Seminar On Non-Commutative Hodge Structures

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Contents

Introduction

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Hodge Structures

Let M be a real C^∞ manifold. Then we have $A^k(M)$, the (real) vector spaces of C^∞ k-forms on M. These piece together to form a cochain complex[1]:

$$A^\bullet(M) := A^0(M) \xrightarrow{d} A^1(M) \xrightarrow{d} \cdots \xrightarrow{d} A^{2n}(M)$$

We then have

Theorem (de Rham)**.** $H^k(A^\bullet(M)) \cong H^k(M, \mathbb{R})$[2]

Now suppose that M has a complex structure J. Let $z_1, z_2, \ldots, z_n$ local complex coordinates, with $z_j = x_j + iy_j$. Then we have a decomposition:

$$\begin{aligned} A^1(M) \otimes_{\mathbb{R}} \mathbb{C} &\cong A^{1,0}(M) \oplus A^{0,1}(M) \\ \omega &= \sum f_j dz_j + \sum g_j d\overline{z_j} \end{aligned}$$

or, more generally:

$$A^k(M) \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{p+q=k} A^{p,q}(M)$$

Hodge theory asks (and) answers the fundamental question:
DOES THIS DECOMPOSITION DESCEND TO COHOMOLOGY?

Definition. $H^{p,q} \subseteq H^k(A^\bullet(M) \otimes_{\mathbb{R}} \mathbb{C})$ is the subspace given by classes represented by closed (p,q)-forms[3]

Theorem (Hodge)**.** *Assume M is a compact Kähler*[4] *manifold. Then*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M)$$

Further, we have that $\overline{H^{p,q}} = H^{q,p}$. This leads us to the *Hodge Diamond*, a way to codify the symmetries of the cohomology of Kähler manifolds.

[1] The *de Rham Complex*, where d denotes the exterior derivative. We take $\dim(M) = 2n$ for agreement with the complex case.

[2] Where $H^k(M, \mathbb{R})$ is the standard singular homology. In some sense the de Rham theorem states that an analytically defined chain complex yields a purely topological invariant.

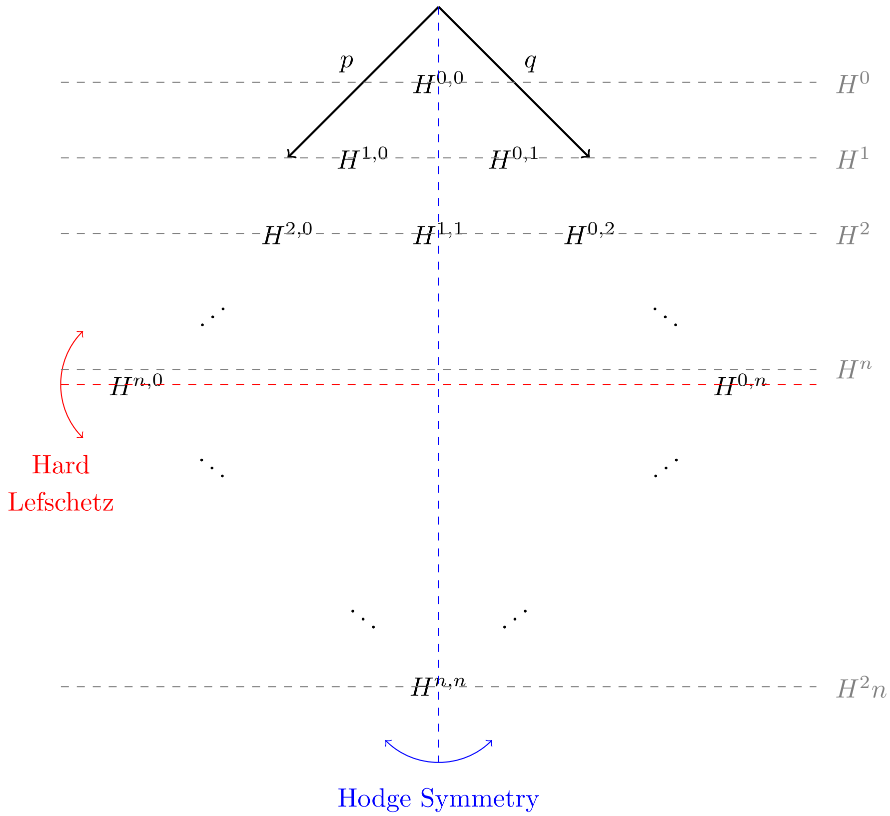
[3] Naïvely, one might expect that, in general

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M)$$

but this is not generally true.

[4] That is, M is equipped with a complex structure J, a symplectic structure ω, and a Riemannian metric g such that

$$g(X, Y) = \omega(X, JY)$$



Application. M compact Kähler. Then the odd Betti numbers are even. For example, $S^1 \times S^3$ cannot be Kähler.

Definition (1)**.** A *Hodge structure of weight k* consists of a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ together with a decomposition

$$H := H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{p+q=k} H^{p,q}$$

such that $\overline{H^{p,q}} = H^{q,p}$.

or, alternatively

Definition. A *Hodge structure of weight k* consists of a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$[5] together with a finite decreasing filtration $F^\bullet H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$[6]

$$\cdots \subset F^p \subset F^{p+1} \subset \cdots$$

such that for all p, q such that $p + q = k +_1$, we have

- $F^p H \cap F^q H = \{0\}$
- $F^p H \oplus F^q H = H$

[5] Called the *rational lattice.*
[6] Called the *Hodge filtration.*

To see that (1)⇒ (2), simply set

$$F^pH = \bigoplus_{1 \geq p} H^{i,k-i}$$

. To see the reverse implication, set

$$H^{p,q} := F^pH \cap \overline{F^qH}$$

Hodge Theory from the Algebraic Perspective

Fact: Every smooth projective variety $X/\mathbb{C}$ is a compact Kähler manifold.

GROTHENDIECK: The Hodge filtration can be constructed in an intrinsically algebraic way. There is an *algebraic de Rham complex*[7]:

$$\Omega_X^\bullet := \Omega_X^0 \to \Omega_X^1 \to \cdots \to \Omega_X^n$$

[7] Which is, in fact, a complex of coherent sheaves on X.

Theorem (Grothendieck)**.** *The hypercohomology*[8] *of the algebraic de Rham complex satisfies:*

$$\mathbb{H}^k(\Omega_X^\bullet) \cong H^k(X(\mathbb{C}), \mathbb{C})$$

[8] The hypercohomology is defined as the derived local section functor

$$\mathbb{H}^k(\Omega_X^\bullet) := R^k\Gamma_X(\Omega_X^\bullet)$$

The complex $\Omega_X^\bullet$ has a filtration

$$F^p\Omega_X^\bullet = 0 \to \cdots \to 0 \to \Omega_X^p \to \cdots \to \Omega_X^n$$

Which induces a filtration on $\mathbb{H}^k(\Omega_X^\bullet)$[9] And a corresponding spectral sequence (the so-called *Hodge to de Rham* spectral sequence):

$$E_2^{p,q} = H^q(X, \Omega_X^p) \Rightarrow \mathbb{H}^{p+q}(\Omega_X^\bullet)$$

[9] This is an elementary fact from homological algebra. See eg Weibel.

Question (Grothendieck)**.** Can we prove, *purely algebraically*, that this spectral sequence degenerates?

Answer (Deligne-Illusie)**.** Yes!

Non-Commutative Geometry

Let $\mathcal{A}$ be a dg category. Then we get invariants

- Hochschild homology $HH_*(\mathcal{A})$
- Periodic Cyclic Homology $HP_*(\mathcal{A})$
- Spectral sequence $HH_*(\mathcal{A}) \Rightarrow HP_*(\mathcal{A})$

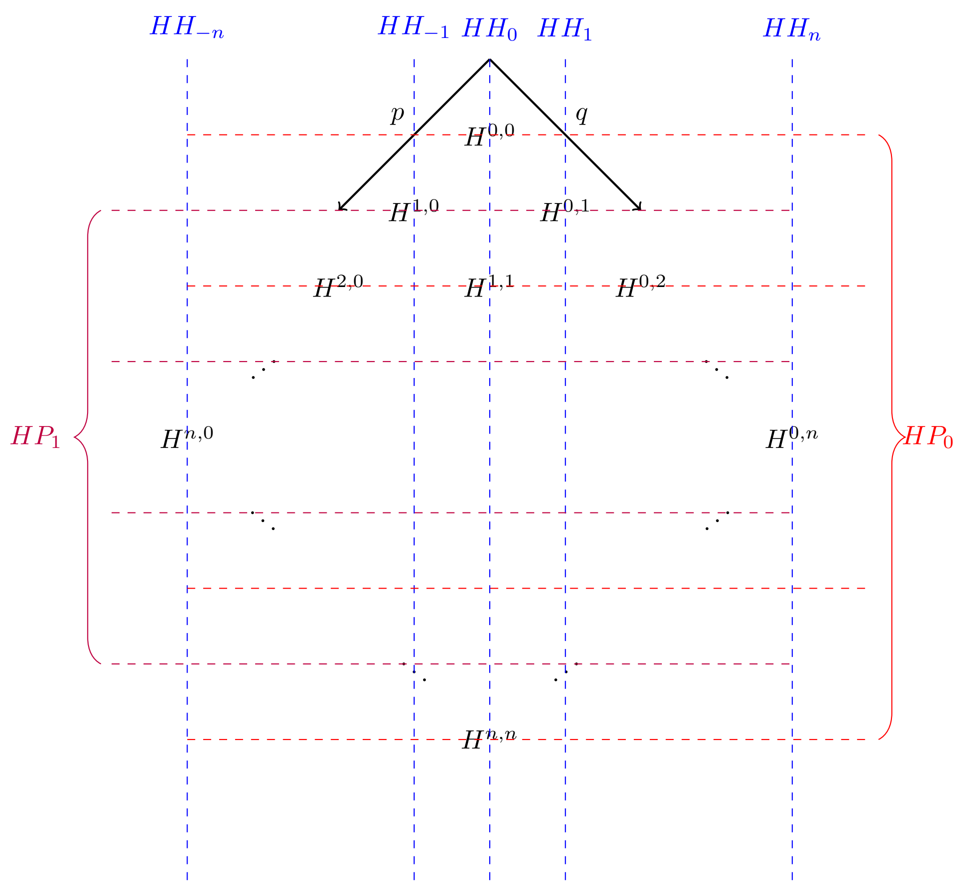
Example (Generalized Hochschild-Kostent-Rosenberg)**.** Let X be a smooth projective variety over $\mathbb{C}$.

$$HH_k(\mathrm{Perf}_X) \cong \bigoplus_{p-q=k} H^q(X, \Omega^p)$$

$$HP_0(\mathrm{Perf}_X) \cong \bigoplus_{k \text{ even}} H^k(X(\mathbb{C}), \mathbb{C})$$

$$HP_1(\mathrm{Perf}_X) \cong \bigoplus_{k \text{ odd}} H^k(X(\mathbb{C}), \mathbb{C})$$

So we have something like a generalization of the Hodge diamond:



Additionally, we have a spectral sequence

$$HH_* \Rightarrow HP_*$$

which in some sense recovers the Hodge to de Rham spectral sequence.

Question (For the seminar)**.** Can we define Non-Commutative Hodge Structures for suitable dg-categories not necessarily of the form Perf_X?

- Can we find a Hodge Filtration?
- Can we find a rational lattice?

HKR for Rings

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Hochschild Homology and Variants

Let k be a commutative ring, and A be a unital associative algebra projective over k.

Definition. The *Hochschild Homology of* A, $HH_*(A)$, is[10]

$$\mathrm{Tor}_*^{A^e}(A, A)$$

[10] Where

$$A^e = A \otimes_k A^{op}$$

is the universal enveloping algebra of A.

To relate this definition to an explicit chain complex, we take the *bar resolution of* A:

$$\underbrace{\boxed{\cdots \to A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2}}}_{C_*^{bar}(A)} \xrightarrow{b'} A$$

With the differential b' given explicitly as[11]

$$b' = \sum_{i=0}^{n-1} (-1)^i d_i$$

[11] The d_i come from the standard simplicial structure on $C_*^{bar}(A)$, and are given explicitly by

$$d_i(a_0, a_1, \ldots, a_n) = \begin{cases} (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) & 0 \leq i < n \\ (a_n a_0, a_1, \ldots, a_{n-1}) & i = n \end{cases}$$

Tensoring A with the bar resolution, we get a chain complex that computes the Hochschild Homology of A: The *Hochschild Chain Complex.*

$$C_*(A) := \cdots \to A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A$$

where

$$b = \sum_{i=0}^{n} (-1)^i d_i$$

On this chain complex, there is a cyclic action $t : C_n(A) \to C_n(A)$

$$t(a_0, a_1, \ldots, a_n) = (-1)^n (a_n, a_0, a_1, \ldots, a_{n-1})$$

The norm of this action is

$$N = \sum_{i=0}^{n} t^i$$

Additionally, there is a map[12]:

$$s : A^{\otimes n} \to A^{\otimes n+1}$$
$$(a_1, \dots, a_n) \mapsto (1, a_1, \dots, a_n)$$

[12] Both this s map and and the cyclic action are contained in the notion of a *cyclic object*, which is an extension of the notion of a simplicial object. See, eg [4],Ch. 6.

From these operators, we can define *Connes B operator*:

$$B = (1 - t)sN$$

Remark. The B operator has the explicit form

$$\begin{aligned} B(a_0, \dots, a_n) = \sum_{i=0}^{n} \big[&(-1)^{ni}(1, a_i, \dots, a_n, a_0, \dots, a_{i-1}) \\ &-(-1)^{ni}(a_i, 1, \dots, a_n, a_0, \dots, a_{i-1}) \big] \end{aligned}$$

has degree 1, and satisfies the identities

$$B^2 = \{B, b\} = Bb + bB = 1$$

If we take the differential graded algebra $k[\epsilon]$, where $\epsilon^2 = 0$ and $|\epsilon| = 1$, then the B operator turns $C_*(A)$ into a graded $k[\epsilon]$-module[13] under the assignment

$$\epsilon \mapsto B$$

[13] This is equivalent to the notion of a *mixed complex* found in the literature

Definition. The *Cyclic Homology* of A is[14]

$$HC_* = \mathrm{Tor}_*^{k[\epsilon]}(k, C_*(A))$$

The *Negative Cyclic Homology* of A is

$$HC_*^- = \mathrm{Ext}_{k[\epsilon]}^*(k, C_*(A))$$

[14] Note that, thoughout this definition, k represents the graded $k[\epsilon]$-module concentrated in degree 0.

We can specify a $k[\epsilon]$-free resolution for k to compute explicit chain complexes for (negative) cyclic homology:

$$\underbrace{\cdots \to k[\epsilon][-2] \xrightarrow{\epsilon} k[\epsilon][-1] \xrightarrow{\epsilon} k[\epsilon]}_{L_\bullet} \to k$$

More precisely:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & k\epsilon & \xrightarrow{\epsilon} & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & k & \xrightarrow{\epsilon} & k\epsilon & \xrightarrow{\epsilon} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & k & \longrightarrow & k \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

$L_\bullet \otimes_{k[\epsilon]} C_*(A)$ then yields a double complex whose total complex computes $HC_*(A)$:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow b & & \downarrow b & & \downarrow \quad \iddots \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\
\downarrow b & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & \\
\downarrow b & & & & \\
A & & & &
\end{array}
$$

We call the resulting complex the *cyclic chain complex of A*, and write[15]:

$$CC_*(A) := \mathrm{Tot}\left(L_\bullet \otimes_{k[\epsilon]} C_*(A)\right)$$

[15] Notice that the cyclic chain complex can also be represented in a much more compact form as a polynomial algebra over $C_*(A)$:

$$CC_*(A) \cong \left(C_*(A)[u^{-1}], b + Bu\right)$$

where $|u| = -2$.

Similarly, we find that $\mathrm{Hom}_{k[\epsilon]}(L_\bullet, C_*(A))$ gives a double complex whose total complex computes $HC_*^-(A)$.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow b & & \downarrow b & & \downarrow b \\
\cdots & \xleftarrow{B} & A^{\otimes 4} & \xleftarrow{B} & A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} \\
 & & \downarrow b & & \downarrow b & & \downarrow b \\
\cdots & \xleftarrow{B} & A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\
 & & \downarrow b & & \downarrow b & & \downarrow b \\
\cdots & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A & \xleftarrow{B} & 0 \\
 & & \downarrow b & & \downarrow b & & \\
\cdots & \xleftarrow{B} & A & \longleftarrow & 0 & & \\
 & & \downarrow b & & & & \\
 & & 0 & & & &
\end{array}
$$

We call the total complex[16] the *Negative cyclic chain complex of A*, and write

$$CC_*^-(A) := \mathrm{Tot}\left(\mathrm{Hom}_{k[\epsilon]}(L_\bullet, C_*(A))\right)$$

[16] As before, there is an expression in terms of polynomials in u. In this case, though, it is important that we are taking the direct product total complex, so that we get

$$CC_*^-(A) \cong (C_*(A)[[u]], b + Bu)$$

As an analogy to better understand (negative) cyclic homology, we can consider the case of group (co)homology[17]:

[17] The circle action in the right hand column cannot be fully explained here, as it requires ∞-categorical notions to make fully accurate. For a more complete exposition, see [5].

Group Homology	Cyclic Homology
G a group, k a field $G \circlearrowright M \in \mathrm{Vect}_k$ $\downarrow$ $C_*^{cell}(G) = kG$ when G is treated as a discrete topological group, giving the induced action $kG \circlearrowright M$ $\downarrow$	A a k-algebra, k a field $S^1 \circlearrowright (C_*(A), b)$ $\downarrow$ $C_*^{cell}(S^1) = k[\epsilon]$, giving the induced action (precisely the action described above) $k[\epsilon] \circlearrowright (C_*(A), b)$ $\downarrow$
$k \otimes_{kG}^L M = M_{hG}$ 'homotopy coinvariants'	$k \otimes_{k[\epsilon]} C_*(A) = C_*(A)_{hS^1}$
$\mathrm{RHom}_{kG}(k, M) = M^{hG}$ 'homotopy invariants'	$\mathrm{RHom}_{k[\epsilon]}(k, C_*(A)) = C_*(A)^{hS^1}$

When A is commutative, we also get a product on Hochschild homology. It is induced by the *shuffle product* on $C_*(A)$

$$- \times - = sh_{p,q} : C_p(A) \otimes C_q(A) \to C_{p+q}(A \otimes A)$$
$$(a_0, \dots, a_p) \times (a'_0, \dots, a'_q) = \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma)\sigma.(a_0 \otimes a'_0, a_1 \otimes 1, \dots, a_p \otimes 1, 1 \otimes a'_1, \dots, 1 \otimes a'_q)$$

where $\mathrm{Sh}(p, q)$ is the set of p, q-shuffles[18].

[18] A p, q-shuffle is a permutation with preserves the ordering of the first p elements it acts on, and of the last q elements it acts on. More intuitively, it is any permutation that can be obtained by shuffling once a deck of cards that has been divided into two parts. The action of the symmetric group on an element $(c_0, c_1, \dots, c_{p+q}) \in C_*(A \otimes A)$ used in the definition is given by

$$\sigma.(c_0, c_1, \dots, c_{p+q}) = (c_0, c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(p+q)})$$

Lemma. *$- \times -$ satisfies graded Leibnitz rule, that is,*

$$b(x \times y) = b(x) \times y + (-1)^{|x|} x \times b(y)$$

for all $x, y \in C_*(A)$.

Sketch of proof. Let

$$x \times y = \sum \pm (c_0, c_1, \dots, c_{p+q})$$

and consider sets

$$X := \{a_1 \otimes 1, \dots, a_p \otimes 1\}$$
$$Y := \{1 \otimes a'_1, \dots, 1 \otimes a'_q\}$$

where $x = (a_0, a_1, \dots, a_p)$ and $y = (a'_0, \dots, a'_q)$.

Now, given an element $(c_0, \dots, c_{p+q})$ in the above sum, notice that if c_i, c_{i+1} are both in X (resp Y), then $d_i(c_0, \dots, c_{p=q})$ is a summand of $b(x) \times y$ (resp. $x \times b(y)$). If c_i, c_j are in different sets, then $(c_0, \dots, c_{i+1}, c_i, \dots c_{p+q})$ is still a shuffle, and appears with opposite sign. Since A is commutative, we then see that

$$d_i(c_0, \dots, c_{i+1}, c_i, \dots c_{p+q}) = d_i(c_0, \dots c_{p+q})$$

so that the terms in the differential cancel. The rest of the proof amounts to checking signs. □

We then look at the product

$$\mu : A \otimes A \to A$$

which induces

$$\mu : C_*(A \otimes A) \to C_*(A)$$

So we are left with a product

$$- \times - : C_p(A) \otimes C_q(A) \to C_{p+q}(A)$$

Which, by the lemma, descends to Hochschild Homology. More precisely

Theorem. *The product*

$$- \times - : HH_*(A) \otimes HH_*(A) \to HH_*(A)$$

equips $HH_*(A)$ withe the structure of a graded-commutative algebra.

Differential forms

Lemma. *Let A be unital and commutative*[19]*. There is a canonical isomorphism*

$$HH_1(A) \cong \Omega^1_{A|k}$$

from Hochschild Homology to Kähler differentials[20]

[19] For the rest of this section, this assumption will remain in place

[20] *Kähler differentials* are in some sense 'universal derivations' on A. More precisely, $\Omega^1_{A|k}$ is generated over k by symbols da satisfying

$$d(\lambda a + \mu b) = \lambda da + \mu db \qquad \lambda, \mu \in A$$
$$d(ab) = (da)b + a(db)$$

Proof. A commutative implies $b : A \otimes A \to A$ trivial. The image of $b : A^{\otimes 3} \to A^{\otimes 2}$ is

$$K = \langle xy \otimes z - x \otimes yz + zx \otimes y \rangle$$

It is then clear that the maps

$$[a \otimes b] \mapsto a\ db$$
$$a\ db \mapsto [a \otimes b]$$

are well-defined inverse module homomorphisms

$$A \otimes A/K \leftrightarrow \Omega^1_{A|k}$$

□

The shuffle product gives us a map

$$\Omega^1_{A|k} \otimes \Omega^1_{A|k} \to HH_2(A)$$

which factors as

$$\Omega^1_{A|k} \otimes \Omega^1_{A|k} \to \bigwedge^2 \Omega^1_{A|k} = \Omega^2_{A|k} \to HH_2(A)$$

More generally, in fact, it provides a homomorphism of graded algebras. We assert that this, is in fact given by the antisymmetrization maps.

Definition. The *antisymmetrization maps*[21] are the maps

$$\epsilon_n : \Omega^n_{A|k} \to HH_n(A)$$

given by

$$(a_0\, da_1 \cdots\, da_n) \mapsto \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma.(a_0, a_1, \ldots, a_n)$$

[21] Sometimes also refered to collectively as the *HKR map* or the *HKR isomorphism*

Lemma. *The antisymmetrization maps form an algebra homomorphism.*[22]

[22] Note that, if this is the case, then it will be the homomorphism induced by the canonical isomorphism $HH_1(A) \to \Omega^1_{A|k}$, since this map is precisely ϵ_1.

Proof. First, we want to see that the maps defined above do indeed take values in cycles. If we set

$$h(u)(a) = \sum_{i=0}^{n} (-1)^i h_i = \sum_{i=0}^{n} (-1)^i (a_0, \ldots, a_i, u, a_{i+1}, \ldots, a_n)$$

then we can compute directly that

$$b \circ h(u) = 0 - h(u) \circ b$$

and that, when $n = 0, 1$

$$b \circ \epsilon_n = 0$$

Assume now, inductively, that this holds up to n. Then

$$\begin{aligned} b \circ \epsilon_{n+1}(\underline{a}, y) &= (-1)^n b \circ h_(y) \circ \epsilon_n(\underline{a}) \\ &= (-1)^n h(y) \circ b \circ \epsilon_n(\underline{a}) = 0 \end{aligned}$$

So that we do, indeed have an induced morphism to Hochschild homology.

To see that this is an graded algebra homomorphism amounts to showing that the diagrams

$$\begin{array}{ccc} \Omega^p_A \times \Omega^q_A & \xrightarrow{\epsilon_p \times \epsilon_q} & HH_p(A) \times HH_q(A) \\ \wedge \downarrow & & \downarrow \times \\ \Omega^{p+q}_A & \xrightarrow[\epsilon_{p+q}]{} & HH_{p+q}(A) \end{array}$$

commute.

This amounts to showing that

$$\sum_{\tau \in S_p} \sum_{\xi \in S_q} \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\mathrm{sgn}(\xi)\sigma(\tau \times \xi) = \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\sigma$$

which follows from the fact that any permutation in S_{p+q} has a unique expression as a composition of a p,q-shuffle with a product of permutations in S_p and S_q respectively. $\square$

HKR Theorem for commutative rings

Definition. For A a commutative unital ring, we say that A is smooth over k if it is flat over k and if, for any maximal ideal $\mathfrak{m} \subset A$, the kernel

$$I = \ker\left(\mu_{\mathfrak{m}} : (A \otimes_k A)_{\mu^{-1}(\mathfrak{m})} \to A_{\mathfrak{m}}\right)$$

is generated by a regular seguence[23] in $(A \otimes_k A)_{\mu^{-1}(\mathfrak{m})}$

[23] Recall that a sequence of elements $(x_1, \ldots, x_m)$ in A is *regular* if multiplication by x_i in $S/\langle x_1, \ldots, x_m\rangle$ is injective.

Definition. Let R be a commutative ring and V an R-module, with

$$x : V \to R$$

a linear form. The *Koszul complex* of x is

$$\mathcal{K}(x) = (\bigwedge_R^* V, d_x)$$

where the differential is given by

$$d_x(v_0 \wedge \cdots \wedge v_n) = \sum_{i=0}^{n} (-1)^i x(v_i) v_0 \wedge \cdots \wedge \hat{v_i} \wedge \cdots \wedge v_n$$

For the remainder of the talk, let us fix R a commutative ring, I an ideal of R generated by a regular sequence $x = (x_1, \ldots, x_m)$ in R. From this setup, we get a form[24]:

[24] This form can be thought of as a sort of scalar product.

$$x(r_1, \ldots, r_m) = \sum_{i=1}^{m} x_i r_i$$

From this, we get a Koszul complex $\mathcal{K}(x)$

Lemma. *The Koszul complex $\mathcal{K}(x)$ is a resolution of R/I*

Proof. By induction on m. Suppose $m = 1$, then we have the complex

$$\mathcal{K}(x) = \mathcal{K}(x_1) = 0 \to R \xrightarrow{x_1} R \to 0$$

So that

$$H_n(\mathcal{K}(x)) = \begin{cases} R/I & n = 0 \\ 0 & \text{else} \end{cases}$$

Suppose this is true for $m - 1$. Then we can fit $\mathcal{K}(x_m)$ into the exact sequence

$$0 \longrightarrow \mathcal{K}_0 \longrightarrow \mathcal{K}(x_m) \longrightarrow \mathcal{K}_1 \longrightarrow 0$$

$$\begin{CD}
0 @>>> 0 @>>> R @>>> R @>>> 0 \\
@VVV @VVV @VV{x_m}V @VVV @VVV \\
0 @>>> R @>>> R @>>> 0 @>>> 0
\end{CD}$$

If we tensor this exact sequence with

$$L := \mathcal{K}(x_1, \dots, x_{m-1})$$

we get the exact sequence

$$0 \to \mathcal{K}_0 \otimes L \to \mathcal{K}(x) \to \mathcal{K}_1 \otimes L \to 0$$

We can then take the LES on homology to see that

$$0 \to \operatorname{coker}((x_m)^n) \to H_n(\mathcal{K}(x)) \to \ker((x_m)^{n-1}) \to 0$$

where

$$(x_m)^n : H_n(L) \overset{x_m}{\to} H_n(L)$$

For $n > 1$, this tells us $H_n(\mathcal{K}(x)) = 0$. When $n = 1$, we get[25]

$$H_1(\mathcal{K}(x)) = \ker\left(x_m : R/\langle x_1, \dots x_{m-1}\rangle \to R/\langle x_1, \dots x_{m-1}\rangle\right) = 0$$

and when $n = 0$

$$H_0(\mathcal{K}(x)) = \operatorname{coker}\left(x_m : R/\langle x_1, \dots x_{m-1}\rangle \to R/\langle x_1, \dots x_{m-1}\rangle\right) = R/I$$

□

[25] Injectivity in this case follows from the regularity of x_m in $R/\langle x_1, \dots, x_{m-1}\rangle$.

Lemma. *The morphism*

$$\epsilon_* \bigwedge_{R/I}^{*} (I/I^2) \to \operatorname{Tor}_*^R(R/I, R/I)$$

induced by

$$\epsilon_1 : I/I^2 \cong \operatorname{Tor}_1^R(R/I, R/I)$$

is an isomorphism of graded algebras.

Proof. We take the Koszul complex of x as a resolution of R/I to compute Tor, and end up with the complex

$$\left(\bigwedge_R^{*}[R^m] \otimes_R R/I, d_x \otimes 1\right)$$

However, d_x takes coefficients in I, so the differential is identically zero. Hence the homology is

$$\bigwedge^{*} ((R/I)^m) \cong \bigwedge_R^{*} (I/I^2)$$

□

Theorem (HKR). *For any smooth algebra A over k, the antisymmetrization map*

$$\epsilon_* : \omega^*_{A|k} \to HH_*(A)$$

is an isomorphism of graded algebras[26].

[26] Though we will not prove it here, this isomorphism also takes the B operator to the differential on forms.

Proof. Firstly, we notice that $A \cong A^{op}$. Additionally, it suffices to prove the proposition for localizations at maximal ideals, so we have to show

$$\Omega^n_{A_\mathfrak{m}|k} \cong (\Omega^n_{A|k})_\mathfrak{m} \to \left(\mathrm{Tor}_n^{A\otimes A}(A,A)\right)_\mathfrak{m}$$

for any maximal ideal $\mathfrak{m} \subset A$.

We can notice that

$$\theta_n : \left(\mathrm{Tor}_n^{A\otimes A}(A,A)\right)_\mathfrak{m} \to \mathrm{Tor}_n^{(A\otimes A)_{\mu^{-1}(\mathfrak{m})}}(A_\mathfrak{m}, A_\mathfrak{m})$$

is a natural transformation of homological functors with θ_0 an isomorphism. Hence, it is a natural isomorphism. If we then let

$$R = (A \otimes A)_{\mu^{-1}(\mathfrak{m})}$$
$$R/I = A_\mathfrak{m}$$

in the terminology of the previous lemma, then the lemma implies the theorem. □

Hochschild Homology of Schemes

Michael Brown

We fix, for the rest of the talk, k a field, and X a quasi-compact separated k-scheme.

Goal: Sketch a proof[27] that, once they are defined,

$$HH_*(\mathrm{perf}_{dg} X) \cong HH_*(X)$$

and similarly for HC_*, HN_*, and HP_*.[28] This establishes a well-defined notion of Hochschild homology on X[29]

[27] Following, among other sources, Keller's paper [6]

[28] In the notation of the last talk, $HN_* = HC_*^-$.

[29] There is a parallel story for Hochschild cohomology. See for example [8] and [9].

Defining invariants of Schemes

Definition. A *mixed complex* of k-vector spaces is a dg-module over the dg algebra $k[x]/x^2$, $|x| = 1$, which has trivial differentials[30].

Further, we set the following notation

$$\mathcal{D}\,\mathrm{Mix}(k) := \mathcal{D}(k[x]/x^2)$$

considered as a dg-algebra. We also define $\mathcal{D}\,\mathrm{Mix}(X)$ to be the derived category of sheaves of dg-modules over $k[x]/x^2$

[30] The differentials here follow chain complex conventions, ie are of degree -1.

Example. The Hochschild complex $C_*(A)$ associated to a k-algebra is a mixed complex equipped with the Connes B-operator, as we saw in the last talk. Call this mixed complex $M(A)$.

We then have a presheaf

$$U \mapsto M(\Gamma(U, \mathcal{O}_x))$$

and can set

$$M(\mathcal{O}_X)$$

to be the sheafification of this presheaf[31]. We then define the Hochschild homology to be[32]

$$HH_*(X) := \mathbb{H}^{-*}(X, M(\mathcal{O}_X))$$

[31] Level-wise in this complex.

[32] Note: it is not immediately obvious that the hypercohomology inherits a differential from the mixed complex structure of the sheaf. As it turns out, it can, in fact, be equipped with a 'Connes B operator', but this is a fact that requires some checking

Now, given a k-algebra A, let $BM(A)$ denote the direct sum totalization of the bicomplex[33]:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
b\downarrow & & b\downarrow & & \downarrow \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \quad \iddots \\
b\downarrow & & b\downarrow & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & \\
b\downarrow & & & & \\
A & & & &
\end{array}
$$

[33] That is, the bicomplex for the negative cyclic homology of A

And let $BM(\mathcal{O}_X)$ denote the sheafification of the presheaf

$$U \mapsto BM(\Gamma(U, \mathcal{O}_X))$$

We then can define the cyclic homology of X to be the hypercohomology

$$HC_*(X) := \mathbb{H}^{-*}(X, BM(\mathcal{O}_X)$$

If we denote by $\mathbb{H}$ the hypercohomology complex corresponding to $HC_*(X)$, then there is a surjection, the *Connes periodicity operator*

$$s : \mathbb{H}[2] \to \mathbb{H}$$

We can define a new complex via the limit[34]

$$L_* := \varprojlim \left(\cdots \xrightarrow{s} \mathbb{H}[2p+2] \xrightarrow{s} \mathbb{H}[2p] \to \cdots \xrightarrow{s} \mathbb{H} \right)$$

[34] Notice that in coordinates one can think of s as multiplication by the formal variable u introduced last talk. In this sense, the inverse limit simply 'inverts' u

Using this complex, we can then define periodic cyclic homology

$$HP_n := H^{-n}(L_*)$$

and, using the map (which exists by universal property)

$$L_* \to \mathbb{H}[-2]$$

we can also define negative cyclic homology

$$HN_n(X) := \ker (L_* \to \mathbb{H}[-2])$$

Theorem (Geller, Weibel. [10])**.** *If* $X = \text{Spec}(A)$*,* $HH_*(X) = HH_*(A)$.

Proof. Let $\mathcal{HH}_n(X)$ be the Sheafification of the presheaf

$$U \mapsto HH_*\Gamma(U, \mathcal{O}_X)$$

Then there exists a bounded spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{HH}_{-q}(X)) \Rightarrow HH_{-p-q}(X)$$

This spectral sequence collapses at $p = 0$[35]. □

[35] To see that $HC_n(\text{Spec}(A)) \cong HC_n(A)$, see the main theorem 2.5 of Weibel, [7].

DG-Categories

Let $\mathcal{C}$ be a dg category. We can associate a bicomplex to $\mathcal{C}$ with columns as follows[36]

$$C_n = \bigoplus_{X_0,\ldots,X_n} \mathcal{C}(X_n, X_0) \otimes_k \mathcal{C}(X_{n-1}, X_n) \otimes_k \cdots \otimes_k \mathcal{C}(X_0, X_1)$$

[36] Where $\mathcal{C}(X,Y)$ here denotes the morphism complex between the two objects, and the sum ranges over all tuples of objects.

The Horizontal differentials are given by alternating sums of the following 'face maps'

$$\begin{aligned} d_i : C_n &\to C_{n-1} & 0 \le i < n \\ (f_n, \ldots, f_0) &\mapsto (f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_0) \end{aligned}$$

$$\begin{aligned} d_n : C_n &\to C_{n-1} \\ (f_n, \ldots, f_0) &\mapsto (-1)^{n+\sigma} (f_0 \circ f_n, \ldots, f_1) \end{aligned}$$

where

$$\sigma = |f_0| \left(|f_1| + \cdots + |f_{n-1}| \right)$$

Let $C(\mathcal{C})$ be the direct sum totalization of this bicomplex, and call it the *Hochschild complex of* $\mathcal{C}$.

Definition. A complex P of $\mathcal{O}_X$-modules is *strictly perfect* if, for every $x \in X$ there exists a neighborhood U of x such that $P|_U$ is isomorphic to a bounded complex of summands of locally free $\mathcal{O}_X$-modules.

A complex P of $\mathcal{O}_X$-modules is *perfect* if, for every $x \in X$ there exists a neighborhood U of x such that P_U is quasi-isomorphic to a strictly perfect complex.

We also write $\operatorname{perf}_{dg} \mathcal{O}_X$ for the dg quotient of the dg category of perfect complexes on X. And analogously for $\operatorname{strperf}_{dg} \mathcal{O}_X$.

Theorem (Keller, [6])**.** *There is an isomorphism in* $\mathcal{D}\operatorname{Mix}(k)$[37]

$$\tau : M(\operatorname{perf}_{dg} \mathcal{O}_X) \xrightarrow{\simeq} \mathbb{R}\Gamma(X, M(\mathcal{O}_X))$$

[37] Where we write $M(\mathcal{C})$ for $C(\mathcal{C})$ as a mixed complex, for a dg category $\mathcal{C}$. And

$$\mathbb{R}\Gamma(X, -) : \mathcal{D}\operatorname{Mix}(X) \to \mathcal{D}\operatorname{Mix}(k)$$

is the total right derived functor of the global sections functor. Note that there is a quasi-isomorphism

$$\mathbb{R}\Gamma(X, M(\mathcal{O}_X)) \xrightarrow{\simeq} \mathbb{H}(X, M(\mathcal{O}_X))$$

as proved in the appendix of [6]

We now write $M(\mathit{perf}_{dg}\mathcal{O}_X)$ for the sheafification of

$$U \mapsto M(\operatorname{perf}_{dg} \mathcal{O}_X)$$

For every $U \subset X$ open, there are maps

$$M(\Gamma(U, \mathcal{O}_X)) \to M(\operatorname{proj}(\Gamma(U, \mathcal{O}_X))) \to M(\operatorname{perf}_{dg} \mathcal{O}_U)$$

therefore, the exists a morphism of sheaves

$$M(\mathcal{O}_X) \xrightarrow{\alpha} M(\mathit{perf}_{dg}\mathcal{O}_X)$$

Lemma (Key Lemma, [6]). *α is an isomorphism in $\mathcal{D}\,\mathrm{Mix}(X)$.*

Sketch of Proof. [38] Our strategy will be to show that α is an quasi-isomorphism on stalks.

First notice that

$$M(\mathit{perf}_{dg}\mathcal{O}_X) \overset{\cong}{\to} \lim_{\to}\big(M(\mathrm{perf}_{dg}\,\mathcal{O}_U)\big)$$

We will show that

$$\beta : \lim_{\to} \mathrm{perf}_{dg}\,\mathcal{O}_U \to \mathrm{perf}_{dg}\,\mathcal{O}_{X,x}$$

is a quasi-isomorphism. This will complete the proof, since the following diagram commutes

$$\begin{array}{ccccc} M(\mathcal{O}_{X,x}) & \xrightarrow{\alpha} & M(\mathit{perf}_{dg}\mathcal{O}_X)_x & \xrightarrow{a\ \simeq} & M(\lim_{\to} \mathrm{perf}_{dg}\,\mathcal{O}_U) \\ & \searrow^{b\ \simeq} & & & \downarrow \\ & & & & M(\mathrm{perf}_{dg}\,\mathcal{O}_X, x) \end{array}$$

where a is a quasi-isomorphism because of the properties of the limit, and b is a quasi-isomorphism by the assumption of Morita invariance.

To see that β is a quasi-isomorphism, notice that we have a commutative diagram

$$\begin{array}{ccc} \lim_{\to} M(\mathrm{perf}_{dg}\,\mathcal{O}_U) & \xrightarrow{\beta} & M(\mathrm{perf}_{dg}\,\mathcal{O}_{X,x}) \\ c\uparrow & & \uparrow d \\ \lim_{\to} M(\mathrm{strperf}_{dg}\,\mathcal{O}_U) & \xrightarrow{e} & M(\mathrm{strperf}_{dg}\,\mathcal{O}_{X,x}) \\ a\uparrow & \nearrow_{b} & \\ \lim_{\to} M(\mathrm{strperf}_{dg}\,\Gamma(U, \mathcal{O}_X)) & & \end{array}$$

Since a and b are quasi-isomorphisms, so is e, since c, e, and d are quasi-isomorphisms[39], we see that β is. □

[38] We here use the assumption of the morita invariance of

$$M(-) : \mathrm{dg} - \mathrm{Cat} \to \mathcal{D}\,\mathrm{Mix}(k)$$

which will be discussed in future talks.

[39] d is a quasi-isomorphism since it is an affine case. a and c are because we can restrict to affines. b is a quasi-isomorphism by Morita invariance.

To define the map τ of the theorem, we use the following square in $\mathcal{D}\,\mathrm{Mix}(k)$

$$\begin{array}{ccc} M(\mathrm{perf}_{dg}\,\mathcal{O}_X) & \longrightarrow & \Gamma(X, M(\mathit{perf}_{dg}\mathcal{O}_X))) \\ \tau \,\vdots\, \downarrow & & \downarrow \\ \mathbb{R}\Gamma(X, M(\mathcal{O}_X)) & \xrightarrow[\text{Key Lemma}]{\simeq} & \mathbb{R}\Gamma(X, M(\mathit{perf}_{dg}\emptyset_X)) \end{array}$$

Claim. τ is an isomorphism when $X = \operatorname{Spec}(A)$.

Proof. We have the following commutative diagram

$$\begin{array}{ccc} M(A) & \xrightarrow{\ b\ } & M(\operatorname{perf}_{dg} \mathcal{O}_X) \\ {\scriptstyle c}\downarrow & \nearrow_{\tau} & \\ \mathbb{R}\Gamma(X, M(\mathcal{O}_X)) & & \\ {\scriptstyle a}\downarrow & & \\ \mathbb{H}(X, M(\mathcal{O}_X)) & & \end{array} \qquad M(A) \xrightarrow{\simeq} \mathbb{H}(X, M(\mathcal{O}_X))$$

Since a is a quasi-isomorphism, so is c, and since c and b are quasi-isomorphisms, τ is as well. ☐

Proposition ([6]). *If $V, W \subset X$ are open and quasi-compact, and $X = V \cup W$, there exists an isomorphism of triangles in $\mathcal{D}\operatorname{Mix}(k)$*[40]

$$\begin{array}{ccc} M(\operatorname{perf}_{dg} \mathcal{O}_X) & \xrightarrow{\ \beta\ } & \mathbb{R}\Gamma(M(\mathit{perf}_{dg}\mathcal{O}_X)) \\ \downarrow & & \downarrow \\ M(\operatorname{perf}_{dg} \mathcal{O}_V) \oplus M(\operatorname{perf}_{dg} \mathcal{O}_W) & \longrightarrow & \mathbb{R}\Gamma(V, M(\mathit{perf}_{dg}\mathcal{O}_V)) \oplus \mathbb{R}\Gamma(W, M(\mathit{perf}_{dg}\mathcal{O}_W)) \\ \downarrow & & \downarrow \\ M(\operatorname{perf}_{dg} \mathcal{O}_{V\cap W}) & \longrightarrow & \mathbb{R}\Gamma(V \cap W, M(\mathit{perf}_{dg}\mathcal{O}_{V\cap W})) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

[40] Note that this can be thought of as a sort of Mayer-Vietoris-type result.

This tells us that

Corollary. *Our two definitions of Hochschild Homology coincide*[41], *ie*

$$HH_*(\operatorname{perf}_{dg} \mathcal{O}_X) \cong HH_*(X)$$

[41] Additionally, we have that

$$HC_*(\operatorname{perf}_{dg} \mathcal{O}_X) \cong H_*(k \otimes^L_{k[x]/x^2} \mathbb{R}\Gamma(X, M(\mathcal{O}_X)))$$

But it is not immediately clear that this is $HC_*(X)$. This is, however, proved by Keller in [6], and can further be used to deduce the same for HN and HP.

Now, let $\mathbb{Q} \subset k$, and

$$\epsilon : M(A) \to (\Omega^*_{A|k}, 0, d)$$

be the antisymmetrization maps from last talk[42]. As we saw before, when $A|k$ is smooth, e is a quasi-isomorphism. However, we have the same result for schemes, which follows from simply sheafifying. That is

[42] We list the target as a triple to emphasize that it is a *mixed* chain complex.

Proposition (HKR for Schemes)**.** *The induced map*

$$e : M(\mathcal{O}_X) \to (\Omega^*_{X|k}, 0, \partial)$$

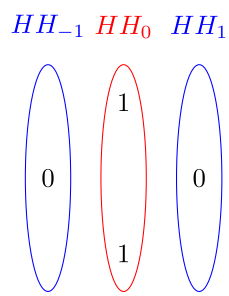
is an isomorphism when $X|k$ is smooth and $\mathbb{Q} \subset k$.

As a result, we get the decomposition

$$HH_i(X) \cong \bigoplus_{q-p=i} H^q(X, \Omega^p_X)$$

Example. We can, for example, compare the Hodge diamond to the dimensions of the Hochschild Homology groups. For example

$HH_*(\mathbb{P}^1)$



Differential Graded Categories

Gustavo Jasso

For the most part, this talk will follow [11], and will try as far as possible to use the same notation.

Let k be a commutative ring, and denote by $C(k)$ the category of complexes of k-modules.

Preliminaries

Definition. A *dg category* $\mathcal{A}$ consists of

- A class $\mathrm{Obj}(\mathcal{A})$ of objects
- For any $x, y \in \mathrm{Obj}(\mathcal{A})$, a complex
$$\mathcal{A}(x,y) \in C(k)$$
of morphisms
- For any $x, y, z \in \mathrm{Obj}(\mathcal{A})$ a morphism of complexes
$$\mathcal{A}(x,y) \otimes_k \mathcal{A}(y,z) \to \mathcal{A}(x,z)$$
satisfying unitality and associativity.

Example. a) Let $\mathcal{A}$ be a dg category such that $\mathrm{Obj}(\mathcal{A}) = \{*\}$. Then
$$A := \mathcal{A}(*,*)$$
inherits the structure of a dg-algebra
$$(A \otimes A, d \otimes 1 + 1 \otimes d) \to (A, d)$$
Recall that
$$(A \otimes A)^n := \bigoplus_{p+q=n} A^p \otimes A^q$$
So for $f \in A^p$, $g \in A^q$, we want that TFDC[43]

[43] Up to Koszul sign convention.

$$\begin{array}{ccc} f \otimes g & \longmapsto & df \otimes g + (-1)^{pq} f \otimes dg \\ \downarrow & & \downarrow \\ fg & \longmapsto & d(fg) \end{array}$$

so that

$$d(fg) = df \cdot g + (-1)^{pq} f \cdot dg$$

b) Let $\mathscr{B}$ be an additive k-category[44], and define $C_{dg}(B)$ to be the category whose objects are complexes in $\mathscr{B}$, and with

$$\mathrm{Hom}(X, Y)^n = \text{degree } n \text{ maps } X \to Y$$

equipped with the differentials

$$f \mapsto df := f \circ d_Y - (-1)^n d_X \circ f$$

for $f \in \mathrm{Hom}(X, Y)^n$.

[44] For example, $\mathrm{Proj}(A)$, $\mathrm{Mod}-A$, $\mathrm{Qcoh}(X)$...

Definition. For a dg category $\mathscr{A}$,

a) $Z^0(\mathscr{A})$ the *cycle category* has

$$\mathrm{Obj}(Z^0(\mathscr{A})) = \mathrm{Obj}(\mathscr{A})$$

$$Z^0(\mathscr{A})(x, y) = Z^0(\mathscr{A}(x, y))$$

b) $H^0(\mathbb{Z})$ has

$$\mathrm{Obj}(H^0(\mathscr{A})) = \mathrm{Obj}(\mathscr{A})$$

$$H^0(\mathscr{A})(x, y) = H^0(\mathscr{A}(x, y))$$

Example. Let $\mathscr{B}$ be an additive category.

a) We have

$$Z^0(C_{dg}(\mathscr{B})) = C(\mathscr{B})$$

Since

$$f \in Z^0(\mathrm{Hom}(X, Y)) \Leftrightarrow \begin{cases} f \in \mathrm{Hom}(X, Y) \\ df = f \circ d_Y - d_X \circ f = 0 \end{cases}$$

b) Moreover

$$H^0(C_{dg}(\mathscr{B})) = K(\mathscr{B})$$

since

$$f \in B^1(\mathrm{Hom}(X, Y)) \Leftrightarrow \begin{cases} \exists h \in \mathrm{Hom}(X, Y)^1 \\ f = dh = h \circ d_Y + d_X \circ h \\ (f \text{ is null-homotopic}) \end{cases}$$

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories. A *dg-functor* $F : \mathcal{A} \to \mathcal{B}$ consists of

- a map $F : \mathrm{Obj}(\mathcal{A}) \to \mathrm{Obj}(\mathcal{B})$
- For any $x, y \in \mathrm{Obj}(\mathcal{A})$ a morphism of complexes

$$F_{x,y} : \mathcal{A}(x, y) \to \mathcal{B}(F(x), F(y))$$

  satisfying unitality and composition.

Example. For a dg category $\mathcal{A}$, and

$$C_{dg}(k) := C_{dg}(\mathrm{Mod}_k)$$

Then for all $x \in \mathrm{Obj}(\mathcal{A})$

$$\mathcal{A}(x, -) : \mathcal{A} \to C_{dg}(k)$$

is a dg functor.

Definition. Let $F, G : \mathcal{A} \to \mathcal{B}$ be dg functors, and

$$\mathrm{Hom}(F, G) = \text{degree } n \text{ natural transformations}$$

that is, the set

$$\{\eta_x \in \mathcal{B}(F(x), G(x)) \mid x \in Obj(\mathcal{A})\}$$

satisfying that for any $f \in \mathcal{A}(x, y)$, TFDC

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

In this case, $Z^0(\mathrm{Hom}(F, G))$ is simply the set of morphisms from F to G.

Definition. Let $\mathcal{A}$ be a small dg category, $\mathcal{B}$ any dg category. Then the category $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ has objects dg functors $\mathcal{A} \to \mathcal{B}$, and

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B})(F, G) := \mathrm{Hom}(F, G)$$

Definition. Let $\mathscr{A}$ and $\mathscr{B}$ be k-categories. The *tensor product* of $\mathscr{A}$ and $\mathscr{B}$ is defined by

$$\mathrm{Obj}(\mathscr{A} \otimes \mathscr{B}) := \mathrm{Obj}(\mathscr{A}) \times \mathrm{Obj}(\mathscr{B})$$

and

$$(\mathscr{A} \otimes \mathscr{B})\left((a,b),(a',b')\right) := \mathscr{A}(a,a')\mathscr{B}(b,b')$$

Before continuing, we fix some notation. We will denote by

$$\text{dg-cat}$$

the category of small dg categories.

Proposition. (dg-cat, $\otimes$) *is a symmetric closed monoidal category. In particular, there is a canonical isomorphism*[45]

$$\mathrm{Hom}(\mathscr{A} \otimes \mathscr{B}, \mathscr{C}) \cong \mathrm{Hom}(\mathscr{A}\,\mathrm{Hom}(\mathscr{B}, \mathscr{C}))$$

[45] This isomorphism determines the tensor product up to isomorphism.

Dg-modules

Definition. Let $\mathscr{A}$ be a dg category. Then the *opposite dg category* $\mathscr{A}^{op}$ is given by

- $\mathrm{Obj}(\mathscr{A}^{op}) = \mathrm{Obj}(\mathscr{A})$
- For any $x, y \in \mathrm{Obj}(\mathscr{A}^{op})$

$$\mathscr{A}^{op}(x,y) = \mathscr{A}(y,x)$$

- For any $x, y, z \in \mathrm{Obj}(\mathscr{A}^{op})$

$$\begin{aligned} \mathscr{A}^{op}(x,y) \otimes \mathscr{A}^{op}(y,z) &\to \mathscr{A}^{op}(x,z) \\ f \otimes g &\mapsto (-1)^{|f||g|} g \circ f \end{aligned}$$

Definition. Let $\mathscr{A} \in$ dg-cat. The *dg category of (right) dg modules* is

$$\mathrm{Mod}_{\mathscr{A}} = C_{dg}(\mathscr{A}) := \mathrm{Hom}(\mathscr{A}^{op}, C_{dg}(k))$$

Proposition. *Let $\mathcal{A}$ be a dg category.*

a) For any $x \in \mathrm{Obj}(\mathcal{A})$ and any $M \in C_{dg}(\mathcal{A})$

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}(-,x), M) \cong M_x$$

b) If $\mathcal{A}$ is small,

$$\begin{aligned} \mathcal{A} &\hookrightarrow C_{dg}(\mathcal{A}) \\ x &\mapsto \mathcal{A}(-,x) \end{aligned}$$

Definition. Let $\mathcal{A} \in$ dg-cat and $M \in C_{dg}(\mathcal{A})$

a) M is *acyclic* if, for any $x \in \mathrm{Obj}(\mathcal{A})$, $M_x \in C_{dg}(\mathcal{A})$ is acyclic.

b) M is *h-projective*[46] if, for any $N \in C_{dg}(\mathcal{A})$ that is acyclic,

$$H^0(\mathrm{Hom}(M,N)) = 0$$

c) M an h-projective object is *compact* if, for any indexing set I and any

$$\{N_i \in C_{dg} \mid i \in I\}$$

the canonical morphism

$$\coprod_{i \in I} H^0\left(\mathrm{Hom}_{\mathcal{A}}(M, N_i)\right) \to H^0\left(\mathrm{Hom}_{\mathcal{A}}\left(M, \coprod_{i \in I} N_i\right)\right)$$

[46] In the context of model categories, this can be thought of as *cofibrant*.

Definition. Let $\mathcal{A} \in$ dg-cat.

a) $\mathcal{D}_{dg}(\mathcal{A})$ is the dg category of h-projective dg $\mathcal{A}$-modules[47].

b) $\mathrm{perf}_{dg}(\mathcal{A})$ is the dg category of compact h-projective dg $\mathcal{A}$-modules.

[47] In the context of model categories, these are the derived/perfect derived categories. This description of them works because every object admits a cofibrant (h-projective) replacement.

Note that

$$\mathrm{perf}(\mathcal{A}) := H^0(\mathrm{perf}_{dg}(\mathcal{A}))$$

Remark. Let $\mathcal{A} \in$ dg-cat. Then we have a commutative square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{Yoneda}} & C_{dg}(\mathcal{A}) \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow \\ \mathrm{perf}_{dg}(\mathcal{A}) & \longrightarrow & \mathcal{D}_{dg}(\mathcal{A}) \end{array}$$

The category Hqe

Definition. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a *quasi-equivalence* if

- For any $x, y \in \mathrm{Obj}(\mathcal{A})$

$$F_{x,y} : \mathcal{A}(x, y) \to \mathcal{B}(F(x), F(y))$$

  is a quasi-isomorphism
- $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is an equivalence.

Remark. $\otimes$ and Hom do not preserve quasi-equivalences.

Theorem (Tabuada)**.** *The category*

$$\mathrm{Hqe} := (\text{dg-cat})[qeq^{-}1]$$

exists and is equivalent to the model category of a cofibrantly generated model category.

Definition. A dg category $\mathcal{A}$ is *h-flat*, if, for all $x, y \in \mathrm{Obj}(\mathcal{A})$,

$$\mathcal{A}(x, y) \otimes - : C(k) \to C(k)$$

preserves quasi-isomorphisms.

Remark. For any $\mathcal{A} \in$ dg-cat, there exists $\mathcal{A}_{cof} \in$ dg-cat which is h-flat, such that

$$\mathcal{A} \cong \mathcal{A}_{cof}$$

in Hqe.

Definition. Let $\mathcal{A}, \mathcal{B} \in$ dg-cat and $X \in C_{dg}(\mathcal{A}^{op} \otimes \mathcal{B})$. X is a *quasi-functor* if, for any $a \in \mathrm{Obj}(\mathcal{A})$ there exists $b \in \mathrm{Obj}(\mathcal{B})$ such that

$${}_aX \cong \mathcal{B}(-, b)$$

in $H^0(\mathcal{D}_{dg}(\mathcal{B}))$.

Remark. Let $\mathcal{A}, \mathcal{B} \in$ dg-cat and $X \in C_{dg}(\mathcal{A}^{op} \otimes \mathcal{B})$ a quasi-functor. Then X induces

$$H^0(\mathcal{A}) \to H^0(\mathcal{B})$$

Definition. Let $\mathcal{A}, \mathcal{B} \in$ dg-cat. Then $\mathrm{rep}_{dg}(\mathcal{A}, \mathcal{B})$ is the dg category of quasi-functors in

$$\mathcal{D}_{dg}(\mathcal{A}^{op} \otimes \mathcal{B})$$

Definition. Let $\mathcal{A} \in$ dg-cat. The *left derived tensor product* is

$$\mathcal{A} \otimes^L - := \mathcal{A}_{cof} \otimes - : \mathrm{Hqe} \to \mathrm{Hqe}$$

Theorem (Drinfeld, Toën)**.** $(\mathrm{Hqe}, \otimes^L)$ *is symmetric closed monoidal with internal hom*

$$R\,\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \cong \mathrm{rep}_{dg}(\mathcal{A}_{cof}, \mathcal{B})$$

in Hqe

Theorem (Toën)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories, then*

$$\mathcal{D}_{dg}(\mathcal{A}^{op} \otimes \mathcal{B}) \to R\,\mathrm{Hom}_c\left(\mathcal{D}_{dg}(\mathcal{A}), \mathcal{D}_{dg}(\mathcal{B})\right)$$

is an isomorphism in Hqe.

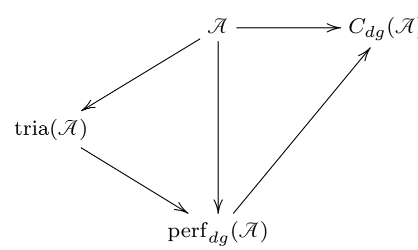
Triangulated dg categories

Definition (Toën)**.** $\mathcal{A} \in$ dg-cat is ~~*triangulated*~~ *perfect*[48] if

$$H^0(\mathcal{A}) \overset{H^0(\mathrm{can})}{\longrightarrow} H^0(\mathrm{perf}_{dg}(\mathcal{A}))$$

is an equivalence

[48] This notion is stronger than the notion of a pretriangulated dg category. To get the definition of pretriangulated, take the subcategory $\mathrm{tria}(\mathcal{A})$ fitting into



in place of $\mathrm{perf}_{dg}(\mathcal{A})$.

Definition (Toën)**.** $F : \mathcal{A} \to \mathcal{B}$ dg functor is called a *Morita equivalence* (mo) is

$$F^* : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$$

is an equivalence.

Remark. a) qeq $\subset$ mo

b) $\mathcal{A} \overset{\mathrm{can}}{\to} \mathrm{perf}_{dg}(\mathcal{A})$ is a Morita equivalence.

Theorem (Tabuada)**.** Hmo := (dg-cat)$[mo^{-1}]$ *exists and is equivalent to the homotopy category of a cofibrantly generated model category.*

Remark. There is a quotient

$$\mathrm{Hqe} \xrightarrow{\pi} \mathrm{Hmo}$$

Proposition (Toën?)**.** *The map* $\mathcal{A} \mapsto \mathrm{perf}_{dg}(\mathcal{A})$ *is right adjoint to* π *and induces an equivalence*

$$\mathrm{Hmo} \cong \{\mathcal{A} \in \mathrm{Hqe} \mid \mathcal{A} \text{ is perfect}\}$$

Theorem (Toën)**.** *Let* $\mathcal{A}, \mathcal{B} \in$ dg-cat*. The map*

$$\mathcal{D}_{dg}(\mathcal{A}^{op} \otimes^L \mathcal{B}) \to R\,\mathrm{Hom}_c(\mathcal{D}_{dg}(\mathcal{A}), \mathcal{D}_{dg}(\mathcal{B}))$$

is an equivalences in Hqe*.*

Remark. Hmo is pointed and has finite direct sums.

Definition. A *short exact sequence* is a bicartesian diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & \mathcal{B} \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & \mathcal{C} \end{array}$$

Theorem (Toën)**.** *Let* $\mathcal{A} = \mathcal{A}_{conf}$ *be a dg category, and* $\mathbb{1}_{\mathcal{A}} : \mathcal{A}^{op} \otimes \mathcal{A} \to \mathcal{A}$ *be given by* $(x, y) \mapsto \mathcal{A}(x, y)$

a) $\mathbb{1}_{\mathcal{A}} \in R\,\mathrm{Hom}(\mathcal{A}, \mathcal{A}) = \mathrm{rep}_{dg}(\mathcal{A}, \mathcal{B})$

b) $HH^*(\mathcal{A}) \cong H^*(\mathrm{Hom}(\mathbb{1}_{\mathcal{A}}, \mathbb{1}_{\mathcal{A}}))$

c) In the above isomorphism, cup product is sent to composition, and vice versa.

Remark. • The morphism

$$HH_*(\mathcal{A}) \xrightarrow{HH_*(can)} HH_*(\mathrm{perf}_{dg}(\mathcal{A}))$$

is a quasi-isomorphism

• HH_* of dg categories preserves short exact sequences in Hmo.

Reduction to Characteristic $p > 0$ for Schemes

Anthony Blanc

We have the Hodge to De Rham spectral sequence for K a field[49], and X a K-scheme.

$$E_1^{p,q} = H^q(X, \Omega^p_{X|K}) \Rightarrow H^{p+q}(X|K) \qquad (*)$$

[49] We will, in general use K for a field of characteristic 0, and k for a field of positive characteristic.

Theorem (1)**.** *If $\operatorname{char} K = 0$ and X is smooth and proper over K, then $(*)$ degenerates at E_1.*

Theorem (2)**.** *Let $\operatorname{char} k = p > 0$, and X is smooth and proper over k. If $\dim(X) < p$ and X admits a lift to $W_2(k)$[50] then $(*)$ degenerates at E_1.*

[50] The only facts about $W_2(k)$ that will be needed for this talk are that $W_2(k) = k^2$ as a k-vector space, that the addition and multiplication are given by

$$(a, b) + (a', b') = \big(a + a', b + b' + \frac{1}{p}\left(a^{p-1} + (a')^{p-1} - (a + a')^p\right)\big)$$
$$(a, b)(a', b') = \left(aa', (a')^p b + b'a^p\right)$$

that there is a SES

$$I \to W_2(k) \to k$$

with $I^2 = 0$, and that $W_2(\mathbb{F}_p) \simeq \mathbb{Z}/(p^2)$.

The main body of this talk will be devoted to proving that:

Claim. Theorem 1 implies Theorem 2.

Before that, we will need

Theorem (Grothendieck)**.** *Let X be a smooth proper K-scheme. There exists a finitely generated ring A and a smooth proper A-scheme Y such that $Y \otimes_A K \simeq X$.*

Proof. $(*)$ The first step we need is that there exists some scheme $Y \to \operatorname{Spec}(A)$ where A has finite type.

Since $X \to \operatorname{Spec}(K)$ has finite type, we have a decomposition into affines

$$X = \bigcup_{i=1}^{s} \underbrace{\operatorname{Spec}(A_i)}_{U_i}$$

where

$$A_i = K[X_1, \ldots, X_{n_i}]/\mathfrak{a}_i$$

and $\mathfrak{a}_i = \left(p_1^i, \ldots, p_{r_i}^i\right)$. Similarly, letting $U_i \cap U_j = \operatorname{Spec} A_{ij}$, we have

$$A_{ij} = K[X_1, \ldots, X_{n_{ij}}]/\mathfrak{a}_{ij}$$

and $\mathfrak{a}_{ij} = \left(p_1^{ij}, \ldots, p_{r_{ij}}^{ij}\right)$.

We can then set

$$A = \mathbb{Z}[\text{coeff. of } p_\ell^i \;\&\; p_\ell^{ij}] \hookrightarrow K$$

so that there exist ideals

$$\begin{aligned} \mathfrak{a}_i' &\subseteq A[X_1, \ldots, X_{n_i}] \quad \mathfrak{a}_i' \cap K = \mathfrak{a}_i \\ \mathfrak{a}_{ij}' &\subseteq A[X_1, \ldots, X_{n_{ij}}] \quad \mathfrak{a}_{ij}' \cap K = \mathfrak{a}_{ij} \end{aligned}$$

If we then set

$$A_i' = A[X_1, \ldots, X_{n_i}]/\mathfrak{a}_i'$$

and let

$$Y = \operatorname*{colim}_i \operatorname{Spec} A_i'$$

we almost definitionally have

$$\begin{aligned} &Y \to \operatorname{Spec}(A) \\ &Y \otimes_A K = X \end{aligned}$$

We can then write

$$K = \operatorname*{colim}_{i \geq 0} A_i$$

where the A_i are finitely generated over Z. As a result, there exists $i_0 > 0$ with

$$\begin{CD} X @>>> Y_{i_0} \\ @VVV @VV{\text{fin. type}}V \\ \operatorname{Spec}(K) @>>> \operatorname{Spec}(A_{i_0}) \end{CD}$$

($*$) The next claim, which we state without proof, is that there exists $i_1 > i_0$ and a *proper* A_{i_1}-scheme Y_{i_1} such that $Y_{i_1} \otimes_{A_{i_1}} K \simeq X$[51]. Using the Chow lemma, we can then reduce to the projective case.

[51] For a complete treatment, see [12]

($*$) Finally, we claim that there exists $i_2 \geq i_1$ and a *smooth* and proper A_{i_2}-scheme Y_{i_2} such that $Y_{i_2} \otimes_{A_{i_2}} K \simeq X$[52].

[52] Once again, see [12] for a full treatment.

$$\begin{CD} X @>{x \mapsto x_{i_2}}>> Y_{i_2} @>>> Y_{i_1} \\ @V{f}VV @VVV @VVV \\ \operatorname{Spec}(K) @>>> \operatorname{Spec}(A_{i_2}) @>>> \operatorname{Spec}(A_{i_1}) \end{CD}$$

Smoothness follows from

- f is smooth at $x \in X$ if and only if X is geometrically regular at x (ie $X \otimes_K X$ is regular at $\overline{x}$).
- f is smooth at $x \in X$ if and only if there exists $i_2 > i_1$ such that f_{i_2} is smooth at $x_{i_2} \in Y_{i_2}$. This is true because X being geometrically regular at x is equivalent to Y_{i_2} being geometrically regular at x_{i_2}.

- By quasi-compactness of X, we then get that the result.

$\square$

Proposition. *If S is a finite type integral scheme over $\operatorname{Spec}\mathbb{Z}$, then the smooth locus of S is a non-empty open subset of S.*

We now consider, for $X \to \operatorname{Spec} K$ as in Theorem 1,

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} A = S \end{array}$$

Let $d \geq$ the dimensions of the fibers of Y[53]. Set

[53] This is possible by quasi-compactness.

$$N = \prod_{p \leq d \text{ prime}} p$$
$$S' = \operatorname{Spec} A\left[\frac{1}{N}\right]$$
$$S' \to S$$

There exists $s' \in S'$ such that $\operatorname{char}(k(s')) > d$, so we can suppose that S has a closed point $s \in S$ such that $\operatorname{char}(k(s)) = p > d$

We can then define coherent[54] sheaves over S:

[54] By the Grothendieck finiteness theorem for proper morphisms.

$$R^j f_* \Omega^i_{Y|S} =: \mathscr{H}^{ij}$$
$$R6n f_* \Omega^*_{Y|S} =: \mathscr{H}^n$$

If we let $\eta = (0) \in S$, we see that $\mathscr{H}^{ij}_\eta$ and $\mathscr{H}^n_\eta$ are finite dimensional K-vector spaces. And K is given by the (filtered) colimit

$$K = \operatorname{Frac}(A) = \operatorname*{colim}_{a \neq 0} A[a^{-1}]$$

Which implies that there exists $a \neq 0$ such that $\mathscr{H}^{ij}|_{D(a)}$ and $\mathscr{H}^n|_{D(a)}$ are locally free sheaves over $D(a)$.

Considering then the diagram

$$\begin{array}{ccccccc} Y_s & \longrightarrow & & \longrightarrow & Y & \longleftarrow & X \\ \downarrow & & & & \downarrow & & \downarrow \\ \operatorname{Spec} k(s) & \longrightarrow & \operatorname{Spec} W_2(k(s)) & \xrightarrow{*} & S & \longleftarrow & \operatorname{Spec} K \end{array}$$

We have that the map $*$ exists by the smoothness of s. Since Theorem 2 applies to Y

$$h^n = \sum_{i+j=n} h^{ij}$$

and the claim from the beginning is proved.

Dg Algebra Analogue

Theorem (1 Toën)**.** *Let k be a commutative ring. Let A be a smooth proper k-dg-algebra. Then there exists a finitely generate ring k_0 and a smooth proper k_0-dg-algebra A_0 such that $A_0 \otimes^L_{k_0} k \simeq A$ (quasi-isomorphic).*

Fix some notation

dgalg_k	cat. of dg algebras
$\mathrm{Ho}(\mathrm{dgalg}_k)$	homotopy cat.
$\mathrm{Ho}(\mathrm{dgalg}_k^{sp})$	full subcat. of smooth+proper

Theorem (2)**.** *let $\{k_i\}_{i \in I}$ be a filtration diagram of commutative rings with*

$$k = \operatorname*{colim}_{i \in I} k_i$$

Then the functor

$$\operatorname*{colim}_{i \in I} Ho(dgalg_{k_i}^{sp}) \overset{\mathrm{colim}_{i \in I}(- \otimes^L_k k_i)}{\longrightarrow} Ho(dgalg_k^{sp})$$

is an equivalence.

As before, we have that

Claim. Theorem 2 implies Theorem 1

The Deligne-Illusie Decomposition

Tobias Dyckerhoff

There are two basic constructions that we will need to prove the degeneration of the Hodge-to-De Rham Spectral Sequence in positive characteristic:

1) The Frobenius endomorphism

- Let S be a scheme of characteristic $p > 0$[55] We then get the *absolute Frobenius* $F_S : S \to S$ given by
  - The identity map on underlying topological spaces.
  - $F_S(f) = f^p$ where $f \in \mathcal{O}_S$.

[55] That is, such that

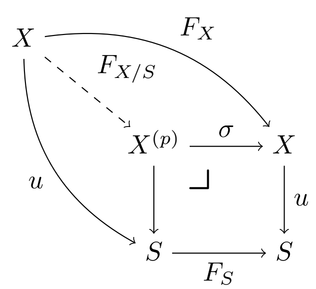
$$p \cdot 1 = 0 \in \mathcal{O}_S$$

over any open.

Assume $S = \operatorname{Spec}(k)$ where k is a field of characteristic $p > 0$. Let

$$X \xrightarrow{u} S$$

be a k-linear scheme. Then we can form the following diagram



We define $X^{(p)}$ to be the pullback, and then the morphism

$$F := F_{X/S}$$

called the *relative Frobenius*, exists by universal property[56].

[56] Notice that F is a homeomorphism of the underlying topological spaces, but it is **not** generally an isomorphism of schemes.

Example. Let $X = \operatorname{Spec} k[t]/(f)$, where $f = \sum a_m t^m$. Then

$$X^{(p)} = \operatorname{Spec} k \otimes_k k[t]/(f)$$

where the morphism $k \to k$ in the definition of the tensor product is F_S. So we see that

$$X^{(p)} \cong \operatorname{Spec} k[t]/(f^{(p)})$$

where

$$f^{(p)} = \sum a_m^p t^m$$

Furthermore, for $a \in k$

$$\sigma^*(at) = 1 \otimes at = a^p t$$

and

$$F^*(a \otimes t) = at^p$$

2) The De Rham Complex

- Let $S = \operatorname{Spec} k$, for k a field. $X \xrightarrow{u} S$ a scheme over k.
- Then we have

$$\Omega^b ullet_{X/S} = \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \to \cdots$$

  the *De Rham Complex* of X over S.

Example. 1) Take $k = \mathbb{C}$, and $X = \mathbb{A}^1_{\mathbb{C}}$. Then we can write down the de Rham complex

$$\Omega^\bullet_{X/S} = \mathbb{C}[t] \xrightarrow{d} \mathbb{C}[t]dt$$
$$t^n \mapsto nt^{n-1}dt$$

So that, computing the homology, we see that

$$\mathcal{H}^0 = \langle 1 \rangle = \mathbb{C}$$
$$\mathcal{H}^1 = 0$$

which is exactly the same as

$$H^*(\mathbb{A}^1(\mathbb{C}), \mathbb{C})$$

2) Take $k = \mathbb{C}$, $X = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. Then the de Rham complex is

$$\Omega^\bullet_{X/S} = \mathbb{C}[t, t^{-1}] \xrightarrow{d} \mathbb{C}[t, t^{-1}]dt$$
$$t^n \mapsto nt^{n-1}dt$$

so the homology is

$$\mathcal{H}^0 = \langle 1 \rangle \cong \mathbb{C}$$
$$\mathcal{H}^1 = \left\langle \frac{dt}{t} \right\rangle \cong \mathbb{C}$$

Which, as before, is

$$H^*(\mathbb{A}^1(\mathbb{C}) \setminus \{0\}, \mathbb{C})$$

the homology of the analytic space[57]

[57] There is, in fact, a more general result:

Theorem (Grothendieck)**.** *For X smooth over $\mathbb{C}$, there is an algebra isomorphism*

$$\mathbb{H}^*(X, \Omega^\bullet_{X/S}) \cong H^*(X(\mathbb{C}), \mathbb{C})$$

3) Now let $k = \mathbb{F}_q$, and $X = \mathbb{A}^1_{\mathbb{F}_q}$[58]. Then the complex is

$$\Omega^\bullet_{X/S} = \mathbb{F}_q[t] \to \mathbb{F}_q[t]dt$$
$$t^n \mapsto nt^{n-1}dt$$

So

$$\mathcal{H}^0 = \langle 1, t^p, \ldots, t^{mp}, \ldots \rangle \cong \mathbb{F}_q[t^p]$$
$$\mathcal{H}^1 = \langle t^{p-1}dt, \ldots, t^{mp-1}dt, \ldots \rangle \cong \mathbb{F}_q[t^p]t^{p-1}dt$$

Returning to the more general case, let $S = \operatorname{Spec} k$, and $\operatorname{char}(k) = p > 0$. Let $X \xrightarrow{u} S$ be a scheme over k. Then for $a \otimes f \in \mathcal{O}_{X^{(p)}}$, we have

$$dF^*(a \otimes f) = d(af^p) = 0$$

This implies that[59]

$$\boxed{F_*\Omega^\bullet_{X/S} \text{ is an } \mathcal{O}_{X^{(p)}} \text{ linear complex.}}$$

further, we have

> **Theorem** (Cartier). *Assume that X is smooth over k. Then there exist isomorphisms of $\mathcal{O}_{X^{(p)}}$-modules*
>
> $$C^{-1} : \Omega^i_{X^{(p)}/S} \xrightarrow{\cong} \mathcal{H}^i(F_*\Omega^\bullet_{X/S})$$
>
> *These isomorphisms are uniquely determined by the properties*[60]
>
> *1.* $C^{-1}(1 \otimes f) = F^*(1 \otimes f) = f^p \in \mathcal{H}^0(F_*\Omega^\bullet_{X/S})$
>
> *2.* $C^{-1}(1 \otimes df) = f^{p-1}df \in \mathcal{H}^1(F_*\Omega^\bullet_{X/S})$[61]
>
> *3.* $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$

Goal: Improve Cartier's result (under additional assumptions) via a more systematic interpretation of $\frac{df^p}{p}$.

Question: How to divide by p in characteristic $p > 0$?

Trick: Let M be a $\mathbb{Z}/(p)$-module, and assume that there exists a *lift* $\tilde{M}$ of M to $\mathbb{Z}/(p^2)$, that is

1. $\tilde{M}/p\tilde{M} \cong M$

2. $\tilde{M}$ is flat over $\mathbb{Z}/(p^2)$, so there exists a short exact sequence

$$0 \to p\tilde{M} \to \tilde{M} \xrightarrow{p} p\tilde{M} \to 0$$

In this situation, we *can* divide by p

[58] Where we assume $q = p^n$, so that $\mathbb{F}_q$ has characteristic p

[59] This is a direct generalization of what we can observe in examples 3), where
$$\mathcal{O}_{X^{(p)}} = \mathbb{F}_q[t^p]$$

[60] Imposed locally over some open affine in X.

[61] Where the term
$$f^{p-1}df$$
can be thought of heuristically as an analogue of
$$\frac{df^p}{p}$$
We will make this notion more precise later on.

$$p\tilde{M} \xleftarrow{\ \cong\ } \tilde{M}/p\tilde{M} \cong M, \qquad p^{-1} : M \to p\tilde{M}$$

Slightly more generally, if M is a module over k of character $p > 0$, there exists a ring $W_2(k)$[62] which is a flat $\mathbb{Z}/(p^2)$-module equipped with an isomorphism

$$W_2(k)/pW_2(k) \cong k$$

replacing $\mathbb{Z}/(p^2)$ by $W_2(k)$ in the above construction gives a k-linear version of p^{-1}.

[62] The so-called *Witt vectors*.

Theorem (Deligne-Illusie)**.** [63]*Let k be perfect of character $p > 0$ and let X be a smooth scheme over k. Assume that X admits a smooth lift $\tilde{X}$ over $W_2(k)$. Then there exists an isomorphism*

$$\phi_{\tilde{X}} : \bigoplus_{i<p} \Omega^i_{X^{(p)}/S}[-i] \xrightarrow{\cong} \tau_{<p} F_* \Omega^\bullet_{X/S}$$

in $\mathcal{D}(X^{(p)})$[64] inducing $\mathcal{C}^{-1}$ on cohomology sheaves.[65]

[63] Called the *Decomposition theorem.*

[64] The derived category of complexes of $\mathcal{O}_{X^{(p)}}$-modules.

[65] Note: Cartier uses fewer assumptions *and* gets a stronger result. However, this is a more refined version of the theorem, which allows us to access what's really going on.

Proof. Assume first that there exists a lift of the relative Frobenius

$$\begin{array}{ccc} X & \xrightarrow{F} & X^{(p)} \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

to[66]

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{F}} & \tilde{X}^{(p)} \\ & \searrow \quad \swarrow & \\ & \tilde{S} & \end{array}$$

[66] Here we use the notations

$$S = \operatorname{Spec} k$$
$$\tilde{s} = \operatorname{Spec} W_2(k)$$

Step 0: We can explicitly write down the map

$$\begin{aligned} \phi^0_{\tilde{X}} : \mathcal{O}_{X^{(p)}} &\to \mathcal{H}^0(F_*\Omega^\bullet_{X/S}) \hookrightarrow F_*\Omega^\bullet_{X/S} \\ 1 \otimes f &\mapsto F^*(1 \otimes f) \end{aligned}$$

Step 1: Note that the map

$$F^* : \Omega^\bullet_{X^{(p)}/S} \to F_*\Omega^1_{X/S}$$

is the zero map. This means that the image of

$$\tilde{F}^* : \Omega^1_{\tilde{X}^{(p)}/\tilde{S}} \to \tilde{F}_*\Omega^1_{\tilde{X}/\tilde{S}}$$

lies in $p\tilde{F}_*\Omega^1_{\tilde{X}/\tilde{S}}$. Then we use the diagram

$$
\begin{array}{ccc}
\Omega^1_{\tilde{X}^{(p)}/\tilde{S}} & \xrightarrow{\tilde{F}^*} & p\tilde{F}_*\Omega^1_{\tilde{X}/\tilde{S}} \\
\downarrow & \nearrow & \cong \uparrow \cdot p \\
\Omega^1_{X^{(p)}/S} & \xrightarrow[\exists! \phi^1_{\tilde{X}}]{} & F_*\Omega^1_{X/S}
\end{array}
$$

Where the diagonal map exists since $\tilde{F}^*$ factors, and the bottom map exists by inverting the multiplication by p[67].

[67] This is where all the additional assumptions come into play. In particular, the requirement of smoothness.

In local coordinates

$$
\begin{aligned}
\tilde{F}^*(1 \otimes f) &= f^p + pu(f) \\
\tilde{F}^*(1 \otimes df) &= pf^{p-1}df + du(f)
\end{aligned}
$$

so that

$$
\phi^1_{\tilde{X}} = \frac{1}{p}\tilde{F}^*(1 \otimes df) = f^{p-1}df + du(f)
$$

is a closed 1-form. Hence,

$$
\phi^1_{\tilde{X}} : \Omega^1_{X^{(p)}/S}[-1] \to F_*\Omega^\bullet_{X/S}
$$

yields the morphism from Cartier's theorem.

Step 2: We can explicitly define

$$
\begin{aligned}
&\phi^i_{\tilde{X}} : \Omega^i_{X^{(p)}/S} \to F_*\Omega^\bullet_{X/S} \\
&\omega_1 \wedge \cdots \wedge \omega_i \mapsto \phi^1_{\tilde{X}}(\omega_1) \wedge \cdots \wedge \phi^1_{\tilde{X}}(\omega_i)
\end{aligned}
$$

Problem: $\tilde{F}$ typically doesn't exist globally, but only locally.

To solve this, we replace Step 1 by choosing an open cover $\mathcal{U} = \{U \subset X\}$ on which F admits a lift

$$
\begin{array}{ccc}
F_*\hat{C}(\mathcal{U}, \Omega^\bullet_{X/S}) & \xleftarrow[b]{\simeq} & F_*\Omega^\bullet_{X/S} \\
\uparrow a & & \\
\Omega^1_{X^{(p)}/S}[-1] & &
\end{array}
$$

Where a is produced from local data coming from lifts of F in addition to carefully chosen homotopies on overlaps. b is known as the *Čech replacement* and is a quasi-isomorphism. Hence, if we pass to the derived category $\mathcal{D}(X^{(p)})$, we get a morphism

$$
\Omega^1_{X^{(p)}/S}[-1] \to F_*\Omega^\bullet_{X/S}
$$

Unfortunately Step 2 was element theoretic, and so also does not generalize. Instead, we take the diagram

$$
\begin{array}{ccc}
\left(\Omega^1_{X^{(p)}/S}[-1]\right)^{\otimes i} & \xrightarrow{\left(\phi^1_{\tilde{X}}\right)^{\otimes i}} & \left(F_*\Omega^\bullet_{X/S}\right)^{\otimes i} \\
\uparrow a[-i] & & \downarrow \\
\Omega^i_{X^{(p)}/S} & & F_*\Omega^\bullet_{X/S}
\end{array}
$$

where a is the *antisymmetrization*[68]

$$a(\omega_1 \wedge \cdots \wedge \omega_i) = \frac{1}{i!} \sum_{\sigma \in S_i} \operatorname{sgn}(\sigma)\omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)}$$

Tracing through this diagram then proves the theorem. □

[68] Note that the use of $\frac{1}{i!}$ implicitly makes the assumption that $i < p$.

Corollary. *Suppose X is smooth and proper over k, with a lift $\tilde{X}$ to $W_2(k)$ and $\dim X < p$. Then the Hodge-to-de Rham Spectral Sequence*

$$E_1^{a,b} = H^b(X, \Omega^a_{X/S}) \Rightarrow \mathbb{H}^{a+b}(\Omega^\bullet_{X/S})$$

degenerates on page 1.

Proof. The proof is based on dimension counting.

$$\begin{aligned} \mathbb{H}^m(X, \Omega^\bullet_{X/S}) &\cong \mathbb{H}^m(X^{(p)}, F_*\Omega^\bullet_{X/S}) \\ &\cong \bigoplus_{i>0} H^{m-i}(X^{(p)}, \underbrace{\Omega^i_{X^{(p)}/S}}_{\sigma^*\Omega^i_{X/S}}) \end{aligned}$$

where the second line follows from Deligne-Illusie.

We can then apply base change to get

$$\mathbb{H}^m(X, \Omega^\bullet_{X/S}) \cong \bigoplus_{i \geq 0} F_S^* H^{m-i}(X, \Omega^i_{X/S})$$

but, since F_S^* is a field automorphism

$$\dim_k F_S^* H^{m-i}(X, \omega^i_{X/S}) = \dim_k H^{m-i}(X, \omega^i_{X/S})$$

so that degeneration happens on page one. □

The Conjugate Spectral Sequence.

Recall that there are two spectral sequences for hypercohomology. Given a resolution:

$$\begin{array}{ccccc}
\mathcal{A}^{0,2} & \longrightarrow & \mathcal{A}^{1,2} & \longrightarrow & \mathcal{A}^{2,2} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{A}^{0,1} & \longrightarrow & \mathcal{A}^{1,1} & \longrightarrow & \mathcal{A}^{2,1} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{A}^{0,0} & \longrightarrow & \mathcal{A}^{1,0} & \longrightarrow & \mathcal{A}^{2,0}
\end{array}$$

$$\Omega^\bullet_{X/S} \qquad \mathcal{O}_X \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^2_{X/S}$$

One can take either the vertical or the horizontal filtration, leading to

I) Horizontal filtration gives us Hodge-to-de Rham

$$E_1^{a,b} H^b(X, \Omega^a_{X/S}) \Rightarrow \mathbb{H}^{a+b}(\Omega_{X/S})$$

II) Vertical filtration gives us the *Conjugate spectral sequence*

$$E_2^{a,b} = H^a(X, \mathcal{H}^b(\Omega^\bullet_{X/S})) \Rightarrow \mathbb{H}^{a+b}(\Omega^\bullet_{X/S})$$

Using Cartier's result, we have

$$\begin{aligned} H^a(X, \mathcal{H}^b(\Omega^\bullet_{X/S})) &\cong H^a(X^{(p)}, \mathcal{H}^b(F_*, \Omega^\bullet_{X/S})) \\ &\cong H^a(X^{(p)}, \Omega^b_{X^{(p)}/S} \\ &\cong F_S^* H^a(X, \Omega^b_{X/S}) \end{aligned}$$

and, under suitable finiteness conditions, the degeneration of I) is equivalent to the degeneration of II).

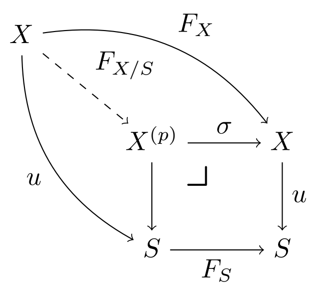
STRATEGY for approaching non-commutative geometry. (Kontsevich, Kaledin)

- Find a non-commutative analogue of the conjugate spectral sequence.
- Show it degenerates.
- Use this to conclude that the Hodge to de Rham spectral sequence degenerates for reasons of dimension.

Non-commutative Cartier Isomorphism, Part I

Tobias Dyckerhoff

As we saw previously, in the commutative case, if X is a scheme that is smooth over $S = \operatorname{Spec} k$ where $\operatorname{char}(k) = p > 0$, k perfect, then we have the relative frobenius



and the Cartier isomorphism

$$\mathcal{C}^{-1}\Omega^i_{X^{(p)}/S} \overset{cong}{\to} \mathcal{H}^i(F_*\Omega^\bullet_{X/S})$$

is an $\mathcal{O}_{X^{(p)}}$-linear isomorphism determined by

1. $\mathcal{C}^{-1}(f) = F^*(f) = f^p$
2. $\mathcal{C}^{-1}(df) = \frac{F^*(df)}{p} = f^{p-1}df$
3. $\mathcal{C}^{-1}(\omega \wedge \tau) = \mathcal{C}^{-1}(\omega) \wedge \mathcal{C}^{-1}(\tau)$

Special Phenomenon in Characteristic $p > 0(*)$: Every function of the germ f^p is constant ($df^p = 0$)

Goal: Let A be an ossociative k-algebra, A smooth over k, and let[69]

$$A^{(p)} = A \otimes_k k$$

We then hope that, for $|u| = -2$ there is an isomorphism

$$HH_*(A^{(p)})((u)) \overset{\cong}{\to} HP_*(A)$$

As we will see, there is an analogous special phenomenon to $(*)$ in the non-commutative case. That is, $(*)_{\text{top}}$:

$$S^1 \to S^1$$
$$Z \mapsto z^p$$

[69] Where the connecting morphism $k \to k$ is given by the Frobenius F.

is 'constant modulo p' or 'factors over the point modulo p' which can be expressed in a heuristic diagram as

$$\begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ & \searrow \;\; \text{mod}\, p \;\; \nearrow & \\ & * & \end{array}$$

Of course, this doesn't make much sense until we explain what is meant by 'modulo p'. What we mean here is precisely that the diagram

$$\begin{array}{ccc} [S^1] & \longmapsto & p \cdot [S^1] = p \\ H_*(S^1, k) & \longrightarrow & S^1 \\ & \searrow \qquad \nearrow & \\ & * & \end{array}$$

commutes[70].

[70] This is not just an analogy. We will make explicit use of precisely this fact in the proof of a key lemma.

Recall: A an associative unital algebra allows us to write down the *bar construction*[71]

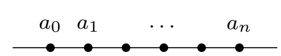
$$C'_\bullet(A) = A \xleftarrow{b'} \underbrace{A^{\otimes 2} \xleftarrow{b'} A^{\otimes 3} \xleftarrow{b'} \cdots}_{A \otimes A^{op} \text{ free resolution}}$$

[71] Throughout this talk, we will make use of the homological grading convention.

where the differential b' is given by[72]

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

[72] We can also view this differential pictorially as



The differential is just the sum over 'contracting intervals'.

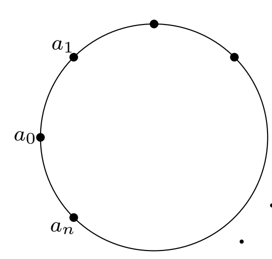
If we tensor the free resolution from the bar construction (over $A \otimes A^{op}$) by A, we get a new complex, the *cyclic bar construction*

$$A \xleftarrow{b} A^{\otimes 2} \xleftarrow{b} A^{\otimes 3} \xleftarrow{b} \cdots$$

where the differential is given by[73]

$$b(a_0 \otimes \cdots \otimes a_n) = b'(a_0 \otimes \cdots \otimes a_n) + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

[73] Again, there is a pictorial representation. As before, the differential is given by a sum over contracted intervals, but now on the circle:



In particular, we have the *Hochschild Homology*

$$HH_*(A) := H_*(C_\bullet(A)) \cong \mathrm{Tor}_*^{A \otimes A^{op}}(A, A)$$

Alan Connes Mad the fundamental observation that

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

defines an action of $\mathbb{Z}/n\mathbb{Z}$ on the n-cells of the cyclic bar construction such that

$$C'_\bullet(A) \stackrel{1-t}{\to} C_\bullet(A)$$

is a map of complexes, ie

$$(1-t)b' = b(1-t)$$

Therefore, the k-vector spaces

$$C_n^\lambda(A) = (C_n(A))_{\mathbb{Z}/(n+1)}$$

organize into a new complex called the *Connes Complex.*

Theorem (Connes)**.** *Let A be commutative and smooth over k, where k has characteristic 0^{74}, then (for $X = \operatorname{Spec} A$)*

$$HC_n^\lambda = \Omega^n_{A|k}/d\Omega^{n-1}_{A|k} \oplus H^{n-2}_{dR}(x) \oplus H^{n-4}_{dR}(X) \oplus \cdots$$

[74] This is significantly simpler than the B-operator/mixed complex picture precisely because it only works in characteristic 0.

Problem: This does not hold in characteristic p. Reason: the functor $(-)_{\mathbb{Z}/p\mathbb{Z}}$ is not exact in characteristic p, rather, we have lots of group homology[75].

[75] For example

$$H_i(\mathbb{Z}/p\mathbb{Z}, \mathbb{F}_p) \cong \mathbb{F}_p$$

for every $i \geq 0$.

To try and address this problem, we can consider the full[76] double complex

[76] In defining $C_n^\lambda(A)$, we were, in effect, merely considering the first two rows.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow b & & \downarrow b' & & \downarrow b & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & \cdots \\
 & & & & & & \\
C_\bullet(A) & & C'_\bullet(A) & & & &
\end{array}$$

We call this the *cyclic bicomplex* $CC_{\bullet,\bullet}(A)$, and write

$$CC_\bullet(A) = \operatorname{tot} CC_{\bullet,\bullet}(A)$$

We then have the *cyclic homology*

$$HC_\bullet(A) := H(CC_\bullet(A))$$

In characteristic 0, $HC_* \cong HC_*^\lambda$. Connes theorem holds verbatim in characteristic $P > 0$ if we replace HC_*^λ by HC_*[77].

[77] See the talks on the HKR theorem for more details.

Observation. The fact that we have lots of group homology for $\mathbb{Z}/p\mathbb{Z}$ in characteristic p tells us that we have lots of deRham cohomology in characteristic p.

Our strategy to move away from complexes and reach a broader definition and construction of the Cartier isomorphism will be to use simplicial methods:

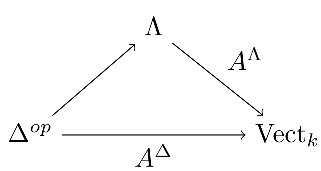
(1) The bar complex arises from a simplicial vector space

$$A^\Delta : \Delta^{op} \to \mathrm{Vect}_k$$
$$[n] \mapsto A^{\otimes(n+1)}$$

whose simplicial structure is given by

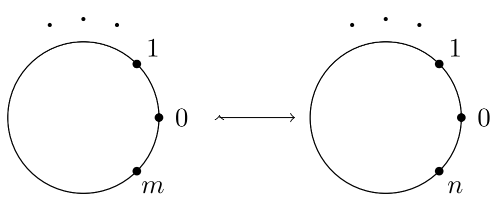
$$\partial_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$\sigma_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes \underbrace{1}_{i} \otimes \cdots \otimes a_n$$

(2) The cyclic symmetries of Connes can be captured in a lift of A^Δ to Connes/Tsygan's *cyclic category* Λ



The structure of the cyclic category is relatively straightforward.

- Like Δ, *Lambda* has one object $\langle n \rangle$ for each $n \geq 0$.
- The morphism sets $\mathrm{Hom}_\Lambda(\langle m \rangle, \langle n \rangle)$ are the sets of maps



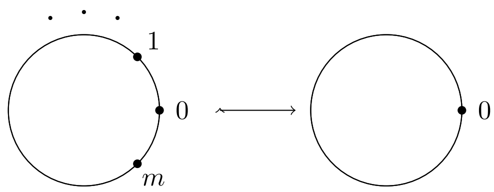
which are continuous, monotone, degree 1, and preserve the sets of marked points. Two such maps are considered equivalent if there is a homotopy through such maps between them.

So, can't we not just consider cyclic order preserving maps?

Example.

$$|\mathrm{Hom}_\Lambda(\langle m \rangle, \langle 0 \rangle)| = m + 1$$

Considering the picture



we see that which segment we choose to collapse determines which morphism we are considering. In particular, Λ has no final object.

Fact. Every morphism in the cyclic category has a unique factorization $\sigma \circ \phi$ where

$$\phi \in \mathbb{Z}/(m+1) = \mathrm{Aut}_\Lambda(\langle m \rangle)$$

and $\sigma \in \Delta$.

The subcategory of Λ consisting of those morphisms preserving 0 can be identified with Δ^{op}, giving us an inclusion

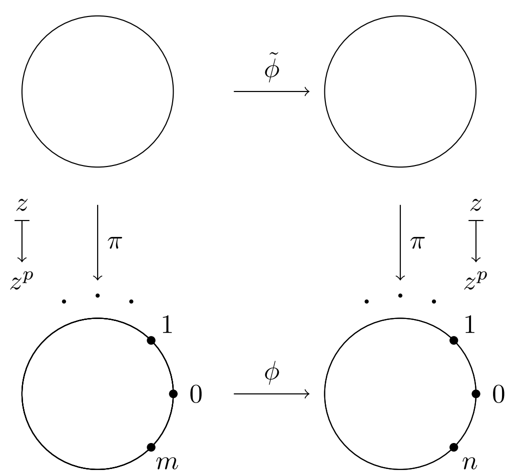
$$\Delta^{op} \hookrightarrow \Lambda$$

Definition. For every $p > 0$, there is a variant of Λ called the *p-cyclic category* Λ_p. It has morphisms

$$(\phi, \tilde{\phi}) \in \mathrm{Hom}_{\Lambda_p}(\langle m \rangle, \langle n \rangle)$$

where $\phi \in \mathrm{Hom}_\Lambda(\langle m \rangle, \langle n \rangle)$ and $\tilde{\phi}$ is a lift to the p-fold cover

Pictorially, we can represent such a morphism as



From this definition, then, we can think of Λ_p as something of a hybrid between Δ and $\mathbb{Z}/p(m+1)$.

Fundamental: There are two canonical functors

$$\langle n \rangle \longmapsto \pi^{-1}(\{0,1,\dots,r\}) = \langle p(n+1)-1\rangle$$

$$\begin{array}{ccccc} \langle n \rangle & & \Lambda_p & \xrightarrow{\quad S \quad} & \Lambda \\ \text{forget} \big\downarrow & & \big\downarrow q & & \\ \langle n \rangle & & \Lambda & & \end{array}$$

Key Fact:

$$|\Lambda| \simeq BS^1 \simeq BB\mathbb{Z} \simeq \mathbb{C}P^\infty$$

and the same is true for Λ_p. Furthermore, the diagram above becomes, after applying $|-|$

$$\begin{array}{ccc} BS^1 & \xrightarrow{\simeq} & BS^1 \\ B(z \mapsto z^p) \big\downarrow & & \\ BS^1 & & \end{array}$$

Proof. Apply Quillen's Theorem B to the functor

$$\Delta^{op} \hookrightarrow \Lambda$$

Using the fiber diagram

$$\begin{array}{ccc} S^1 & \longrightarrow & |\Delta^{op}| \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & |\Lambda| \end{array}$$

we then see the result. □

To see the relation to cyclic homology, consider the adjunction

$$\operatorname*{colim}_{\Lambda} : \mathrm{Fun}(\Lambda, \mathrm{Vect}_k) \leftrightarrow \mathrm{Vect}_k : \mathrm{const}$$

then the cyclic homology is given by[78]

$$CC_\bullet(A) \simeq L \operatorname*{colim}_{\Lambda}(A^\Lambda) \in \mathscr{D}\,(\mathrm{Vect}_k)$$

We can however, obtain a refined understanding of this colimit, using the machinery of Kan extensions.

[78] As an aside: if we do the same thing for the simplex category

$$\operatorname*{colim}_{\Delta} : \mathrm{Fun}(\Delta^{op}, \mathrm{Vect}_k) \leftrightarrow \mathrm{Vect}_k : \mathrm{const}$$

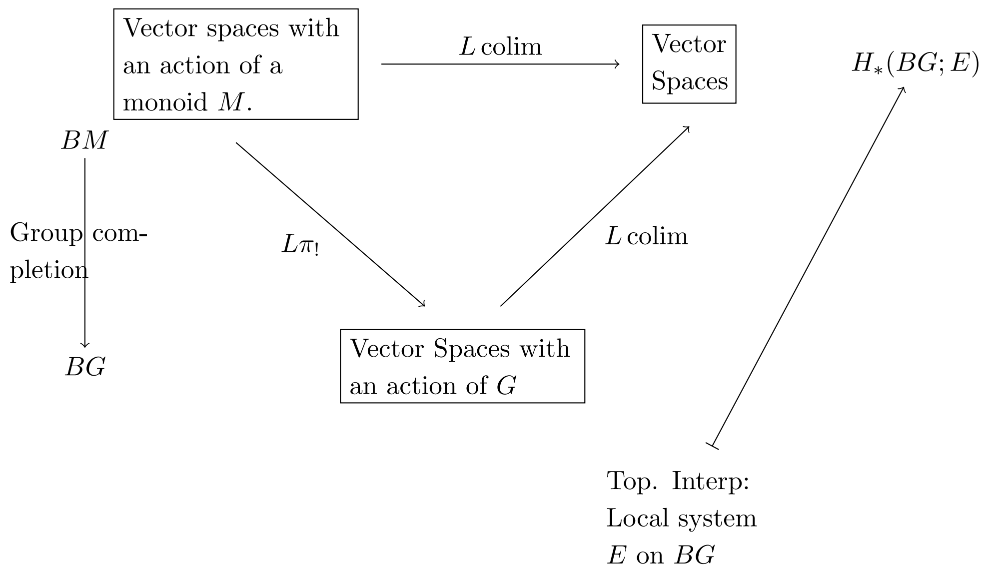
We get that

$$L \operatorname*{colim}_{\Delta}(X_\bullet)$$

is the complex associated with X via

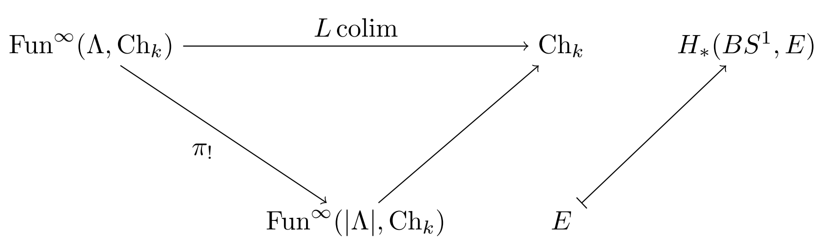
$$d = \sum (-1)^i d_i$$

A prototypical example of this sort of use of Kan extensions is the case of Vector spaces equipped with a monoid action:



In the case, for example $M = \mathbb{N}$, we then have $G = \mathbb{Z}$ and $BG = S^1$.

In our case, we want an infinity-categorical variant on this, so we take:



where we interpret $\mathrm{Fun}^{\infty}(|\Lambda|, \mathrm{Ch}_k)$ as "infinity-local systems of complexes" on $|\Lambda|$. This computation gives us

$$H_*(BS^1) = k[u^{-1}]$$

Non-commutative Cartier Isomorphism, Part II

Tobias Dyckerhoff

We can refine our understanding of $L\,\mathrm{colim}_\Lambda(-)$ via Kan extensions:

$$\begin{array}{ccc} \mathrm{Fun}(N(\Lambda), N_{dg}(\mathrm{Ch}(k))) & \xrightarrow{\mathrm{colim}_\Lambda(-) := H_\bullet(\Lambda,-)} & N_{dg}(\mathrm{Ch}(k)) \\ & {\scriptstyle \pi_!}\searrow \qquad \nearrow{\scriptstyle H_\bullet(\overbrace{|N(\Lambda)|}^{BS^1}, -)} & \\ & \mathrm{Fun}(\mathrm{Sing}\,|N(\Lambda)|, N_{dg}(\mathrm{Ch}(k))) & \end{array}$$

Definition. An ∞-*category* is a simplicial set $\mathcal{C} \in \mathrm{Set}_\Delta$ such that every inner horn $\Lambda^n_i \to \mathcal{C}$ $(0 < i < n)$ has a filler $\Delta^n \to \mathcal{C}$.

Examples.

(1) If C is a category, then $N(C)$ is an ∞-category (Every inner horn has a **unique** filler, in fact).

(2) If X is a topological space, then $\mathrm{Sing}\,X$ is an ∞-category (every horn has a filler, that is, $\mathrm{Sing}\,X$ is an ∞-groupoid).

(3) For $C \in \mathrm{Cat}_{\mathrm{Top}}$, $N_{\mathrm{Top}}(C)$ is an infinity category.

(4) For $C \in \mathrm{Cat}_{dg}(k)$ a k-linear dg-category, N_{dg} is an infinity category.

(5) Given $I \in \mathrm{Set}_\Delta$ and $\mathcal{C}$ and ∞-category, we can define the functor category to be the internal Hom

$$Fun(I, \mathcal{C}) := \underline{\mathrm{Hom}}_{\mathrm{Set}_\Delta}(I, \mathcal{C})$$

which is, itself, an ∞-category.

All of these constructions can be understood in terms of (Quillen) adjunctions, for example:

(2) We have an adjunction

$$|-|:\mathrm{Set}_\Delta \leftrightarrow \mathrm{Top}:\mathrm{Sing}$$

For any simplicial set X, the map

$$X \to \mathrm{Sing}\,|X|$$

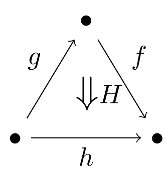
is given by the counit of the adjunction, eg

$$\pi : N(\Lambda) \to \mathrm{Sing}\,|N(\Lambda)|$$

(4) We have an adjunction

$$\begin{aligned} dg:\mathrm{Set}_\Delta &\leftrightarrow \mathrm{Cat}_{dg}(k):N_{dg} \\ [2] &\mapsto dg[2] \end{aligned}$$

where, for a diagram



The dg category has $|f|=|g|=|h|=0$ are cycles, and has $|H|=1$ with

$$dH = f\circ g - h$$

Definition. If X is a topological space, then[79]

$$\mathscr{L}oc(X,k) := \mathrm{Fun}(\mathrm{Sing}\,X, N_{dg}(\mathrm{Ch}(k)))$$

is called the ∞-*category of* ∞-*local systems* on X with values in $\mathrm{Ch}(k)$.

[79] Often, in the notation that follows, we will drop the k when it is clear what field we are working over.

Via (4), if X is a connected topological space, we have, in some sense[80], a quasi-isomorphism

$$\mathrm{Fun}_\infty\left(\mathrm{Sing}\,X, N_{dg}(\mathrm{Ch}(k))\right) \simeq \mathrm{Fun}_{dg}\left(dg\,\mathrm{Sing}\,X, \mathrm{Ch}(k)\right)$$

[80] To make this rigorous, we need to be quite careful. We are working with **Quillen** adjunctions, so in some sense the proper functor categories to consider are those defined via bimodules.

We can also compute that

$$dg(\mathrm{Sing}(X)) \simeq C_*(\Omega_x X,,k)$$

the differential graded algebra of singular chains.

Examples.

(1) Let $X = BG$ where G is a discrete group. Then

$$C_*(\Omega_x X, k) \simeq kG$$

so that

$$\mathscr{L}oc(X, k) \simeq \mathscr{D}(\mathrm{Mod}_{kG})$$

(2) Let $X = BS^1$. Then

$$C_*(\Omega_x BS^1, k) \simeq k[\epsilon]$$

where $|\epsilon| = 1$ and $\epsilon^2 = 0$. Then we have

$$\mathscr{L}oc(BS^1, k) \simeq \mathscr{D}(\mathrm{Mod}_{k[\epsilon]})$$

To relate this to cyclic homology, consider the diagram

$$\begin{array}{ccc}
N(\Delta^{op}) & \xrightarrow{\ j\ } & N(\Lambda) \\
\Big\downarrow r & \lrcorner & \Big\downarrow q \\
\mathrm{Sing}\,|N(\Delta^{op})| & \xrightarrow[\ i\ ]{} & \mathrm{Sing}\,|N(\Lambda)| \\
\wr| & & \wr| \\
\mathrm{pt} & & BS^1
\end{array}$$

This is a pullback diagram of infinity categories[81].

[81] As argued in the previous lecture, this result follows by first noting that Quillen's Theorem B implies that i is a fibration.

In this context we have a notion of base change: for

$$E \in \mathrm{Fun}(N(\Lambda), N_{dg}(\mathrm{Ch}(k)))$$

we have that

$$i^* \pi_! \simeq r_! J^* E$$

Therefore, we have an object

$$\pi_! A^\Lambda \in \mathscr{L}oc(BS^1)$$

with

$$i^* \pi_! A^\Lambda \simeq r_! \underbrace{j^* A^\Lambda}_{A^\Delta} \simeq C_\bullet(A)$$

To illustrate how this perspective is natural, we take the example of a mixed complex. Let V be a vector space over k with an action of $\langle t \rangle = \mathbb{Z}/p\mathbb{Z}$, then we get a complex

$$\underset{1}{V} \overset{1-t}{\to} \underset{0}{V}$$

This complex has a $k[\epsilon]$-structure (a mixed complex structure) given by the diagram

$$\begin{array}{ccccc} V & \xrightarrow{1-t} & V & \longrightarrow & 0 \\ \downarrow & & \downarrow N & & \downarrow \\ 0 & \longrightarrow & V & \xrightarrow[1-t]{} & V \end{array}$$

where

$$N = \sum_{i=0}^{p-1} t^i$$

The topological explanation is that, taking

$$B(\mathbb{Z}/p\mathbb{Z}) \stackrel{i}{\hookrightarrow} BS^1$$

We can consider

$$V \in \mathscr{L}oc(B\mathbb{Z}/p\mathbb{Z}) \stackrel{i_!}{\to} \mathscr{L}oc(BS^1) \stackrel{\pi_!}{\to} \mathscr{L}oc(pt)$$

Exercise. Check that $i_! V$ yields the constructed $k[\epsilon]$-module.

Now that we have dealt with the background, we can return to the non-commutative Cartier isomorphism

Proof Strategy

We can consider the diagram

$$\langle n \rangle \longmapsto \pi^{-1}(\{0, 1, \dots, r\}) = \langle p(n+1) - 1 \rangle$$

$$\begin{array}{ccccc} \langle n \rangle & & \Lambda_p & \xrightarrow{s} & \Lambda \\ \text{forget} \big\downarrow & & \big\downarrow q & & \\ \langle n \rangle & & \Lambda & & \end{array}$$

from last lecture. Under geometric realization, as we remarked, it leads to

$$\begin{array}{ccc} BS^1 & \xrightarrow{\simeq} & BS^1 \\ B(z \mapsto z^p) \big\downarrow & & \\ BS^1 & & \end{array}$$

Let A be an associative k-algebra

Step ① Show that, for the map s above[82]

$$H_*(\Lambda_p, S^* A^\Lambda) \simeq CC_*(A)$$

Step ② In $\operatorname{char} k = p > 0$, show that we have a quasi-isomorphism

$$H_*(\Lambda_p, q^* A^\Lambda) \simeq \big(C_*[u^{-1}], b\big)$$

Step ③ Construct a map

$$q^*(A^{(p)})^\Lambda \to s^* A^\Lambda$$

which induces an equivalence

$$\underbrace{\lim_{\substack{\leftarrow \\ u}} H_*\left(\Lambda_p, q^*(A^{(p)})^\Lambda\right)}_{C_*(A^{(p)})((u))} \overset{\simeq}{\to} \underbrace{\lim_{\substack{\leftarrow \\ u}} H_*\left(\Lambda_p, s^* A^\Lambda\right)}_{CP_*(A)=(C_*(A), b+uB)}$$

The proof of Step② is based on the folowing:
Let $E \in \mathscr{L}oc(BS^1, k)$, $\operatorname{char} k = p > 0$, and consider the diagram $(*)$

$$\begin{array}{ccc} BS^1 & & \\ \downarrow{\scriptstyle q)} & \searrow{\scriptstyle r} & \\ & & * \\ & \swarrow{\scriptstyle j} & \\ BS^1 & & \end{array}$$

which commutes "modulo p" in the sense of the previous talk.

We then claim that

$$q_! \simeq j_! \circ r_!$$

Why is this true? We can check the adjoint statement

$$q^* \simeq r^* \circ j^*$$

Considering

$$\mathscr{L}oc(BS^1) \overset{q_*}{\to} \mathscr{L}oc(BS^1)$$

If we restrict along this map, we get the commutative diagram

$$\begin{array}{ccc} & C) * (*, k) & \\ \nearrow & & \searrow \\ C_*(S^1, k) & \longrightarrow & C_*(S^1, k) \end{array}$$

[82] Where, for example,

$$s^* A^\Lambda(\langle n \rangle) = A^{\otimes p(n+1)}$$

This step is actually relatively easy to show. We consider the diagram

$$\begin{array}{ccc} \Delta^{op} & \xrightarrow{sd} & \Delta^{op} \\ \downarrow & & \downarrow \\ \Lambda_p & \longrightarrow & \Lambda \end{array}$$

where sd is given by

$$[n] \mapsto \overbrace{[n] * \cdots * [n]}^{p}$$

Then apply the usual tricks to show that we have a pullback square, and thus a weak equivalence.

which proves the claim.

Then we can make the computation[83]

$$\begin{aligned} H_*(BS^1, q^*E) &\overset{K}{\simeq} H_*(BS^1, q_! q^* E) \\ &\overset{P}{\simeq} H_*(BS^1, E \otimes q_! k) \\ &\overset{(*)}{\simeq} H_*(BS^1, \overbrace{E \otimes j_! H_*(BS^1, k)}^{r_! k}) \\ &\overset{P}{\simeq} H_*(BS^1, j_!(j^*E \otimes H_*(BS^1, k)) \\ &\simeq j^*E \otimes H_*(BS^1, k) \\ &\simeq C_*(A) \otimes k[u^{-1}] \end{aligned}$$

[83] Where the quasi-isomorphism marked with a K follows from the functorality of the Kan extension, that marked with $(*)$ follows from the diagram $(*)$, and those marked with a P follow from the projection formula.

To show Step ③, we need to find a non-commutative analogue of the Frobenius.

$$\begin{aligned} F : A^{(p)} &\to A^{\otimes p} \\ \text{“}a &\mapsto \underbrace{\otimes \cdots \otimes a}_{p}\text{”} \end{aligned}$$

This doesn't make sense, so instead we consider the necessary equivariance

$$\mathbb{Z}/p \circlearrowleft A^{(p)} \to A^{\otimes p} \circlearrowright \mathbb{Z}/p \qquad (*)$$

and the morphism on Tate homology

$$\check{H}_* \left(\mathbb{Z}/p\mathbb{Z}, A^{(p)}\right) \overset{\cong}{\to} \check{H}_* \left(\mathbb{Z}/p\mathbb{Z}, A^{\otimes p}\right) \qquad (**)$$

We therefore want a morphism $(*)$ inducing $(**)$. If we assume that such a morphism exists, the proof follows.

Problem: Such a morphism basically never exists.

Non-commutative Cartier Isomorphism, Part III

Thomas Poguntke

Let k be a perfect field of characteristic p, and A a smooth k-algebra. We want to construct a (non-commutative inverse Cartier) isomorphism

$$HH_*(A^{(p)})((u)) \to HP_*(A)$$

Last time, we saw that we expect this to be induced by a map of p-cyclic objects[84]

$$q^*(A^{(p)})^\Lambda \to s^* A^\Lambda$$

[84] Where, as before, $q : \Lambda_p \to \Lambda$ forgets the lift to the p-fold cover of the circle, and $s : \Lambda_p \to \Lambda$ sends $\langle n \rangle \mapsto \pi^{-1}\langle n \rangle$.

In a (very) special case, the desired map of p-cyclic objects will be $(-)_*$ applied to some 'NC-frobenius' map

$$A^{(p)} \to A^{\otimes p}$$
$$\text{`}a \mapsto a^{otimesp}\text{'}$$

which is $\mathbb{Z}/p$-equivariant, and induces an isomorphism on Tate homology.

Lemma. *For any vector space W, the map*

$$W^{(p)} \cong \hat{H}_\bullet(\mathbb{Z}/p, W^{(p)}) \to \hat{H}_\bullet(\mathbb{Z}/p, W^{\otimes p})$$

given by

$$a \mapsto a^{\otimes p}$$

is an isomorphism. In particular, it is additive.

Proof. Cyclic groups have cyclic Tate homology with differentials

$$d_i : M \to M$$

where M is a $\mathbb{Z}/p$-module, given by

$$d_i = \begin{cases} 1 - \sigma & i \text{ odd} \\ 1 + \sigma + \cdots + \sigma^{p-1} & i \text{ even} \end{cases}$$

Now, choose a basis I of $W = kI$. Then[85]

[85] This equality is $\mathbb{Z}/p$-equivariant.

$$W^{\otimes p} = k\Delta \oplus k(I^{\times p} \setminus \Delta)$$

Where $\mathbb{Z}/p$ acts on Δ trivially and $\mathbb{Z}/p$ acts on $I^{\times p} \setminus \Delta$ freely.

Therefore, we can decompose the homology into[86]

$$\hat{H}_i(\mathbb{Z}/p, W^{\otimes p}) \cong \underbrace{\hat{H}_i(\mathbb{Z}/p, k\Delta)}_{\cong W} \oplus \underbrace{\hat{H}_i(\mathbb{Z}/p, k(I^{\times p} \setminus \Delta))}_{=0}$$

□

[86] This is a morphism of vector spaces. It is not at all clear that it lifts to an additive algebra morphism $A^{(p)} \to A^{\otimes p}$ for an algebra A.

Now assume A is commutative. Spec(A) is a commutative group scheme (a cocommutative Hopf algebra)[87]. Then

$$V : A \xrightarrow{c^p} (A^{\otimes p})^{\mathbb{Z}/p} \to (A^{\otimes p})^{\mathbb{Z}/p}/d_1(A^{\otimes p}) \cong A^{(p)}$$

is called the *Verschiebung* which is, in fact, an algebra homomorphism[88].

Then

$$\begin{array}{ccccc} A & \xrightarrow{V} & A^{(p)} & \xrightarrow{F} & A \\ & \searrow^{c} & \downarrow{\scriptstyle\exists} & \nearrow_{\text{mult}} & \\ & & A^{\otimes p} & & \end{array}$$

ie $F \circ V = p \circ id_A$. That is, V is an isomorphism if and only if Spec$(A) \subseteq \mathbb{G}_m^N$ is a subgroup of the multiplicative group, which itself holds if and only if $A \cong kG$ over a separable extension (where G is commutative, although the 'NC-frobenius' exists for any G)[89].

[87] ie there exists an algebra homomorphism

$$\begin{array}{ccc} A & \xrightarrow{c} & A \otimes A \\ & \searrow \quad \nearrow & \\ & (A^{\otimes 2})^{\mathbb{Z}/2} & \end{array}$$

[88] The map

$$(A^{\otimes p})^{S_p}/d_1(A^{\otimes p}) \cong A^{(p)}$$

is clearly an algebra isomorphism, with inverse

$$a \mapsto a^{\otimes p}$$

so that

$$(a+b)^{\otimes p} = \sum_{r=0}^{p} \binom{p}{r} a^{\otimes r} b^{\otimes (p-r)}$$

in symmetric tensors.

[89] In this case,

$$\begin{aligned} V : kG &\to kG^p \\ g &\mapsto g \otimes 1 \end{aligned}$$

so

$$\begin{aligned} kG^{(p)} &\to k\left[G^{\times p}\right] \\ g &\mapsto \underbrace{(g, \ldots, g)}_{p} \end{aligned}$$

Now let $k = \mathbb{F}_p$, and A perfect. Then there exists a p-adically complete ring $W(A)$ together with a residue isomorphism

$$W(a) \twoheadrightarrow W(A)/pW(A) \cong A$$

Moreover, the Frobenius on A lifts to $F : W(A) \to W(A)$, and induces

$$F : W(A)/p^n W(A) =: W_n(A) \to W_{n+1}(A)$$

Additionally, there is another map

$$V : W_{n-1}(A) \to W(A)$$

for any n such that $FV = p \circ id_{W(A)} = VF$.

Finally, there exists a Teichmüller map (which is multiplicative)

$$A \to W(A)$$

$W(A)$ is the collection of *Witt vectors*, reminiscent of

$$\mathbb{Z}_p = W(\mathbb{F}_p)$$

where, in $\mathbb{Z}_p[X]$[90]

$$Z^p - X = \prod_{a \in \mathbb{F}_p} (X - [a])$$

[90] *Hensel's Lemma.*

Construction

Consider $A = (A, \cdot)$ (A commutative) as a multiplicative monoid. The Teichmüller map should be

$$\approx \binom{A \to \mathbb{Z}A}{a \mapsto [a]}$$

There is a Frobenius lift

$$[a] \mapsto [a^p]$$

and augmentation sequence

$$0 \to I \to \mathbb{Z}A \to A \to 0$$

where

$$I = \operatorname{span}([a+b] - [a] - [b])$$

so that

$$W(a) = \lim_{\leftarrow} \mathbb{Z}A/I^n$$

is the I-adic completion. It remains to show p-adic completeness.

If $F(I) = I$, the Frobenius descends, and Teichmüller yields

$$A \hookrightarrow \mathbb{Z}A \twoheadrightarrow W(A)$$

In this case, F is an isomorphism, so we can just set $V = pF^{-1}$.[91]

[91] In fact, $W_n(A) \cong \mathbb{Z}A/I^n$

Proof Sketch of p-adic completeness. There is a short exact sequence[92]

[92] This is because $x \in I$ implies that

$$F(x) = x^p \bmod p\mathbb{Z}A$$

and hence

$$x \equiv F^{-n}(x)^{p^n} \bmod p\mathbb{Z}A$$

for any n. Therefore

$$I = I^n + p\mathbb{Z}A$$

$$0 \to p^{-1}I^n/I^n \to \mathbb{Z}A/I^n \xrightarrow{p} I/I^n \to 0$$

The transition maps

$$p^{-1}(I^n)/I^n \to p^{-1}(I^{n-1})/I^{n-1}$$

are trivial, ie if $p \cdot x \in I^n$ then $x \in I^{n-1}$.

For this case, we can define a 'derivation'

$$\begin{aligned} \delta : \mathbb{Z}A &\to \mathbb{Z}A \\ z &\mapsto p^{-1}(F(x) - x^p) \end{aligned}$$

Then

$$\delta(x+y) \overset{(*)}{=} \delta(x) + \delta(y) - \sum_{r=1}^{p-1} p^{-1} \binom{p}{r} x^r y^{p-r}$$

and[93]

[93] Note also that

$$\delta([a]) = 0$$

$$\delta(xy) = \delta(x)F(y) + x^p \delta(y)$$

This implies that

$$\delta(x_1 \cdots x_n) \overset{(**)}{=} \sum_{r=1}^{p-1} F(x_{r+1}) \cdots F(x_n)$$

which, in turn, implies

$$\delta(I^n) \subseteq I^{n-1}$$

by $(*)$.

Thus, for $px \in I^n$ as above, $\delta(px) \in I^{n-1}$ and

$$\delta(px) \overset{\text{def}}{=} F(x) - p^{p-1}x^p \equiv F(x) \bmod I^n$$

and hence, $F(x) \in I^{n-1}$ as well. Since F is an automorphism of I, we get

$$x \in I^{n-1}$$

Therefore, in fact,

$$\lim_{\leftarrow} \left(p^{-1}I^n/I^n\right) = 0$$

and so p is injective on $W(A)$, with

$$p \cdot W(A) = \lim_{\leftarrow} I/I^n \subset W(a)$$

with respect to which $W(A)$ is complete. □

Remark. If A is commutative,

$$W_n(A) \cong \pi_0\left(\mathrm{THH}(A)^{\mathbb{Z}/p^{n-1}}\right)$$

Witt's Original Construction

Example. Consider $W_2(A) \xrightarrow{\sim} A^2$ (with ring structure to be defined) given by

$$x \mapsto (\overline{x}, \delta(x))$$

where $\overline{x}$ denotes the image of x under

$$W_2(A) \twoheadrightarrow A$$

Namely, on the RHS,

$$(x_0, x_1) + (y_0, y_1) = \left(x_0 + y_0,\ x_1 + y_1 - \sum_{r=1}^{p-1} \frac{1}{p}\binom{p}{r} x_0^r y_0^{p-r}\right)$$

and

$$(x_0, x_1)(y_0, y_1) = (x_0y_0,\ y_0^p x_1 + y_1 x_0^p - p x_1 y_1)$$

So where does this come from? We can think of $W(a) = \prod A$ as power series in p, and then define addition.

Question. If

$$\sum_{n\geq 0}[a_n]p^n + \sum_{n\geq 0}[b_n]p^n = \sum_{n\geq 0}[c_n]p^n$$

what is $[c_n]$?

INCOMPLETE

Non-commutative Cartier Isomorphism, Part VI

Tobias Dyckerhoff

Recall. We are trying to construct

$$C^{-1} : HH_*(A)((u)) \overset{\cong}{\to} HP_*(A)$$

for A smooth over k, a field a characteristic $p > 0$

The perspective on C^{-1} from p-adic Hodge theory:

Definition. Let k be perfect of characteristic $p > 0$. A *filtered Dierdonné module* over $W(k)$ consists of

- A $W(k)$-module M
- A decreasing filtration
$$\{F^i M \mid i \in \mathbb{Z}\}$$
with
$$\bigcap F^i M = 0 \ , \ \bigcup F^i M = M$$
- Frobenius-semilinear maps
$$\phi_i : F^i M \to M$$
satisfying
(1) $\phi_i|_{F^{i+1}M} = p \cdot \phi_{i+1}$
(2) The sequence
$$0 \to \bigoplus F^i M \overset{t-p \cdot id}{\to} \bigoplus F^i M \overset{\sum \phi_i}{\to} M \to 0$$
is exact.

Note. If M is annihilated by p, then condition (2) says

$$gr_F^\bullet M \underset{\cong}{\overset{\sum \phi_i}{\to}} M$$

Example. Let X be a smooth variety over $W(k)$ with $\dim(X) < p$. Then each $H^n(\Omega^\bullet_{X/W(k)})$ carries a filtered Dierdonné module. Reduction modulo p yields the isomorphism

$$gr_F^\bullet H^n_{dR}(X_k) \overset{\cong}{\to} H^n_{dR}(X_k)$$

showing the degeneration of the Hodge-to-de Rham Spectral sequence[94].

[94] Construction due to Faltings.

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