Topological Background

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This is a chapter excerpted from a book draft, covering some background material in general topology. It is being provided in its current form for background reading for the 2021 REU in topology at the University of Virginia. The pace of the presentation is somewhat rapid, and so students may wish to supplement this document with other standard resources in general topology.

1

Continuity and Topologies

In the overture, we defined metric spaces associated to simplicial complexes, which allowed us to talk about notions like continuity, as well as open and closed sets.[1] In analysis, one additionally makes use of notions like limit points, compactness, the interior of a set, and so on.

The fundamental observation of topology is that none of these continuity-related notions really require a metric. We can have satisfactory definitions of continuous, compact, etc. if we forget about the metric (the distance between points) and only remember which sets were open. This insight leads to the notion of a topology, which allows one to think about continuous spaces in a much more general context.

[1] For those of you who don't recall: an *open* subset $U \subset X$ of a metric space (X, d) is a set such that, for every $x \in U$, there is a radius $r > 0$ such that the open ball $B_r(x)$ around x is entirely contained within U. A *closed* subset $C \subset X$ is a set such that the complement C^c is open.

1.1 Continuity

We begin our exploration of topology with the following proposition.

Proposition 1.1. *Let (X, d) and (Y, s) be metric spaces. Then a function $s : X \to Y$ is continuous if and only if, for every open subset $U \subset Y$, the subset $f^{-1}(U) \subset X$ is open.*

Proof. First suppose that s is continuous and let $U \subset Y$ be an open set. Suppose $x \in f^{-1}(U)$. Since U is open, we can find an $\epsilon > 0$ such that the ball $B_\epsilon(f(x)) \subset U$. By continuity, there exists a $\delta > 0$ such that $f(B_\delta(x))) \subset B_\epsilon(f(x))$, and thus $B_\delta(x) \subset f^{-1}(U)$, proving that $f^{-1}(U)$ is open.

Now suppose that every preimage under f of an open set is open. In particular, given $x \in X$, and $\epsilon > 0$, the set $f^{-1}(B_\epsilon(f(x)))$ is open. Therefore, there is a $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$, and thus $f(B_\epsilon(x)) \subset B_\epsilon(f(x))$. Hence, f is continuous. □

This tells us that if we know the collection of open sets of (X, d), we can check whether a map is continuous regardless of whether we know the metric. The idea of a topological space is to forget the metric, and just remember the open sets:

Definition 1.2. Let X be a set. A *topology on X* is a subset $\tau_X \subset \mathbb{P}(X)$ of the power set such that

1. $X, \emptyset \in \tau_X$.

2. Let $\{U_i\}_{i\in I}$ be a (possibly infinite) collection of sets in τ_X. Then

$$\bigcup_{i\in I} U_i \in \tau_X$$

3. Let $\{U_i\}_{i\in 1}^n$ be a finite collection of sets in τ_X. Then

$$\bigcap_{i=1}^n U_i \in \tau_X.$$

We call the elements $U \in \tau_X$ the *open sets* of the topology on X. We refer to a pair (X, τ_X), where τ_X is a topology on X, as a topological space.[2] We call a subset $C \subset X$ of a topological space *closed* if

$$C^c := X \setminus C$$

is an open set, i.e. is in τ_X.

[2] We will often abuse notation and write X for a topological space in cases where the choice of topology is clear from context.

Proposition 1.3. *Let (X, d) be a metric space. Denote by τ_d the collection of d-open subsets of X. Then τ_d is a topology on X.*[3]

[3] This, together with Proposition 1.1 effectively tells us that we can study continuous functions between metric spaces in terms of the associated topological spaces.

Proof. It is immediate that X and $\emptyset$ are elements of τ_d. We now check the remaining properties.

Suppose that $\{U_i\}_{i\in I}$ is a collection of open sets of X, and let $x \in \bigcup_{i\in I} U_i$. Then, in particular, there exists a $j \in I$ such that $x \in U_j$. Since U_j is open, there is a radius r such that $B_r(x) \subset U_j$. However, this implies that $B_r(x) \subset \bigcup_{i\in I} U_i$, so the latter is open.

Now suppose that $\{U_i\}_{i=1}^n$ is a collection of open sets of X. Let $x \in \bigcap U_i$. For each $1 \leq i \leq n$, choose a radius $r_i > 0$ such that $B_{r_i}(x) \subset U_i$. Set $r = \min_i(r_i)$. Then $B_r(x) \subset U_i$ for any $q \leq i \leq n$. Consequently $B_r(x) \subset \bigcap U_i$, and such $\bigcap U_i$ is an open set. □

Note that Proposition 1.3 implies that for any finite simplicial complex K, the realization $|K|$ is canonically a topological space.

Having now established our general definitions, we can now proceed to the concepts necessary to study continuous maps

Definition 1.4. A map $f : X \to Y$ between topological spaces (X, τ_X) and (Y, τ_Y) is called a *continuous map* if, for every open subset $U \in \tau_Y$, the preimage $f^{-1}(U)$ is an element of τ_X. We say that f is a *homeomorphism* if it is bijective and both f and f^{-1} are continuous.

Lemma 1.5. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a map between topological spaces. The following are equivalent:*

1. *The map f is continuous.*

2. *For every closed set C of Y, $f^{-1}(C)$ is closed in X.*

Proof. First suppose that f is continuous, and let $C \subset Y$ be closed. Then $f^{-1}(C^c)$ is open in X. We can then compute

$$f^{-1}(C)^c = X \setminus f^{-1}(C) = f^{-1}(Y) \setminus f^{-1}(C) = f^{-1}(Y \setminus C) = f^{-1}(C^c)$$

and thus, $f^{-1}(C)^c$ is open, so $f^{-1}(C)$ is closed.

Now suppose that f satisfies condition 2. The same computation as above shows that for $U \in \tau_Y$, $f^{-1}(U)$ is open. □

Lemma 1.6. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a map between topological spaces. Then f is a homeomorphism if and only if f is continuous, bijective, and, for all $U \in \tau_X$, $f(U) \in \tau_Y$.*

Proof. Left as an exercise to the reader. □

Definition 1.7. Let τ_1 and τ_2 be two topologies on a set X. If $\mathrm{id}_X : (X, \tau_2) \to (X, \tau_1)$ is continuous, we say that τ_1 is *coarser* than τ_2 or that τ_2 is *finer* than τ_1.

Examples. Some interesting examples of coarseness/fineness are the most extreme. Let X be a set

1. Define a topology τ_{dis} on X by declaring *every* subset of X to be an element of τ_{dis}. We call this the *discrete topology* on X. This is the finest possible topology on X, and it has a very interesting property. Let (Y, τ_Y) be any topological space, and let $f : X \to Y$ be any map of underlying sets. Then $f : (X, \tau_{dis}) \to (Y, \tau_Y)$ is continuous.
2. Define a topology τ_{ind} on X by $\tau_{ind} := \{\emptyset, X\}$. We call this the *indiscrete topology* on X – the coarsest possible topology on X. For any topological space (Y, τ_Y) and any map of sets $f : Y \to X$, the map $f : (Y, \tau_Y) \to (X, \tau_{ind})$ is continuous.

These two examples are *dual* to one another, in a sense that the categorical language we will explore later makes clear.

Exercise 1.8. Show that the following statements are equivalent:

1. $\tau_1 \subset \tau_2$
2. τ_1 is coarser than τ_2.

1.1.1 Bases

We now want a way to uniquely specify a topology on X by giving a simpler collection of sets. We will introduce two such notions, one stronger than the other.

Lemma 1.9. *Let X be a set and let I be a set of topologies on X. Then*

$$\gamma := \bigcap_{\tau \in I} \tau$$

is a topology on X.

Proof. Left as an exercise. □

Definition 1.10. Let X be a set, and $\mathcal{B} \subset \mathbb{P}(X)$ be a subset of the power set. Set

$$I := \{\tau \subset \mathbb{P}(X) \mid \mathcal{B} \subset \tau \text{ and } \tau \text{ is a topology on } X\}.$$

We define

$$\tau_{\mathcal{B}} := \bigcap_{\tau \in I} \tau$$

to be the *topology generated by* $\mathcal{B}$.

Definition 1.11. Let X be a set, and let τ_X be a topology on X. We call a subset $\mathcal{B} \subset \tau_X$ a *basis* of τ_X if every element of τ_X is a (possibly empty) union of elements of $\mathcal{B}$.

Proposition 1.12. *Let (X, τ_X) be a topological space, and let $\mathcal{B} \subset \tau_X$. Then $\mathcal{B}$ is a basis for τ_X if and only if, for every $U \in \tau_X$ and every $x \in U$, there is a $V \in \mathcal{B}$ such that $x \in V$ and $V \subset U$.*

Proof. First suppose that $\mathcal{B}$ is a basis for τ. Let $U \in \tau_X$ and $x \in U$. Then, in particular, there is a set $\{V_i\}_{i\in I}$ of elements in $\mathcal{B}$ such that

$$\bigcup_{i\in I} V_i = U$$

So, for at least one $i \in I$, $x \in V_i$, and every $V_i \subset U$. Therefore, our criterion is fulfilled.

Now suppose our criterion is fulfilled. Let $U \in \tau_X$. For each $x \in U$, let $V_x \in \mathcal{B}$ be a set such that $x \in V_x \subset U$. It is then immediate from the definition that

$$U = \bigcup_{x\in U} V_x$$

so $\mathcal{B}$ is a basis for τ_X. $\square$

Examples.

1. Consider $\mathbb{R}^n$ with the *standard topology* τ, i.e. the topology defined by the Euclidean metric. Then consider the set
$$\mathcal{B} := \{B_r(x) \mid x \in \mathbb{R}^n \text{ and } r > 0\}$$
of open balls. It is easy to check using the preceding proposition that $\mathcal{B}$ is a basis for τ.
2. More generally, let (X, d) be a metric space. Then the set of open balls in X forms a basis of the topology induced by d.
3. $\mathbb{R}^n$ actually has an even smaller basis: The set of open balls with rational radii about points with rational coordinates. That this is so boils down to the statement that the rationals are dense in the reals.

We can also make use of bases to more efficiently check when maps are continuous:

Proposition 1.13. *Suppose (X, τ_X) and (Y, τ_Y) are topological spaces, $\mathcal{B}$ is a basis of τ_Y, and $f : X \to Y$ is a map of sets. Then f is continuous if and only if, for every $U \in \mathcal{B}$ $f^{-1}(U)$ is open.*

Proof. Left as an exercise for the reader. $\square$

We conclude with a criterion for determining when a collection of sets $\mathcal{B}$ is a basis for *some* topology:

Proposition 1.14. *Let X be a set, and $\mathcal{B} \subset \mathbb{P}(X)$. Suppose X can be written as a union of elements of $\mathcal{B}$ (we say $\mathcal{B}$* covers *X) and that, for every $U, V \in \mathcal{B}$ and $x \in U \cap V$, there is a set $W \in \mathcal{B}$ with $W \subset U \cap V$ such that $x \in W$. Then $\mathcal{B}$ is the basis of a topology.*

Proof. Left as an exercise for the reader. $\square$

1.2 Building spaces

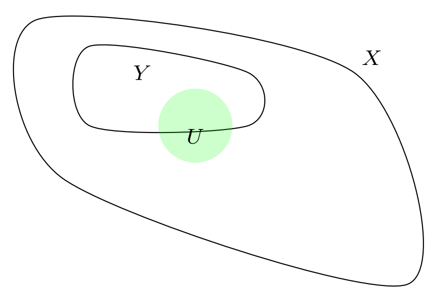
Having established our basic definitions, we now make a brief interlude to discuss some ways of constructing new topological spaces from old ones. We already have quite a large class of topological spaces — those which arise as metric spaces — however, for more general applications, we will want to construct topological spaces directly from other topological spaces.

Construction 1.15 (Subspace topology)**.** Let (X, τ_X) be a topological space and let $Y \subset X$ be a subset. The topology τ_X on X induces a topology τ_Y on Y called the *subspace topology* as follows.

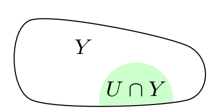
We define

$$\tau_Y := \{U \cap Y \mid U \in \tau_X\}.$$

A schematic depiction of the open sets in the supspace topology is as follows. In the first drawing we have a space X, a subset Y, and an open set U of X.



In the second, we have the corresponding open subset $Y \cap U$ of Y in the subspace topology



To see that (Y, τ_Y) is a topological space, we first note that $Y = X \cap Y \in \tau_Y$ and $\emptyset = \emptyset \cap Y \in \tau_Y$. Suppose we have a set of open sets $\{V_i\}_{i\in I}$ in τ_Y. For each i, choose[4] a $U_i \in \tau_X$ such that $U_i \cap Y = V_i$. Then $U := \bigcup_{i\in I} U_i$ is in τ_X, and thus

$$\bigcup_{i\in I} V_i = \bigcup_{i\in I} Y \cap U_i = Y \cap U \in \tau_Y.$$

Finally, for a finite collection $\{V_i\}_{i=1}^n$ of sets in τ_Y, we again choose[5] $U_i \in \tau_X$ with $U_i \cap Y = V_i$, and note that

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (Y \cap U_i) = Y \cap \bigcap_{i=1}^n U_i \in \tau_Y.$$

Thus, τ_Y is a topology on Y.

[4] This requires the axiom of choice.

[5] This doesn't.

The subspace topology is, in a sense, the coarsest topology on X such that $Y \to X$ is continuous, as the next lemma makes clear.

Lemma 1.16. *Let (X, τ_X) be a topological space, let $Y \subset X$, and let τ_Y denote the subspace topology on Y. Then for any topology γ on Y such that the inclusion $\iota : (Y, \gamma) \to (X, \tau_X)$ is continuous, the identity map $\mathrm{id}_Y : (Y, \gamma) \to (Y, \tau_Y)$ is continuous.*

Proof. Let $V \in \tau_Y$. Then there is a $U \in \tau_X$ with $U \cap Y = V$. However, $\iota^{-1}(U) = Y \cap U$. Thus, since $\iota : (Y, \gamma) \to (X, \tau_X)$ is continuous, $V \in \gamma$. Therefore $\tau_X \subset \gamma$. $\square$

Lemma 1.17. *Let (X, τ_X) be a topological space, $Y \subset X$, and τ_Y the subspace topology on Y. Denote the inclusion $\iota : Y \hookrightarrow X$. Let (Z, τ_Z) be a topological space, and $f : Z \to Y$ a map of sets. Then f is continuous if and only if $\iota \circ f$ is continuous.*

Proof. Left as an exercise to the reader. $\square$

Example 1.18. Consider the unit circle $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$. The Euclidean metric on $\mathbb{R}^2$ induces a topology on $\mathbb{R}^2$ (called the *standard topology* on $\mathbb{R}^2$), and we can equip S^1 with the subspace topology.

Construction 1.19 (Quotient topology)**.** Let (X, τ_X) be a topological space, and $\sim$ an equivalence relation on X. There is a canonical map of sets $\pi : X \to X_{/\sim}$ from X to the quotient set. We now define the *quotient topology*

$$\tau_\sim := \left\{U \subset X_{/\sim} \mid \pi^{-1}(U) \in \tau_X\right\}.$$

We claim that $(X_{/\sim}, \tau_\sim)$ is a topological space. We leave the verification that this is, in fact a topology to the reader

Exercise 1.20. Rigorously formulate and prove the statement that '$\tau_\sim$ is the finest topology on $X_{/\sim}$ such that $\pi : X \to X_{/\sim}$ is continuous'.

Exercise 1.21. Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $\sim$ be an equivalence relation on X. Suppose $f : X \to Y$ is a continuous map such that $x \sim y \Rightarrow f(x) = f(y)$. Then there is a unique continuous map $\tilde{f} : (X_{/\sim}, \tau_\sim) \to (Y, \tau_Y)$ such that $\tilde{f} \circ \pi = f$, where $\pi : X \to X_{/\sim}$ is the quotient map.

Examples 1.22.

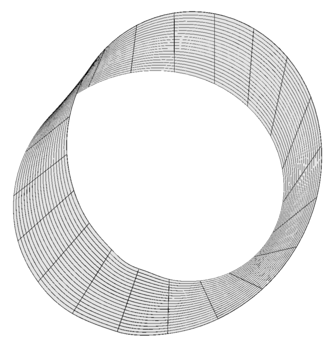
1. Consider $[0,1] \subset \mathbb{R}$ equipped with the subspace topology, and define an equivalence relation on $[0,1]$ by setting $0 \sim 1$. We then get a quotient topological space $S = [0,1]_{/\sim}$ with topology $\tau_\sim$. Consider the map

$$\begin{aligned} f : [0,1] &\to \mathbb{C} = \mathbb{R}^2 \\ t &\mapsto \exp(2\pi i t) \end{aligned}$$

   We know from analysis that this map is well-defined, continuous, and has image $S^1 \subset \mathbb{R}^2$. Moreover, f is a bijection onto it's image, and one can check (using a basis for the standard topology) that it is a homeomorphism. Therefore, $(S, \tau_\sim)$ is homeomorphic to S^1.

2. Let $X = [0,1] \times [-1,1] \subset \mathbb{R}^2$, equipped with the subspace topology. Define an equivalence relation on X by setting $(0,x) \sim (1,-x)$. The quotient space $(X_{/\sim}, \tau_\sim)$ is called the *Möbius band*.[6]

[6] The Möbius band is a rare example of a *non-orientable* surface which is easy to visualize:



Indeed, one can easily construct a Möbius band from paper or fabric.

We now come to a slightly more subtle construction. We want to define topologies on the cartesian products of topological spaces $\prod_{i\in I} X_i$. However, in the case where the product has an infinite number of factors, care must be taken to get a sensible definition.

Construction 1.23. Let I be a set, and $\{(X_i, \tau_i)\}_{i\in I}$ be a collection of topological spaces indexed by I. We define a topology on the set

$$X := \prod_{i\in I} X_i$$

as follows. Define a set

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i \;\middle|\; \substack{U_i\in\tau_i \text{ and} \\ U_i = X \text{ for all but a finite} \\ \text{number of } i\in I} \right\}$$

One can easily check that $\mathcal{B}$ satisfies the criteria from Proposition 1.14, and thus, defines a topology τ_X on X. We call this topology the *product topology*.

Example 1.24. Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be equipped with the standard topologies, and denote the product topology on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ by $\tau_{n\times m}$. Denote the standard topology on $\mathbb{R}^{n+m}$ by γ. It is immediate from the definitions that the identity map defines a homeomorphism $(\mathbb{R}^{n+m}, \tau_{n\times m}) \xrightarrow{\cong} (\mathbb{R}^{n+m}, \gamma)$, so $\tau_{n\times m} = \gamma$.

Exercise 1.25. Formulate and prove the statement that the product topology is the coarsest topology on $\prod_{i\in I} X_i$ such that the projections

$$\pi_j : \prod_{i\in I} X_i \to X_j$$

are all continuous.

Construction 1.26 (Disjoint unions)**.** Let $\{(X_i, \tau_i)\}_{i\in I}$ be a collection of topological spaces, we define a topology τ on $X := \coprod_{i\in I} X_i$ called the *coproduct topology* or *disjoint union topology* by setting

$$\tau = \{U \subset X \mid U \cap X_i \in \tau_i \forall i \in I\}.$$

The verification that this is indeed a topology is left as an exercise to the reader.

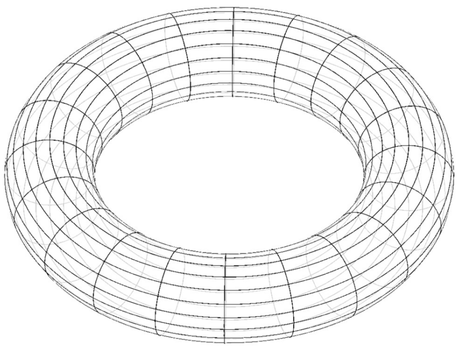
Exercise 1.27. Formulate and prove the statement that the coproduct topology is the finest topology on $\coprod_{i\in I} X_i$ such that all of the inclusions

$$\iota_j : X_j \to \coprod_{i\in I} X_i$$

are continuous.

We now provide an application: realizing infinite simplicial complexes.

A nice example of the product topology is the *torus*. This is the product $S^1 \times S^1$, equipped with the product topology. One can also view the torus as a subspace of $\mathbb{R}^3$, and the subspace topology agrees with the product topology in this case. When drawn, the torus is a surface which looks a bit like a doughnut:



Construction 1.28. Define $(\mathbb{R}^\infty, \tau_\infty)$ to be the space $\prod_{i\in\mathbb{N}} \mathbb{R}$ with the product topology. Suppose that K is an ordered simplicial complex with a countable set X of vertices. We construct a topological space $|K|$ called *the geometric realization of* K as follows. The underlying set of $|K|$ is a set of formal *finite* linear combinations

$$\Lambda := \sum_{x\in X} \lambda_x x$$

with $\lambda_x \in \mathbb{R}$. To each such Λ, we associate a set $S_\Lambda := \{x \in X \mid \lambda_x \neq 0\}$. We can then define the underlying set

$$|K| := \left\{ \sum_{x\in X} \lambda_x x \;\middle|\; \lambda_x \geq 0, \ \sum_{x\in X} \lambda_x = 1, \ \text{ and } S_\Lambda \in \mathrm{Sim}(K) \right\} \subset \mathbb{R}^{|X|}$$

Where we consider the coefficient vector $(\lambda_x)_{x\in X}$ as an element of $\mathbb{R}^{|X|}$. Since X is a set with countable number of vertices, we can identify it with $\mathbb{N}$. Consequently, we can identify $\mathbb{R}^{|X|}$ with $\mathbb{R}^\omega$, and take the subspace topology on $|K|$.

One final example of quotient spaces.

Construction 1.29. Consider $\mathbb{R}^{n+1} \setminus \{0\}$ with the subspace topology coming from $\mathbb{R}^{n+1}$. Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by $x \sim \lambda x$ for every $\lambda \in \mathbb{R} \setminus \{0\}$. We define the *$n$-dimensional real projective space* $\mathbb{R}P^n$ to be the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})_{/\sim}$.

Note that we can also consider $\mathbb{R}P^n$ as the quotient of the unit sphere S^n by the equivalence relation $x = -x$.

Lemma 1.30. *The spaces S^1 and $\mathbb{R}P^1$ are homeomorphic.*

Proof. Let $p : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}P^1$ be the quotient map, and let $q : [0, 1] \to S^1 = [0, 1]_{/0\sim 1}$ be the quotient map.

We first define a map $f : [0, 1] \to \mathbb{R}P^1$ by $x \mapsto p(\exp(\pi i x))$. It is clear that, as a composite of continuous maps, f is continuous. Note that, if $x \in (0, 1)$, then there is no $y \in [0,1]$ such that $-\exp(\pi i x) = \exp(\pi i y)$, so f is injective on $[0,1]$. Moreover, $f(0) = f(1)$. Consequently, f descends to an injective continuous map $f : S^1 \to \mathbb{R}P^1$. Since every element of $\mathbb{R}P^n$ has a representative in the upper half-circle, this map is surjective, and thus is a bijection.

Denote the inverse of f by f^{-1}. The map f^{-1} sends an element in the upper half-circle $x \in C$ to $q(\frac{\ln(x)}{\pi i})$. The assignment $x \mapsto \frac{\ln(x)}{\pi i}$ is a continuous assignment from the upper half-circle to $[0,1]$, and therefore, f^{-1} is continuous. $\square$

1.3 Hausdorff Spaces

Now that we have the basic tools necessary to construct topological spaces, we can begin exploring their properties. Of particular interest is the degree to which our concepts and intuitions from analysis carry over to topologies.

Definition 1.31. Let (X, τ_X) be a topological space. An *open cover* of X is a collection $\mathcal{U} := \{U_i\}_{I \in I} \subset \tau_X$ of open subsets of X such that $\bigcup_{i \in I} U_i = X$. We say that a cover $\mathcal{V}$ is a *subcover* of a cover $\mathcal{U}$ if $\mathcal{V} \subset \mathcal{U}$

Definition 1.32. Let (X, τ_X) be a topological space. We say that X is *compact* if every cover $\mathcal{U}$ of X admits a finite subcover $\mathcal{V}$.

Intuitively, compact sets should be thought of as playing the role of 'closed and sufficiently small' in topology. However, this intuition is significantly complicated by some pathological counterexamples. We do, however, have the following nice property.

Lemma 1.33. *Let (X, τ_X) be a compact topological space, and let $Y \subset X$ be a closed set. Then Y is compact.*

Proof. Let $\mathcal{U} := \{U_i\}_{i \in I}$ be an open cover of Y. Since Y is closed, the collection $\mathcal{V} := \mathcal{U} \cup \{X \setminus Y\}$ is an open cover of X, and therefore admits a finite subcover $U_1, \ldots, U_k, X \setminus Y$. Since $(X \setminus Y) \cap Y = \emptyset$, this means that $U_1, \ldots, U_k$ is a finite subcover of Y. $\square$

It is a classical theorem of analysis that, in a metric space (X, d), every compact subset is closed and bounded. In the case of $\mathbb{R}^n$ with the Euclidean metric, this can be strengthened to an 'if and only if' statement (the *Heine-Borel Theorem*). However, in topological spaces, things become rather stranger.

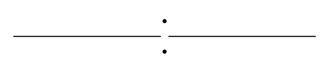
Example 1.34. Let X be a set, and $\tau_X := \{\emptyset, X\}$ be the indiscrete topology on X. Let $x \in X$. Then $\{x\}$ is clearly a compact subset of X (the only open covers are finite), however, if X has more than one point, then $\{x\}$ is not the complement of either X or $\emptyset$, and therefore cannot be closed.

To avoid this particular pathology, we need to impose some condition on our topological spaces to make them better match our intuition.

Definition 1.35. A topological space (X, τ_X) is called a *Hausdorff space* (or just *Hausdorff*) if, for every two distinct points $x, y \in X$, there exist open sets $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$. We say that a Hausdorff space *separates points.*

Example. Every metric space is Hausdorff.

Example (Non-example)**.** Let (Y, τ_Y) be the topological space $\mathbb{R} \times \{0,1\}$, where $\{0,1\}$ is equipped with the discrete topology. Note that Y can also be identified with $\mathbb{R} \amalg \mathbb{R}$. Define an equivalence relation on Y by $(x,0) \sim (x,1)$ for all $x \neq 0$. The quotient space $(Y_{/\sim}, \tau_\sim)$ is a standard counterexample in topology, called the *line with two origins*. Schematically, it looks like



If we label the copies of the origin 0_1 and 0_2, it is not hard to see that every open ball $B_r(0_1)$ of 0_1 must intersect every open ball $B_R(0_2)$, and thus that any open sets $0_1 \in V$ and $0_2 \in U$ with have non-empty intersection.

Corollary 1.36. *Let (X, τ_X) be a Hausdorff space. Then every compact subset of X is closed.*

Proof. Let $Y \subset X$ be a closed subset of X. We show the equivalent statement that $X \setminus Y$ is open. Fix a point $x \in X \setminus Y$; for each point $y \in Y$ choose open sets $y \in V_y$ and $x \in U_y$ such that $V_y \cap U_y = \emptyset$. The collection $\{V_y\}_{y \in Y}$ is an open cover of Y, and therefore admits a finite subcover $V_{y_1}, \ldots, V_{y_n}$. By construction the intersection $U_x := \bigcap_{i=1}^n U_{y_i}$ has empty intersection with $\bigcup_{i=1}^n (V_{y_i})$ and thus has empty intersection with Y. But, as a finite intersection of open sets, U_x is an open set, and since $x \in U_{y_i}$ for all $1 \leq i \leq n$, $x \in U_x$. Therefore, U_x is an open neighborhood of x in X, and $U_x \subset X \setminus Y$.

Construct such a U_x for every $x \in X \setminus$. Then $X \setminus Y = \bigcup_{x \in X \setminus Y} U_x$ is open. □

Proposition 1.37. *Let (X, τ_X) be a Hausdorff space, and $Y \subset X$ a subspace. Then Y is Hausdorff.*

Proof. Left as an exercise for the reader. □

Proposition 1.38. *Let $f : X \to Y$ be a continuous map of topological spaces such that X is compact. Then $f(X)$ is compact.*

Proof. Left as an exercise for the reader. □

1.4 Connectedness and path-connectedness

Above, we constructed the *disjoint union* of topological spaces, $X \amalg Y$, which can be viewed as consisting of two separate 'parts': X and Y. However, given a topological space (X, τ_X), we do not yet have any way of testing whether it has been built in this way. Such a criterion is provided by notions of *connectedness*.

Definition 1.39. Let (X, τ_X) be a topological space. If, for every pair of non-empty open sets $U, V \in \tau_X$ such that $U \cup V = X$, the intersection $U \cap V \neq \emptyset$, we say that X is connected. A *connected component* of X is a maximal connected subspace $Y \subset X$.

Lemma 1.40. *A topological space (X, τ_X) is connected if and only if it has a single connected component: X.*

Proof. Left as an exercise for the reader. □

In a sense made precise by the following proposition, connectedness measures 'discreteness of maps out of X'.

Proposition 1.41. *Let (X, τ_X) be a topological space. The following are equivalent*

1. *(X, τ_X) is connected.*

2. *Every continuous map $f : X \to Y$ to a discrete space is constant.*

Proof. We first show 2. $\Rightarrow$ 1. Suppose that (X, τ_X) is not connected. Then there are two non-empty sets $U, V \in \tau_X$ with $U \cup V = X$ and $U \cap V = \emptyset$. Define a map to $f : X \to \{0,1\}$ by sending every element of U to 0, and every element of V to 1. It is immediate from the definition that f is continuous with respect to the discrete topology on $\{0,1\}$ and non-constant.

We now show 1. $\Rightarrow$ 2. Suppose that there is a continuous, non-constant map $f : X \to Y$, where Y is equipped with the discrete topology. In particular, choose two distinct elements y_0 and y_1 in Y such that both are in the image of f. Choose any map of sets $p : Y \to \{0,1\}$ such that $p(y_0) = 0$ and $p(y_1) = 1$. Since this is continuous with respect to the discrete topologies, we get a non-constant continuous map $p \circ f : X \to \{0,1\}$. Since this is continuous, the sets $U := (p \circ f)^{-1}(0)$ and $V := (p \circ f)^{-1}(1)$ are open. Since $p \circ f$ is non-constant, both U and V are non-empty. By definition $U \cup V = X$ and $U \cap V \neq X$. □

Definition 1.42. Let (X, τ_X) be a topological space, and let $A \subset X$. We define the *closure* of A to be the subset $\overline{A} \subset X$ which is the intersection of all closed subsets of X which contain A.

Lemma 1.43. *Let $f : X \to Y$ be a continuous map, and let X be connected. Then $f(X) \subset Y$ is connected.*

Proof. Left as an exercise for the reader. □

Lemma 1.44. *Let (X, τ_X) be a topological space, and $A \subset X$. For every element $x \in \overline{A}$ and every open $x \in U$, $U \cap A \neq \emptyset$.*

Proof. Left as an exercise to the reader. □

Proposition 1.45. *Let (X, τ_X) be a topological space, and A a connected subset. If $B \subset X$ such that $A \subset B \subset \overline{A}$, then B is connected.*

Proof. Suppose that there were two open sets $U, V \subset X$ such that $U \cup V = B$ and $U \cap V \cap B = \emptyset$. Since A is connected, we must then have that $A \subset U$ or $A \subset V$. WLOG, assume $A \subset U$. But then, for $b \in B$, we have that $b \in \overline{A}$ and V is an open subset containing b. Therefore by lemma 1.44, $V \cap A \neq \emptyset$, and thus, $V \cap U \cap B \neq \emptyset$, which is a contradiction. □

So if connectedness measures the discreteness of maps *out* of X, can we also measure the discreteness of maps *into* X?

Definition 1.46. A *path* in a topological space (X, τ_X) is a continuous map $p : [0,1] \to X$, where $[0,1]$ is equipped with the subspace topology inherited from $\mathbb{R}$. We say that p is a *path from x to y* if $p(0) = x$ and $p(1) = y$.

We define an equivalence relation on X by $x \sim y$ if and only if there exists a path in X from x to y. A *path component of* x is an equivalence class $[x] \in X_{/\sim}$, viewed as a subspace $[x] \subset X$. We say that X is *path connected* if $X_{/\sim}$ is the one-point space.

Exercise 1.47. Show that $\sim$ is indeed an equivalence relation on X.

Exercise 1.48. Show that the interval $[0,1]$ is connected.

Proposition 1.49. *Let (X, τ_X) be a path-connected topological space. Then X is connected.*

Proof. Suppose X is not connected. Then there exists a continous map $f : X \to \{0,1\}$ (where $\{0,1\}$ is equipped with the discrete topology) such that f is non-constant. Let $x \in f^{-1}(0)$ and $y \in f^{-1}(1)$. A path $p : [0,1] \to X$ from x to y would then yield a continuous, non-constant map $f \circ p : [0,1] \to \{0,1\}$. Since $[0,1]$ is connected, this cannot occur, and thus, X is not path connected. $\square$

A standard counterexample in topology is the *topologist's sine curve*. Let $X \subset \mathbb{R}^2$ be the collection of all points $(x, \sin(\frac{1}{x}))$ for $x > 0$, together with the point $(0,0)$. This inherits a topology from $\mathbb{R}^2$ (indeed, this topology is even Hausdorff).

Lemma. The topologist's sine curve is connected.

Proof. We simply need note that the subspace $A := \{(x, \sin(\frac{1}{x})) \mid x > 0\}$ is path-connected, and thus connected. Moreover $A \subset X \subset \overline{A}$. Therefore, by Proposition 1.45, X is connected. $\square$

Lemma. The topologist's sine curve is not path connected.

Proof. Suppose we have a path $p : [0,1] \to X$ going from $(1, \sin(1))$ to $(0,0)$. Consider the component functions $p_x, p_y : [0,1] \to \mathbb{R}$. Since p_x is continuous, its image is connected, and therefore is the interval $[0,1]$. But then, p is the map $t \mapsto (t, \sin(\frac{1}{t}))$. But for every $\delta > 0$, there is a $0 < t < \delta$ such that $\sin(\frac{1}{t}) = 1$, i.e $|p(t) - (0,0)| > 1$. Therefore, p cannot be continuous. $\square$

Warning 1.50. The converse of Proposition 1.49 is *not* true. There are connected spaces which are not path-connected. We need to make additional assumptions about our space X if we want connectedness and path-connectedness to be equivalent.

Definition 1.51. We call a topological space (X, τ_X) *locally path-connected* if, for every $x \in X$ and every open U containing x, there is an open V with $x \in V \subset U$ such that V is path connected.

Proposition 1.52. *If X is a connected, locally path-connected topological space, then X is path-connected.*

Proof. Left as an exercise for the reader. $\square$

1.5 *Homeomorphism, homotopy, and a first look at the fundamental group*

As we saw with $\mathbb{R}P^1$ and S^1, it is often not too hard to explicitly write down a homeomorphism between two spaces. However, to be able to make meaningful statements about topological spaces, it is necessary for us to be able to say when two spaces are *not* the same (i.e. homeomorphic). At first blush this may seem easy. After all, it should be obvious that two spaces are different. Once one starts looking at an example, however, it is not at all clear how one should go about distinguishing two spaces.

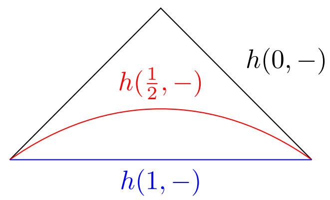
As an example, consider $\mathbb{R}^2$, equipped with the topology induced by the Euclidean metric. and $\mathbb{R}^2 \setminus \{0\}$, equipped with the subspace topology. By inspection, it should be fairly intuitive that these are not homeomorphic spaces, but how do we prove it? The two underlying sets have the same cardinality, and it's not possible to write down every possible continuous map between $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0\}$. So we seem to be stuck.

Paradoxically, the answer comes by considering an even weaker notion of equivalence: *homotopy equivalence*. In loose, intuitive terms, two spaces are homotopy equivalent if one can be 'stretched' or 'shrunk' into another.

Definition 1.53. Let X and Y be topological spaces, and $f, g : X \to Y$ continuous maps. A *homotopy* from f to g is a continuous map $h : [0,1] \times X \to Y$ (where $[0,1]$

is equipped with the subspace topology inherited from the Euclidean topology on $\mathbb{R}$) such that $h(0,-): X \to Y$ is the map f and $h(1,-): X \to Y$ is the map g. If there is a homotopy from f to g, we say that f and g are *homotopic*.

Example 1.54. The image below shows a homotopy $h : [0,1] \times [0,1] \to \mathbb{R}^2$ between paths $[0,1] \to \mathbb{R}^2$.



The idea is that we continuously morph one path into another.

Lemma 1.55. *Let $[a,b] \subset \mathbb{R}$ be an interval equipped with the subspace topology, and let $f, g : X \to Y$ be continuous maps of topological spaces. The following are equivalent:*

1. *There is a homotopy h from f to g.*
2. *There is a map $H : [a,b] \times X \to Y$ such that $H(a,-) = f$ and $H(b,-) = g$.*

Proof. It is immediate that 1. ⇒ 2. Suppose that 2. holds, and we have such a map H. Define a map

$$q : [0,1] \times X \to [a,b] \times X; \quad (t,x) \mapsto (\rho(t), x)$$

where

$$\rho(t) := a + t(b-a)$$

It is easy to verify that q is continuous, and therefore $H \circ q : [0,1] \times X \to Y$ is a continous map. However, by definition $(H \circ q)(0,-) = H(a,-) = f$ and $(H \circ q)(1,-) = H(b,-) = g$. Thus $H \circ q$ is a homotopy from f to g. □

Lemma 1.56. *Denote by $C^0(X,Y)$ the set of continuous maps between two topological spaces X and Y. Then the relation*

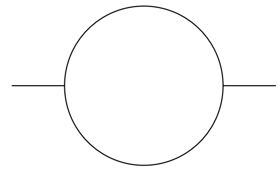
$$f \sim g \Leftrightarrow f \text{ is homotopic to } g$$

is an equivalence relation.

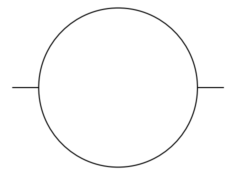
Proof. First, we show reflexivity. Let $f : X \to Y$ be a continuous map. Since the projection $p_2 : [0,1] \times X \to X$ is continuous, the composite $f \circ p_2$ is as well, and provides a homotopy from f to f.[7]

Second, we show symmetry. Let $h : [0,1] \times X \to Y$ be a homotopy. Define $p : [0,1] \to [0,1]$ to send $t \mapsto 1-t$. This is a homeomorphism (as one can easily verify), and exchanges 0 and 1. Therefore the map $\tilde{h} : [0,1] \times X \to Y$ given by $\tilde{h}(t,x) = h(p(t),x)$ is a homotopy from g to f.

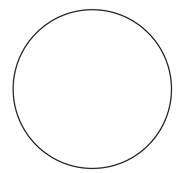
A heuristic depiction of homotopy equivalence might be the following. We consider the space:



This sort of looks like the circle, but they are not homeomorphic. If we remove one of the intersections of the line and the circle, then we get a space with three different path components, but if we remove any single point from the circle, we get a space with only one path component. However, If we are allowed to shrink without tearing our space, we can shrink it to



and then to



yielding the circle. The aim of this section will be to prove rigorous results about this kind of a process.

[7] This is sometimes referred to as the *constant homotopy*.

Finally, supose that h is a homotopy from f to g, and k is a homotopy from g to ℓ. We define a map

$$k * h : [0,2] \times X \to Y$$

via

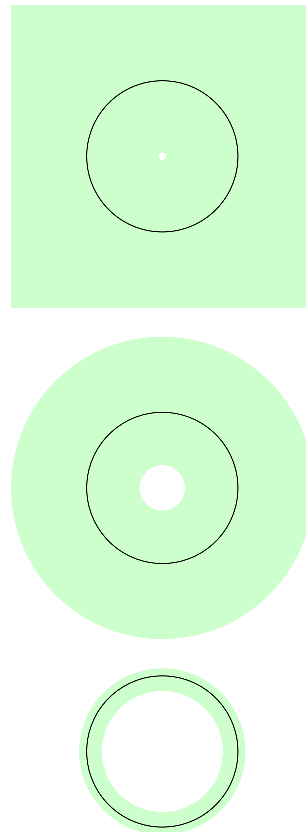
$$(k * h)(t,x) = \begin{cases} h(t,x) & 0 \leq t \leq 1 \\ k(t-1,x) & 1 \leq t \leq 2 \end{cases}$$

It is straightforward to verify that this is well-defined and continuous, and therefore, by Lemma 1.55, f is homotopic to ℓ. $\square$

This now allows us to define our notion of homotopy equivalence:

Definition 1.57. Two continous maps $f : X \to Y$ and $g : Y \to X$ are said to be *homotopy inverses* if $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. In this situation, we call f (or g) a *homotopy equivalence*, and say that X and Y are *homotopy equivalent*.

Remark 1.58. It is immediate from the definitions that every homeomorphism is a homotopy equivalence.



Example 1.59. Consider $X := \mathbb{R}^2 \setminus \{0\}$ and S^1, both with the subspace topology inherited from $\mathbb{R}^2$. There is a canonical inclusion $\iota : S^1 \to X$. We now claim that ι is a homotopy equivalence. Define a map

$$r : X \to S^1; \quad x \mapsto \frac{x}{|x|}$$

Since $x \neq 0$, this is well-defined, and it is easy to check that it is continuous. We note that $r \circ \iota : S^1 \to S^1$ is *equal to* the identity on S^1.

In the other direction, we wish to define a homotopy between $\iota \circ r$ and id_X. Define a continuous map

$$H : [0,1] \times X \to X; \quad (t,x) \mapsto \frac{x}{|x|^t}$$

Note $H(0,x) = \frac{x}{|x|^0} = x$, so $H(0,-) = \mathrm{id}_X$, and that $H(1,x) = \frac{x}{|x|^1} = \frac{x}{|x|} = (\iota \circ r)(x)$. Thus, H is a homotopy from id_X to $\iota \circ R$, and ι is a homotopy equivalence.

Examples 1.60. The following examples are quite straightforward, and you should attempt to verify for yourself that they hold:

1. $\mathbb{R}^n$ is homotopy equivalent to the one-point topological space $*$.

2. The Möbius band is homotopy equivalent to S^1.

Definition 1.61. If, as in Example 1.60 (1), a space X is homotopy equivalent to the one-point topological space $*$, we will call X *contractible*.

Lemma 1.62. *Let $h : [0,1] \times X \to Y$ be a homotopy from f to g, and let $p : Y \to Z$ be a continuous map. Then $p \circ h$ is a homotopy from $p \circ f$ to $p \circ g$.*

Proof. Immediate from the definitions. $\square$

So, how can we use homotopies and homotopy equivalences to show that $\mathbb{R}^2 \setminus \{0\}$ is not homeomorphic to $\mathbb{R}^2$? The answer lies in a speciallized invariant: the fundamental group.

Definition 1.63. We call a continuous map $f : [a, b] \to X$ a *loop in* X with *basepoint* $x \in X$ if $f(a) = f(b) = x$. Denote the set of loops in X with basepoint x by $L(X, x)$.

We say that two loops $f, g : [a, b] \to X$ with basepoint x are *based-homotopic* if there is a homotopy $h : [0, 1] \times [a, b] \to X$ from f to g such that $h(s, a) = h(s, b) = x$ for all $s \in [0, 1]$.

We say that two loops $f : [a, b] \to X$ and $g : [c, d] \to X$ are *equivalent* if there is a homeomorphism $p : [a, b] \to [c, d]$ with $p(a) = c$ and $p(b) = d$ such that f is based-homotopic to $g \circ p$. We write $f \simeq g$ is f and g are equivalent loops.

Exercise 1.64. Show that equivalence of loops is an equivalence relation.

Definition 1.65. Let X be a topological space, and $x \in X$ a basepoint. We define a subset $L^1(X, x) \subset L(X, x)$, consisting of those loops in X with basepoint x which are defined on the unit interval $[0, 1]$. We call such loops *unit loops* in X with basepoint x. We will write $f \sim g$ if the unit loops f and g are based-homotopic.

Proposition 1.66. *For any topological space X with basepoint x, the inclusion $L^1(X, x) \hookrightarrow L(X, x)$ induces a bijection*

$$L(X, x)_{/\simeq} \cong L^1(X, x)_{/\sim}.$$

Before we can prove this result, we need the following lemma.

Lemma 1.67. *Let $f : [0, 1] \to [0, 1]$ be a homeomorphism preserving* 0 *and* 1*. Then there is a homotopy h from f to* id_x *such that $h(t, 0) = 0$ and $h(t, 1) = 1$ for all $t \in [0, 1]$*

Proof. We define a continuous map

$$h : [0, 1] \times [0, 1] \to [0, 1], \quad (s, t) \mapsto sf(t) + (1 - s)g(t).$$

This is well defined since, for all $(s, t) \in [0, 1]^2$, we have

$$sf(t) + (1 - s)g(t) \geq 0 \cdot 0 + 0 \cdot 0 = 0$$

and

$$sf(t) + (1 - s)g(t) \leq 1 \cdot 1 + 1 \cdot 1 = 1.$$

It is immediate from the definitions that $h(0, -) = g$ and $h(1, -) = f$. Moreover,

$$h(s, 0) = sf(0) + (1 - s)g(0) = 0 + 0 = 0$$

and

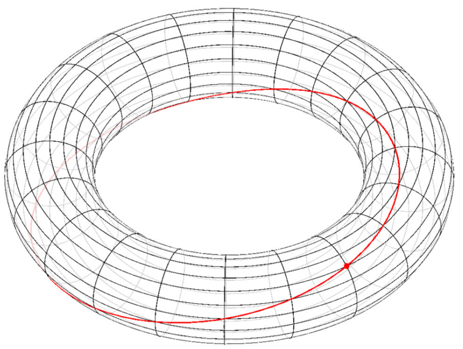
$$h(s, 1) = sf(0) + (1 - s)g(0) = s + (1 - s) = 1$$

proving the lemma. □

A loop in X is pretty easy to visualize, since it matches our intuition precisely. Let's consider the example of the torus $S^1 \times S^1$. We can define a loop in $S^1 \times S^1$ by

$$\begin{aligned} [0, 1] &\longrightarrow S^1 \times S^1 \\ t &\longmapsto (e^{2\pi i t}, e^{2\pi i t}) \end{aligned}$$

If we draw this, we get something like:



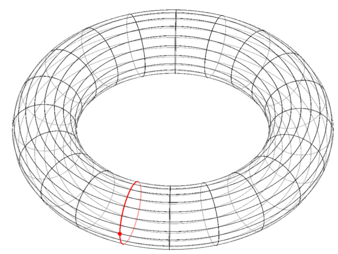
Where the path is drawn in red, and the basepoint is represented by a red dot.

Proof of Proposition 1.66. The proof of Lemma 1.55 can be used to show that every loop in X with basepoint x is equivalent to a unit loop in X with basepoint x. Therefore, it suffices to show that two unit loops are homotopic if and only if they are equivalent. By definition, if $f \sim g$ is a homotopy, then $f \simeq g$, so it suffices to show that any two equivalent unit loops are homotopic.

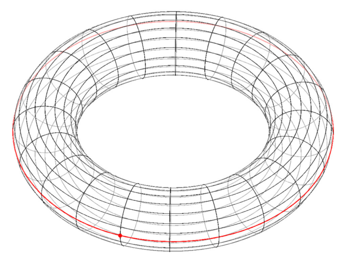
Suppose $f \simeq g$. By definition this means that there is a homeomorphism $p : [0,1] \to [0,1]$ which preserves 0 and 1, such that $f \circ p \sim g$. However, by Lemma 1.67, we have a homotopy h from $\mathrm{id}_{[0,1]}$ to p which respects endpoints. Composing h with f thus yields a based homotopy $f = f \circ \mathrm{id}_{[0,1]}$ to $f \circ p$. Thus, there is a based homotopy between f and g. □

We can define additional structure on $L^1(X,x)_{/\sim}$. In fact, by tracing through loops one after another, we can define a group structure on $L^1(X,x)_{/\sim}$:

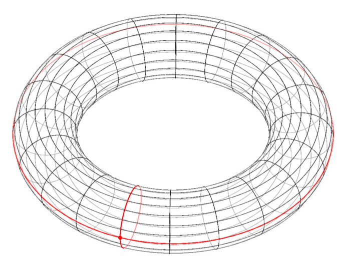
The concatenation of paths is straightforward, but can be a bit tricky to draw. Lets visualize two paths in the torus $S^1 \times S^1$. The first we call α:



and the second, β:



The concatenation $\beta * \alpha$ is a path that looks like



where we first trace through α, and then β.

Construction 1.68. Given two loops $\alpha : [a,b] \to X$ and $\beta : [c,d] \to X$ with basepoint $x \in X$, we define the *concatenation* of α and β to be the based path

$$\beta * \alpha : [a, b + (d - c)] \to X$$

given by

$$(\beta * \alpha)(t) = \begin{cases} \alpha(t) & t \in [a,b] \\ \beta(t - b + c) & t \in [b, b + (d - c)] \end{cases}$$

Exercise 1.69. Show that $\alpha * \beta$ yields a well-defined map on equivalence classes

$$L(X,x)_{/\simeq} \times L(X,x)_{/\simeq} \to L(X,x)_{/\simeq}.$$

Given two unit loops α and β in X, find a unit loop representing the equivalence class of $\beta * \alpha$.

Proposition 1.70. *The binary operation*

$$* : L(X,x)_{/\simeq} \times L(X,x)_{/\simeq} \to L(X,x)_{/\simeq}$$

defines a group structure on $L(X,x)_{/\simeq}$.

Proof. It is immediate from the definitions that $*$ is associative, so we need only define a unit element and inverses.

Let $e_x : [0,1] \to X$ be the constant loop at the basepoint x, i.e. $e_x(t) = x$ for all $t \in [0,1]$. Let $\alpha : [a,b] \to X$ be a loop with basepoint x. We claim that $e_x * \alpha : [a, b+1] \to X$ is equivalent to α.

Define a endpoint-preserving homeomorphism $p : [a, b+1] \to [a,b]$ via

$$p(t) = \frac{b-a}{b+1-a} t + \frac{a}{b+1-a}.$$

And a continuous map $q : [a, b+1] \to [a,b]$ via

$$q(t) = \begin{cases} t & t \in [a,b] \\ b & t \in [b, b+1] \end{cases}$$

Note that $\alpha \circ q = e_x * \alpha$. The construction of Lemma 1.67 can be used to define an endpoint-preserving homotopy $q \sim p$, which then gives rise to a based homotopy $\alpha \circ q \sim \alpha \circ p$. This shows $\alpha \simeq e_x * \alpha$, so e_x is a left unit for $*$. The proof that e_x is a left unit is totally analogous.

We now need only show that every path α has an inverse up to homotopy. Since every equivalence class can be represented by a unit loop, we may assume without loss of generality that $\alpha : [0,1] \to X$ is a based unit loop. Define $p : [0,1] \to [0,1]$ by $p(t) = 1-t$, and set $\alpha^{-1} = \alpha \circ p$. We define a based homotopy $h : [0,1] \times [0,2] \to X$ from $\alpha * \alpha^{-1}$ to the constant loop $e_x^2 : [0,2] \to X$ as follows:

$$h(s,t) = \begin{cases} (\alpha * \alpha^{-1})(t - st) & 0 \leq t \leq 1 \\ (\alpha * \alpha^{-1})((2-t)s + t) & 1 \leq t \leq 2 \end{cases}$$

We leave it to the reader to verify that this provides the desired homotopy, and to check the analogous case of $\alpha^{-1} * \alpha$. □

Proposition 1.71. *Let $f : X \to Y$ be a continuous map of topological spaces such that $f(x) = y$. Then composing with f defines a group homomorphism*

$$f_* : L(X,x)_{/\simeq} \to L(Y,y)_{/\simeq}$$

Proof. There are two things to check: First, that composition defines a well-defined map on equivalence classes, and second, that this map preserves the group structure.

The map in question sends a loop $\ell : [a,b] \to X$ based at x to the loop $f \circ \ell : [a,b] \to X$. Since reparameterization by a homeomorphism operates on the interval $[a,b]$, it will suffice to show that this map sends based homotopy class to based homotopy classes. This follows (with some extra care paid to the basepoint) from Lemma 1.62.

To see that the map preserves the group structure, we first note that $f \circ e_x$ is clearly e_y. For two loops $\beta : [a,b] \to X$ and $\alpha : [c,d] \to X$ based at x, we have

$$f \circ (\beta * \alpha)(t) = \begin{cases} f \circ \alpha(t) & t \in [a,b] \\ f \circ \beta(t - b + c) & t \in [b, b + (d-c)] \end{cases} = ((f \circ \beta) * (f \circ \alpha))(t)$$

completing the proof. □

Remark 1.72. Note that, while we have worked with $L(X,x)$ the propositions above hold true for $L^1(X,x)$ via the canonical isomorphism $L^1(X,x)_{/\sim} \cong L(X,x)_{/\simeq}$.

Definition 1.73. The set $L(X,x)_{/\simeq}$ together with the group structure constructed above is denoted by $\pi_1(X,x)$, and is called *the fundamental group* of X at x.

We will now sketch how the fundamental group can be used to distinguish spaces. We will not prove our claims here, deferring them to later chapters. Instead, we will try to give the idea of how the fundamental group might be used.

Claim 1.74. Suppose that $f : X \to Y$ is a homotopy equivalence with $f(x) = y$ then $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism of groups.

Claim 1.75. For $x, x' \in X$ in the same path component, there is an isomorphism $\pi_1(X, x) \cong \pi_1(X, x')$.

This tells us something very important, namely that if two spaces have different fundamental groups, they cannot be homotopy equivalent, and thus cannot be homeomorphic. We now have almost everything we need to distinguish $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0\}$.

Claim 1.76. We have

$$\pi_1(\mathbb{R}^2, 1) = \{e\}$$

and

$$\pi_1(\mathbb{R}^2 \setminus \{0, \}, 1) = \pi_1(S^1, 1) = \mathbb{Z}$$

Consequently, $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0\}$ cannot be homeomorphic.

Remark 1.77. In general, if a space X is contractible (i.e. homotopy equivalent to the one-point space), then X is path-connected, and so Claim 1.74 implies that $\pi_1(X, x) \cong \pi_1(*, *) \cong \{e\}$ is a trivial group for any basepoint $x \in X$.