartesian products' cannot apply in all cases.

[22] In category theory, many standard algebraic structures — groups, algebras, etc. — have a "many-object" generalization. Often, one calls the many-object generalization of a *thing* and *thingoid*. In this sense, a groupoid is "like a group, but with many objects" as the next exercise helps to make precise. However, this terminological convention is not uniformly observed. For instance, a category is a multi-object version of a monoid, but no-one calls categories *monoidoids*.

[23] The topological analogy here will become clearer in the next chapter, where we will see that every topological space X gives rise to a groupoid in which the morphisms are paths in X. The space X is then path-connected if and only if this groupoid is.

a maximal path-connected full subgroupoid. We write $\pi_0(\mathcal{G})$ for the set of path components of a groupoid $\mathcal{G}$.

Lemma 3.7.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. Then the canonical maps $\pi_0(\mathcal{G} \times \mathcal{H}) \to \pi_0(\mathcal{G})$ and $\pi_0(\mathcal{G} \times \mathcal{H}) \to \pi_0(\mathcal{H})$ display $\pi_0(\mathcal{G} \times \mathcal{H})$ as a product. Explicitly, the map*

$$\begin{aligned} \pi_0(\mathcal{G} \times \mathcal{H}) &\longrightarrow \pi_0(\mathcal{G}) \times \pi_0(\mathcal{H}) \\ [(x,y)] &\longmapsto ([x],[y]) \end{aligned}$$

is a bijection.

Proof. This is immediate from the fact that (x,y) is isomorphic to (z,w) in $\mathcal{G} \times \mathcal{H}$ if and only if x is isomorphic to z and y is isomorphic to w. ☐

Construction 3.7.5. Given a collection $\{\mathcal{G}_i\}_{i \in I}$ of groupoids, we can explicitly construct their coproduct $\coprod_{i \in I} \mathcal{G}_i$ as follows.

- The set of objects is simply the disjoint union of the object sets of the individual groupoids:
$$\mathrm{Ob}\left(\coprod_{i \in I} \mathcal{G}_i\right) = \coprod_{i \in I} \mathrm{Ob}(\mathcal{G}_i).$$
- For any two objects $x, y \in \mathrm{Ob}\left(\coprod_{i \in I} \mathcal{G}_i\right)$ the hom-sets are given by
$$\mathrm{Ob}\left(\coprod_{i \in I} \mathcal{G}_i\right)(x,y) := \begin{cases} \mathcal{G}_i(x,y) & \exists i \text{ s.t. } x, y \in \mathrm{Ob}(\mathcal{G}_i) \\ \varnothing & \text{else.} \end{cases}$$
- The composition and unit morphisms are given by the composition and unit maps of the individual groupoids $\mathcal{G}_i$ when they are not trivial.

It is straightforward, if tedious, to check that this is, indeed, a coproduct in the category Grpd of groupoids.

Lemma 3.7.6. *Let $\mathcal{G}$ be a path-connected groupoid. Then there is a group G and an equivalence of categories $\mathcal{G} \simeq BG$. More generally, if $\mathcal{G}$ is any groupoid, then there are groups $\{G_i\}_{i \in \pi_0(X)}$ and an equivalence $\mathcal{G} \simeq \coprod_{i \in \pi_0(X)} BG_i$.*

Proof. We prove the second statement, as the first is a special case. Let $\mathcal{G}$ be a groupoid. For each path component $i \in \pi_0(\mathcal{G})$ choose a object x_i lying in that path component. Then by Lemma 3.1.10, $\mathrm{End}_{\mathcal{G}}(x_i) = \mathrm{Aut}_{\mathcal{G}}(x_i)$ is a group under composition. We define a functor

$$J : \coprod_{i \in \pi_0(\mathcal{G})} B\mathrm{Aut}_{\mathcal{G}}(x_i) \longrightarrow \mathcal{G}$$

which sends the unique object of $B\mathrm{Aut}_{\mathcal{G}}(x_i)$ to x_i, and sends an element $f \in \mathrm{Aut}_{\mathcal{G}}(x_i)$ to itself (viewed as a morphism from x_i to x_i in $\mathcal{G}$). This preserves composition and identities by construction. Moreover, since every object in $\mathcal{G}$ is isomorphic to one of the x_i (since it lies in the same path component as one of them), the functor J is essentially surjective. Since the induced map on hom-sets is simply the identity

$$\mathrm{End}_{\mathcal{G}}(x) \longrightarrow \mathrm{End}_{\mathcal{G}}(x)$$

the functor J is fully faithful, and thus, by Proposition 3.4.10, is an equivalence of categories. ☐

Remark 3.7.7. We can, in fact, strengthen the statement of Lemma 3.7.6. If for every object $y \in \mathrm{Ob}(\mathcal{G})$, we choose an isomorphism $f_y : y \to x_{[y]}$ to the chosen representative $x_{[y]}$ of the path component of y, we can define a functor

$$P : \mathcal{G} \longrightarrow \coprod_{i\in\pi_0(\mathcal{G})} B\mathrm{Aut}_{\mathcal{G}}(x_i)$$

which sends every object y to $x_{[y]}$, and sends a morphism $g : y \to z$ to the composite $f_z \circ g \circ f_y^{-1}$. If we let f_{x_i} be the identity, then $P \circ J$ is the identity functor, and so, in fact, we have a *retract* of groupoids which is also an equivalence.

What this proposition tells us is that, in effect, we can think of groupoids as being something like “collections of groups”. This is not fully rigorous, but is a quite useful intuition to keep in mind.

In the final section of this course, we will make heavy use of pushouts of groupoids. These are however, computationally challenging to work with, and so we briefly digress to show that in special cases, these agree with ordinary groups.

Construction 3.7.8. Let G and H be groups. We want to define a group $G * H$ which is sort of like a disjoint union of G and H, but such that elements of G do not commute with elements of H. The idea is to let the elements be *formal* products

$$a_1 \cdot a_2 \cdot a_3 \cdots a_k$$

of elements in G and H. This, however, forgets the original group structures of G and H. To add them back in, we identify

$$a_1 \cdot a_2 \cdots a_i \cdot a_{i+1} \cdots a_k = a_1 \cdot a_2 \cdots (a_i a_{i+1}) \cdots a_k$$

whenever a_i and a_{i+1} are both in G, or both in H. We similarly identify

$$a_1 \cdots a_{i-1} \cdot a_i \cdot a_{i+1} \cdots a_k = a_1 \cdots a_{i-1} \cdot a_{i+1} \cdots a_k$$

whenever a_i is the identity element in either G or H.[24]

We call the resulting group the **free product** of G and H, and denote it by $G * H$.

Exercise 3.7.9. Show that the free product is coproduct in the category Grp.

To this definition, we add a little bit of extra structure.

Definition 3.7.10. Suppose we are given homomorphisms of groups

$$\begin{tikzcd} G & K \arrow[l, "\phi"'] \arrow[r, "\psi"] & H \end{tikzcd}$$

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[24] Note that this formally identifies e_G and e_H with the empty product, which is the identity of the free group.

We define the **pushout** of these groups to be

$$G *_K H := (G * H)_{/\sim}$$

where we declare $\phi(k) \sim \psi(k)$ for any $k \in K$.[25]

Remark 3.7.11. It is worth noting that, while the notation $G *_K H$ does not include ϕ and ψ, the nature of the maps ϕ and ψ is *very* important. For example, consider two diagrams

$$\begin{tikzcd} \mathbb{Z} & \mathbb{Z} \arrow[l, "0"'] \arrow[r, "0"] & \mathbb{Z} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{Z} & \mathbb{Z} \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & \mathbb{Z} \end{tikzcd}$$

The pushout of the first diagram is simply the free product $\mathbb{Z} * \mathbb{Z}$, since ψ and ϕ do not give us any new relations.

In the second case, however, we get that the pushout is $(\mathbb{Z} * \mathbb{Z})_{/\sim}$, where $\sim$ is a relation which identifies elements from the second copy of $\mathbb{Z}$ with elements from the first copy of $\mathbb{Z}$. Consequently, we see that the pushout is simply $\mathbb{Z}$ again.

Exercise 3.7.12. Show that the pushout of groups described in Definition 3.7.10 is the pushout in the category Grp.

Proposition 3.7.13. *The functor*

$$B : \mathsf{Grp} \longrightarrow \mathsf{Grpd}$$

which sends a group G to the groupoid BG preserves pushouts.

Proof. Suppose given a pushout square

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\psi"] & H \arrow[d, "\delta"] \\ K \arrow[r, "\gamma"'] & P \end{tikzcd}$$

of groups. Consider a groupoid $\mathcal{A}$ and a cone

$$\begin{tikzcd} BG \arrow[r, "B\phi"] \arrow[d, "B\psi"] & BH \arrow[d, "\eta"] \\ BK \arrow[r, "\mu"'] & \mathcal{A} \end{tikzcd}$$

in Grpd. The commutativity of this square implies that there is an object $x = \eta(*_H) = \mu(*_K)$ in $\mathcal{A}$ such that the diagram factors through the inclusion

$$B\mathrm{Aut}_{\mathcal{A}}(x) \longrightarrow \mathcal{A}$$

as

$$\begin{tikzcd} BG \arrow[r, "B\phi"] \arrow[d, "B\psi"] & BH \arrow[d] \arrow[ddr, bend left, "\eta"] & \\ BK \arrow[r] \arrow[drr, bend right, "\mu"'] & B\mathrm{Aut}_{\mathcal{A}}(x) \arrow[dr] & \\ & & \mathcal{A} \end{tikzcd}$$

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[25] Technically, we are taking the smallest normal subgroup of $G * H$ containing the elements $\phi(k) * \phi(k^{-1})$, and quotienting by that.

Similar considerations show that any morphism $BP \to \mathcal{A}$ which could fulfill the universal property of the pushout would also have to factor through $B\mathrm{Aut}_{\mathcal{A}}(x)$. However, since $\mathrm{Aut}_{\mathcal{A}}(x)$ is a group, and P is a pushout in the category of groups, there is a unique such homomorphism, and thus a unique functor $BP \to \mathcal{A}$ making the diagram commute. As such, the square

$$\begin{tikzcd} BG \arrow[r, "B\phi"] \arrow[d, "B\psi"] & BH \arrow[d, "B\delta"] \\ BK \arrow[r, "B\gamma"'] & BP \end{tikzcd}$$

is a pushout square, as desired. $\square$

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