PERSPECTIVES ON THE 2-SEGAL CONDITIONS

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Abstract. In this article we explain and motivate the 2-Segal conditions, focusing on the underlying intuition. We first discuss higher associativity, and show how contemplating the associativity of partially- and multiply-defined multiplications or compositions leads naturally to the 2-Segal conditions. We then explain how these conditions may be reinterpreted geometrically in terms of a kind of state sum. We make both of these perspectives precise, and indicate some of the many directions in which they may be generalized. Throughout, we attempt to emphasize the elementary nature of these intuitions, focusing on simplicial sets, 1-categories, and pictorial and diagrammatic arguments wherever possible.

1. Background, or what to expect from this paper

What we here term the *2-Segal conditions* were, in fact, independently arrived at by two different groups: by Dyckerhoff and Kapranov in [8] using the name "2-Segal spaces"; and by Galvez-Carrillo, Kock, and Tonks in [10] under the name "decomposition spaces". As indicated by the choice of terminology, this article is primarily focused on the perspectives which arise in the first of these sources, leaving discussions of the second to papers later in this volume (in particular, [15] will focus on exploring the perspective of [10]).

In their greatest generality, the 2-Segal conditions pertain to simplicial objects

$$X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$$

with values in an ∞-category $\mathcal{C}$. However, the technicalities necessitated by this (and by the closely-related model-categorical perspective) tend to greatly obscure the intuitions involved. As such, we will focus exclusively on the case in which $\mathcal{C}$ is a 1-category. Indeed, throughout this article, we will limit ourselves to *simplicial sets*.

In very broad strokes, there are two directions one can take to understand the 2-Segal conditions. On the one hand, the 2-Segal conditions can be arrived at by taking the *spines* — 1-dimensional simplicial subsets of Δ^n — which give rise to the Segal conditions, and replacing them with 2-dimensional simplicial subsets corresponding to triangulations of polygons. On the other hand, the 2-Segal conditions can be arrived at as the minimal set of conditions necessary to require the associativity of a not-necessarily-well-defined operation. We will refer to the former intuition as *the geometric perspective* and the latter as *the algebraic perspective*.

These two perspectives are closely linked. The algebraic perspective focuses on associativity, which, in turn, can be viewed as the requirement that all n-ary composites of a binary multiplication corresponding to rooted binary trees are equal. However, rooted binary trees are precisely dual to triangulations of polygons with ordered vertex sets, allowing for an easy graphical connection between these two conditions.

In this article, we begin with an intuitive exploration of the algebraic perspective, and situate the associativity guaranteed by the 2-Segal conditions in the context of nerves of categories. This leads to the interpretation of the 2-Segal conditions as a relaxation of the 1-Segal conditions which forgets that composition operation in the nerve of a category must be a map of sets, but remembers that it must be associative. From there we explore the geometric perspective, explaining the connection between associativity, trees, and polygonal subdivisions.

The last two sections are devoted to making these perspectives rigorous. In section 4, we define membrane sets and rigorously formulate the geometric version of the 2-Segal conditions. We then show that it is equivalent to the original definition given in section 2. After a brief digression on the coskeletalness of 2-Segal simplicial sets, we then turn to the algebraic perspective in section 5. There, we prove that 2-Segal objects are equivalent to a particular coherently associative structure — algebras in the bicategory of spans of sets.

While this article is intended to act as a gentle introduction to the 2-Segal conditions, assuming some technical background is inevitable. In particular, the entire article will assume some familiarity with basic 1-category theory, the simplex category Δ, and the basics of simplicial sets. This is the only technical background truly required to read §2, §3, and §4. Occasional remarks and footnotes may point to higher categorical intuitions or the associated literature, but these can be safely ignored should the reader so wish. The one exception to this principle is section 4.1, which requires some knowledge of truncation and coskeleton functors. Textbook treatments of the coskeleton can be found in [13, §VII.1], [24, Section 0AMA], and [20, Tag 051Z].

The final section, §5, amounts a recapitulation, in a drastically simplified setting, of one of the main results of [25]. While it is, in principle, possible to read §5 without knowledge beyond what is listed above, the meanings of all of the constructions and definitions in the section will be somewhat opaque to a reader without some understanding of 2-categories and bicategories.

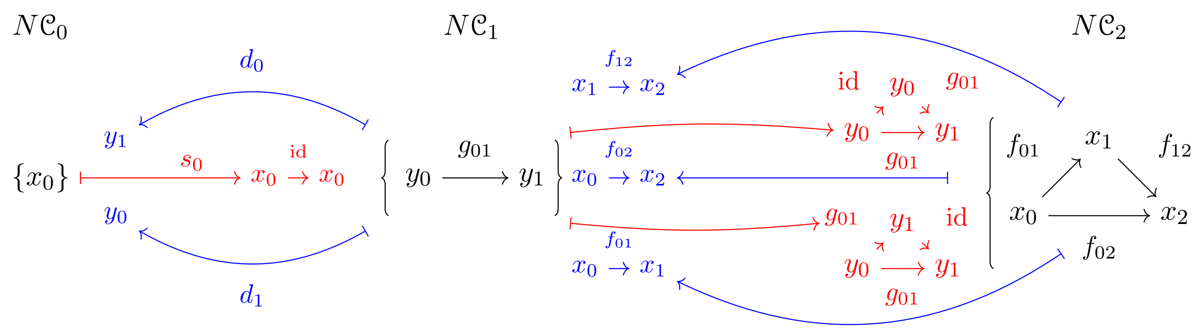
2. The algebraic perspective: (higher) associativity

The first perspective which will lead us to the 2-Segal conditions is the study of associativity. In particular, the consideration of operations which are not well-defined functions (i.e., which may be partially defined, multiply defined, or both) will lead inexorably to the 2-Segal conditions as an avatar of associativity. Before reaching for this high level of generality, however, let us take a step back, and look at associativity in a more familiar context: that of the composition in a category.

Given a category $\mathcal{C}$, we may define its *nerve* to be the simplicial set

$$N(\mathcal{C}) : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

which sends the ordinal $[n]$ to the set $\mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C})$. Conventionally, we picture the first few sets in the nerve of a category as follows



In this section, we will attempt to untangle the reasons for which the nerve of a category comes equipped with an *associative* operation.

However, any contemplation of why the nerve of a category comes equipped with a associative composition must first address the question of why the nerve of a category comes equipped with any kind of composition at all. The starting point in our answer to this question is the well-known Nerve Theorem.[1]

Theorem 2.1. *The nerve functor*

$$N : \mathsf{Cat} \longrightarrow \mathsf{Set}_\Delta$$

is fully faithful. A simplicial set X is isomorphic to the nerve of a category if and only if, for every $n \geq 2$, the map

$$X_n \longrightarrow \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{\times n}$$

which sends a simplex to its spine are bijections.

A variant of this theorem first appeared in [14, Prop. 4.1], as part of Grothendieck's studies in descent theory. We will henceforth follow the convention that we call the maps

$$X_n \longrightarrow \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{\times n}$$

the *1-Segal*[2] *maps* and the condition that the 1-Segal maps be bijections as the *1-Segal conditions*. We will call a simplicial set which satisfy the 1-Segal conditions a *Segal simplicial set*. The reason for this naming convention is the use, by Graeme Segal, of the conditions which now bear his name in [23].

To understand the 1-Segal conditions, we must first specify what the 1-Segal maps are, and unpack what their relation to the defining features of categories are. Let us start by describing

[1]Not to be confused with another celebrated theorem called the Nerve Theorem, which relates to the realization of Čech nerves of open covers. For a version of that nerve theorem, see, e.g., [23, Proposition 4.1].

[2]We amend the convention that these be called simply the *Segal maps* because we will, by necessity, need to discuss *2-Segal* maps later in this work.

the *spine* of a standard n-simplex. We define a 1-dimensional simplicial subset $\mathrm{Sp}(\Delta^n) \subset \Delta^n$ to consist of those simplices $[k] \to [n]$ which factor through one of the subsets $\{i, i+1\} \subset [n]$ for $0 \le i \le n-1$. More intuitively, this simplicial set consists of precisely n 1-simplices, glued end to end, i.e.

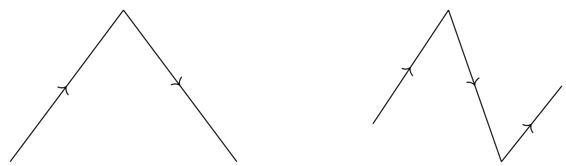
$$\mathrm{Sp}(\Delta^n) \cong \Delta^1 \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^1.$$

To make more explicit the inclusion of the spine into Δ^n, we can write

$$\mathrm{Sp}(\Delta^n) \cong \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}.$$

implicitly identifying the symbol Δ^I, where I is an ordinal isomorphic to $[m]$, with Δ^m. This notational convention will be of great use to us later.

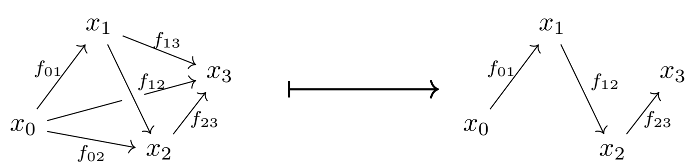
The spines of the 2- and 3-simplices can be easily pictured.



The 1-Segal maps are then given by restricting a simplex $\sigma \in X_n$ to its spine. More formally, they are the maps

$$X_n \cong \mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, X) \longrightarrow \mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, X) \cong \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{\times n}.$$

For instance, on a 3-simplex in $N\mathcal{C}$, the 1-Segal map acts as



To understand the nerve theorem, we must unpack how a category arises from a 1-Segal simplicial set X. We can easily see that to identify X with the nerve of a category $\mathcal{C}$, we must define the sets of objects and morphisms in $\mathcal{C}$ to be $\mathrm{Ob}(\mathcal{C}) = X_0$ and $\mathrm{Mor}(\mathcal{C}) = X_1$, respectively. Similarly, the source and target maps from $\mathrm{Mor}(\mathcal{C})$ to $\mathrm{Ob}(\mathcal{C})$ must be identified with d_1 and d_0, respectively, and the identity morphism on an object $x \in X_0$ must be given by $s_0(x)$. However, two important questions remain.

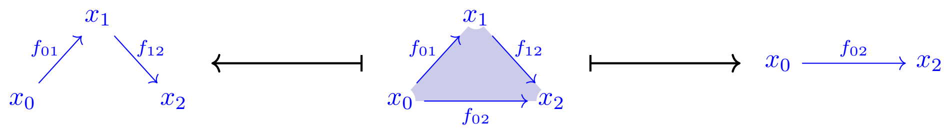
Firstly: How does composition arise from the 1-Segal conditions? Given two "morphisms" $f, g \in X_1$, they are composable precisely when the source of f matches the target of g, i.e., when $d_1(f) = d_0(g)$. Thus, precisely when they define an element $(g, f) \in X_1 \times_{X_0} X_1$. We then obtain a *span* of sets

$$X_1 \times_{X_0} X_1 \xleftarrow{(d_2,d_0)} X_2 \xrightarrow{d_1} X_1.$$

The 1-Segal conditions require that the leftward arrow is a bijection, and so, inverting it, we get a well-defined composition map

$$X_1 \times_{X_0} X_1 \xleftarrow[\cong]{(d_2,d_0)} X_2 \xrightarrow{d_1} X_1.$$

We can also interpret the span which defines the composition pictorially:



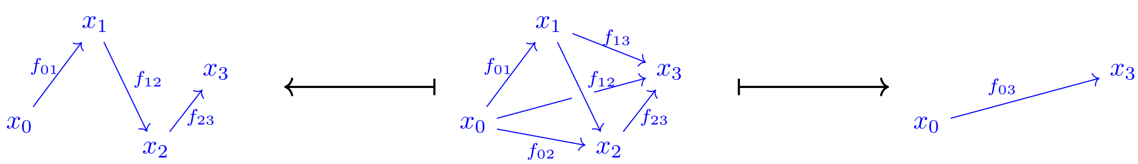
$$X_1 \times_{X_0} X_1 \xleftarrow{\quad (d_2, d_0) \quad} X_2 \xrightarrow{\quad d_1 \quad} X_1$$

More informally, we complete a pair of composable morphisms (the spine of a 2-simplex) to a unique 2-simplex, and then forget everything about that 2-simplex except the unique maximal edge.

More generally, the 1-Segal conditions provide n-fold composition maps[3]

$$\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{\times n} \longrightarrow X_1$$

defined by completing a spine to the corresponding n-simplex, and then forgetting everything except the maximal edge. In dimension 3, this looks like



Now that we understand how composition maps arise from the 1-Segal conditions, let us try to untangle associativity. The simplest expression of the associativity condition is that, writing κ for the composition map

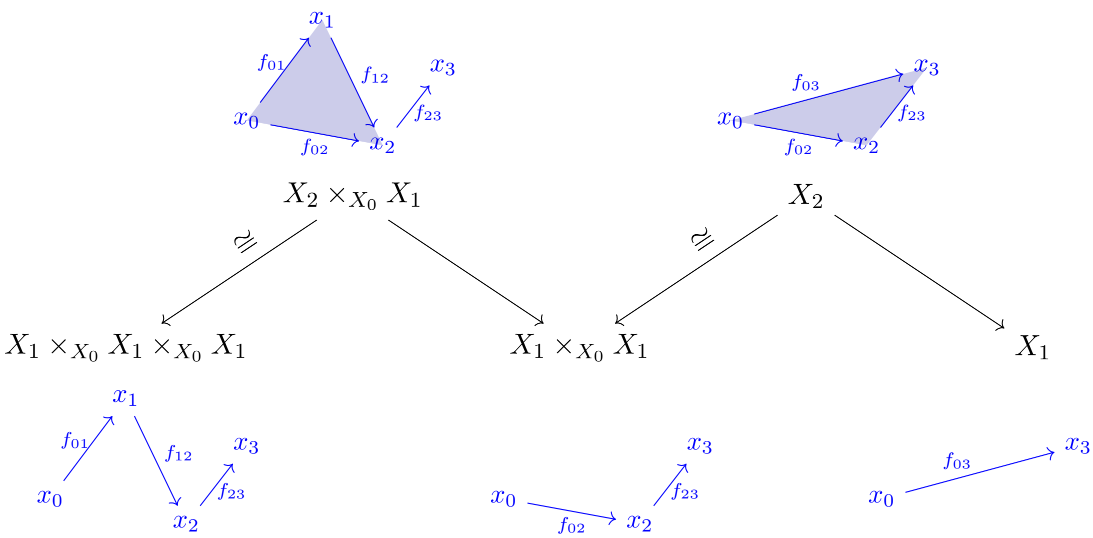
$$\kappa : X_1 \times_{X_0} X_1 \longrightarrow X_1,$$

we require

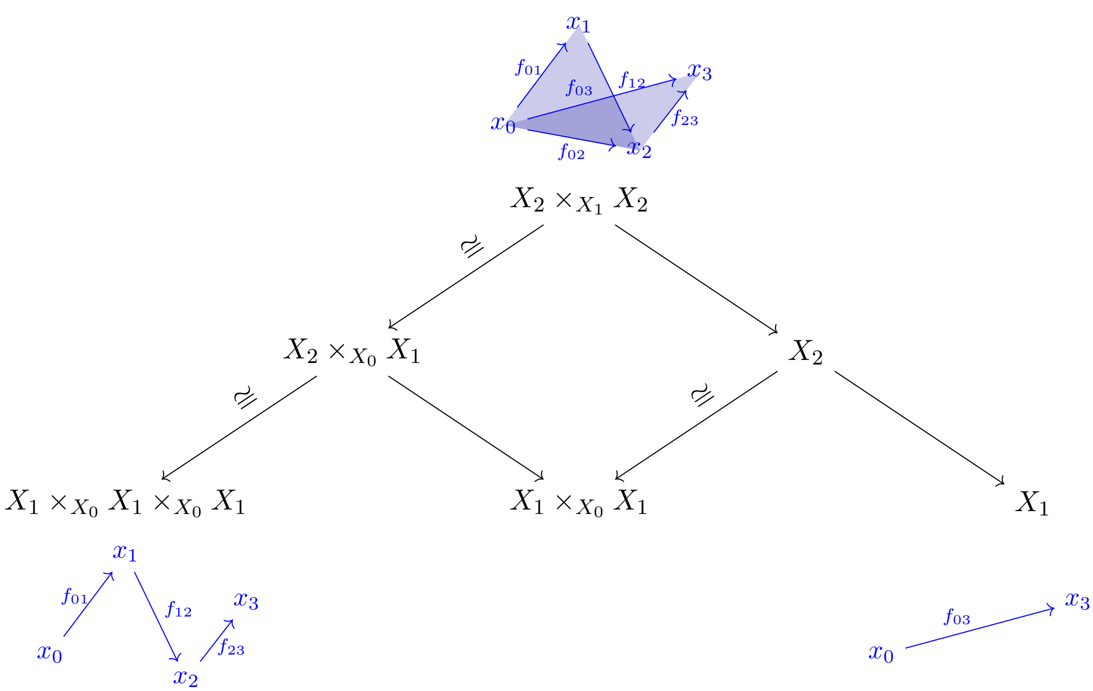
$$\kappa \circ (\kappa \times_{X_0} \mathrm{id}) = \kappa \circ (\mathrm{id} \times_{X_0} \kappa). \tag{1}$$

Since each copy of κ is defined by a span of sets, let's draw what happens in the composition $\kappa \circ (\kappa \times_{X_0} \mathrm{id})$.

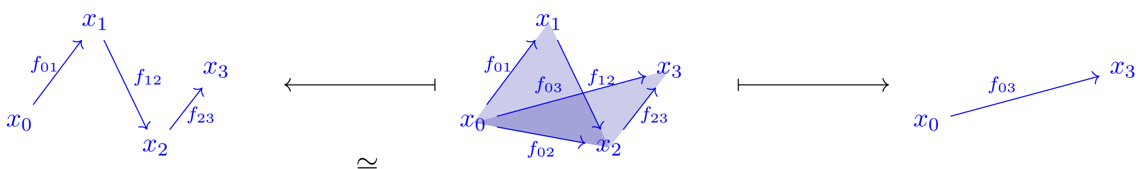
[3]In homotopical algebra and higher category theory, these higher compositions allow 1-Segal *spaces* to encode $(\infty, 1)$-categories and A_∞-algebras. This perspective will reappear later on in the present work, as the 2-Segal conditions relate to *coherent* associativity, even in the Set-valued setting.



As we examine this picture, we might notice something interesting: we could skip the middle "forgetting" step. Instead, we could first complete the 2-simplex $x_0 \to x_1 \to x_2$, then complete the 2-simplex $x_0 \to x_2 \to x_3$, and then forget anything except the maximal edge. Categorically, we can interpret this by replacing the pair of spans above with a single span given by the pullback of the three sets in the middle of the diagram. That is,

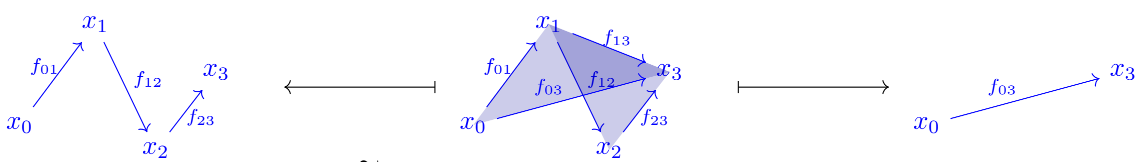


Repeating this procedure for the composite $\kappa \circ (\mathrm{id} \times_{X_0} \kappa)$, we obtain two spans, representing the two composites in our associativity conditions. These are



$$X_1 \times_{X_0} X_1 \times_{X_0} X_1 \xleftarrow{\cong} X_2 {}^{d_1}\!\times^{d_2}_{X_1} X_2 \longrightarrow X_1$$

for $\kappa \circ (\kappa \times_{X_0} \mathrm{id})$, and

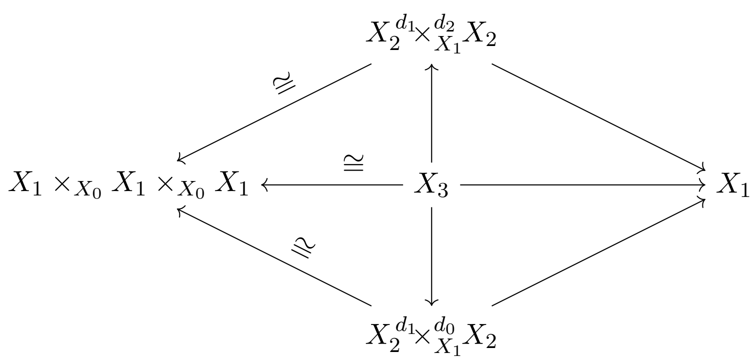


$$X_1 \times_{X_0} X_1 \times_{X_0} X_1 \xleftarrow{\cong} X_2 {}^{d_1}\!\times^{d_0}_{X_1} X_2 \longrightarrow X_1$$

for $\kappa \circ (\mathrm{id} \times_{X_0} \kappa)$.

So why do the 1-Segal conditions guarantee that the maps defined by these spans equal? Simply put, the roofs of the two spans we have computed are isomorphic, in a way compatible with the maps defining the spans. That is, each of the "tacos"[4] pictured above can be uniquely completed to a full 3-simplex of X.

More precisely, each of these spans is isomorphic to the span defining the 3-fold composition, and the diagram



commutes. Thus, this associativity diagram is equivalent to requiring that both legs of the span

$$X_2 {}^{d_1}\!\times^{d_2}_{X_1} X_2 \longleftarrow X_3 \longrightarrow X_2 {}^{d_1}\!\times^{d_0}_{X_1} X_2$$

[4]A wonderfully evocative term introduced by Contreras, Keller, and Mehta in [3].

are isomorphisms. Equivalently, this amounts to requiring that the squares

$$\begin{array}{ccc} X_3 & \xrightarrow{d_3} & X_2 \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ X_2 & \xrightarrow[d_2]{} & X_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} X_3 & \xrightarrow{d_0} & X_2 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_1} \\ X_2 & \xrightarrow[d_0]{} & X_1 \end{array} \tag{2}$$

are pullback squares.

We now, after a half-dozen pages of exploration, are ready to give a slogan explaining what we want the 2-Segal conditions to be.

SLOGAN: A 2-Segal set $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ is a set X_1 equipped with a composition operation which *may not be well-defined*[5], but *is* associative.

We thus remove the 1-Segal condition entirely, and *define* our n-fold compositions to *be* the spans

$$\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{\times n} \longleftarrow X_n \longrightarrow X_1.$$

We then require that the two squares of (2) are pullback.

However, this is not quite sufficient, for a rather subtle reason. If we write κ_n for the n-fold composition operation, then combining the 1-Segal conditions with the pullback squares of (2) implies that the relations

$$\kappa_m \circ (\mathrm{id} \times \kappa_n \times \mathrm{id}) \cong \kappa_{n+m-1}. \tag{3}$$

should hold no matter which input of the m-fold composition we plug the n-fold composition into.[6]

If we forget the 1-Segal conditions, however, the pullback squares (2) no longer suffice to prove these additional associativity conditions. We must, therefore, impose explicit associativity conditions for each such equation (3).

If we unwind the conditions exactly as we did above, each equation (3) yields a square

$$\begin{array}{ccc} X_{n+m-1} & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & X_1 \end{array}$$

which we must require to be pullback. We thus obtain the full *2-Segal conditions.*

[5]By this, we mean that the composition may be partially-defined, multiply-defined, or both. That is, the composition is a span.

[6]Notice that, replacing equalities with (2-)isomorphisms, these are are the conditions defining the data of an A_∞-algebra.

Definition 2.2. A simplicial set $X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is called *2-Segal* if, for every square $n \geq 1$ and every $0 \leq i < j \leq n$, the square of ordered sets

$$\begin{array}{ccc} \{i,j\} & \longrightarrow & \{i, i+1, \ldots, j-1, j\} \\ \downarrow & & \downarrow \\ \{0,1,\ldots,i,j,j+1,\ldots,n\} & \longrightarrow & [n] \end{array}$$

(when interpreted as a square in Δ) is sent by X to a pullback square. Equivalently, X is *2-Segal* if for any $n, m \geq 1$, and and $0 \leq i < n$, the squares

$$\begin{array}{ccc} X_{n+m-1} & \longrightarrow & X_n \\ \downarrow & & \downarrow{\scriptstyle \{i,i+1\}} \\ X_m & \xrightarrow[\{0,m\}]{} & X_1 \end{array}$$

are pullback.[7]

These pullback conditions are collectively referred to as the *2-Segal* conditions.

Remark 2.3. There are many sets of pullback conditions which are equivalent to the 2-Segal conditions (we will see at least one other when we discuss the geometric interpretation). The formulation of *decomposition spaces* in [10] is another way of describing the same conditions.

Remark 2.4. As with the 1-Segal conditions, the 2-Segal conditions make sense in a much broader context than that of simplicial sets. Indeed, one can easily impose the 2-Segal conditions on any simplicial object X in a category $\mathcal{C}$ with pullbacks. More generally, if $\mathcal{C}$ is an ∞-category, one can require that the squares in question be ∞-categorical pullback squares. Since, as we are in the process of showing, 2-Segal objects are intimately connected to associativity in higher algebra, it is these ∞-categorical versions which are often of the greatest use.

Since the 2-Segal conditions are meant to encode unitality, and we arrived at them by contemplating the associativity of composition in the nerves of categories, we might expect that standard associative composition laws also give rise to 2-Segal simplicial sets. It is precisely this that the next condition makes precise.

Theorem 2.5 ([8], 2.3.3 and 2.5.3)**.** *If* $X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ *is 1-Segal, then it is 2-Segal.*

[7] This second description does not actually uniquely specify the squares in question as stated. One must, additionally, require that the underlying maps in Δ are injective.

Proof. Suppose that X is 1-Segal, let $n \geq 2$, and $0 \leq i < j \leq n$. We can use the spine map to extend the associated 2-Segal square to a diagram

$$\begin{array}{ccc} X_n & \longrightarrow & X_{\{i,\ldots,j\}} \\ \downarrow & & \downarrow \\ X_{\{0,\ldots,i,j,\ldots,n\}} & \longrightarrow & X_{\{i,j\}} \\ \cong\downarrow & & \downarrow \mathrm{id} \\ X_{\{0,1\}} \times_{X_{\{1\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}} & \longrightarrow & X_1 \end{array}$$

The lower square is pullback, since two of its parallel legs are isomorphisms, and thus we can identify the map from X_n to pullback of the upper square with the map

$$X_n \longrightarrow X_{\{0,1\}} \times_{X_{\{1\}}} \cdots X_{\{i-1,i\}} \times_{X_{\{i\}}} X_{\{i,\ldots,j\}} \times_{X_{\{j\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}$$

Composing with the spine isomorphism

$$X_{\{i,\ldots,j\}} \longrightarrow X_{\{i,i+1\}} \times_{X_{\{i+1\}}} \cdots \times_{X_{\{j-1\}}} X_{\{j-1,j\}}$$

yields precisely the spine map for X_n. Thus, by the 2-out-of-3 property, the map from X_n to the pullback is an isomorphism, proving the proposition. □

2.1. **Addendum: unitality.** Having now understood how associativity leads us naturally to the 2-Segal conditions, it is natural to ask whether there are similar conditions governing associativity. It turns out that there are. Indeed, the resulting conditions are called the *unitality conditions* in [8]. However, as we will see, the story behind unitality becomes a bit stranger.

First, let us unwind the meaning of unitality in our framework. Within the rubric of our "categories with multiply- and/or partially-defined composition," let's require a fairly strict definition. For $x \in X_0$, the element[8] $s_0(x)$ should be the left unit in the sense that, for any $f \in X_1$ with $d_0(f) = x$, the composition $f \circ s_0(x)$ should have precisely one value, f. We can interpret this diagrammatically, as we did with the first associativity condition.

The requirement of strict (left) unitality can be expressed by writing that $\kappa_2 \circ (\mathrm{id} \times s_0) = \mathrm{id}_{X_1}$. Expressing the left-hand side of this equation as a pair of spans, we obtain

$$\begin{array}{ccccccc} & X_0 \times_{X_0} X_1 & & & X_2 & & \\ {\scriptstyle\cong}\swarrow & & \searrow{\scriptstyle s_0 \times id} & {\scriptstyle (d_2, d_0)}\swarrow & & \searrow & \\ X_0 \times_{X_0} X_1 & & & X_1 \times_{X_0} X_1 & & & X_1 \end{array}$$

[8]One might reasonably object at this point that there is no reason that the unit map $X_0 \longrightarrow X_1$ should not also be multiply and/or partially defined like our multiplications. While this is reasonable, we will set it aside until we reinterpret the 2-Segal conditions in terms of algebras.

The condition then requires that, reformulating this as a single span by the use of a pullback, the result must be isomorphic to the identity span

$$X_1 \xleftarrow{\mathrm{id}} X_1 \xrightarrow{\mathrm{id}} X_1.$$

The right unitality conditions can be analyzed similarly, and we arrive at the conclusion that unitality amounts to the squares

$$\begin{array}{ccc} X_1 & \xrightarrow{s_0} & X_2 \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_2} \\ X_0 & \xrightarrow[s_0]{} & X_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X_1 & \xrightarrow{s_1} & X_2 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_0 & \xrightarrow[s_0]{} & X_1 \end{array}$$

being pullback.

As we did with associativity, we must also include the considerations of unitality for the n-fold compositions. This yields the following conditions.

Definition 2.6. A 2-Segal simplicial set $X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is called *unital* if, for every $n \geq 2$ and every $0 \leq i < n$, it sends the square

$$\begin{array}{ccc} [n-1] & \xleftarrow[s_i]{} & [n] \\ \uparrow & & \uparrow \\ \{i\} & \longleftarrow & \{i, i+1\} \end{array}$$

to a pullback square.

Surprisingly, though, it turns out that this condition is redundant. That is, the 2-Segal conditions which guarantee unitality *also* guarantee associativity! This is less strange than it first appears, because the degeneracy maps in a simplicial set already encode a lot of information pertaining to units. Indeed, If one traces carefully through our work on associativity above, we will see that the associativity makes no use of *any* degeneracy maps.

As a result, we can think of a partially-/multiply-defined multiplication which is associative, but not unital, as a 2-Segal *semi-simplicial* set. That is, a functor

$$X : \Delta_{\mathrm{inj}}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

satisfying the 2-Segal pullback conditions, where Δ_{inj} denotes the wide subcategory of Δ which contains only the injective maps.

The remarkable result, however, is that imposing unitality consists only of extending to a simplicial structure, and does not require new conditions.

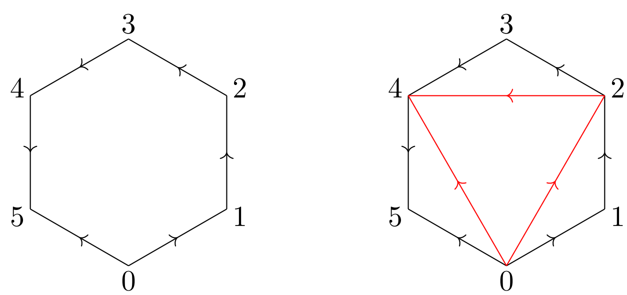
Theorem 2.7 ([9])**.** *If* $X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ *is 2-Segal, then* X *is unital.*

This need not, however, come as a complete shock. Simplicial sets, as opposed to semi-simplicial sets, inherently encode some degree of unitality through the presence of degeneracy maps. In general, the nerve of a non-unital category is a semi-simplicial set, and semi-simplicial sets satisfying an analogue of the 1-Segal condition are often used to model non-unital or weakly unital higher categories. See, e.g., [27] and [17] for some discussion of the role of semi-simplicial sets in modeling weak units.

3. The geometric perspective: polygonal decompositions

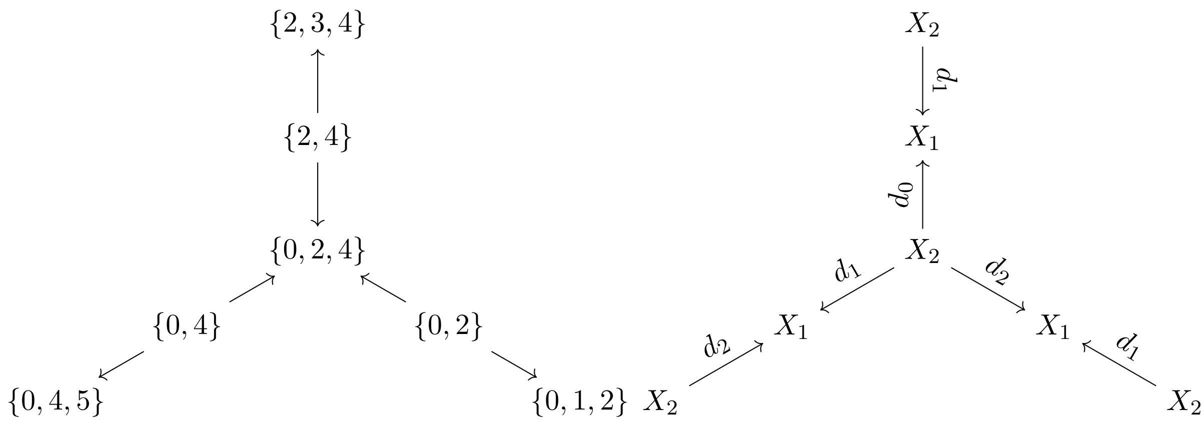
In addition to the purely algebraic intuition discussed above, there is a purely geometric way of thinking about 2-Segal spaces in terms of triangulations of polygons. In this section, we will first develop this perspective, and then explain how it relates to the previous, algebraic perspective.

The basic idea is to try and define a kind of "state-sum" construction for polygons.[9] More precisely, let us start with a simplicial set $X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ and an $(n+1)$-gon P_{n+1} with vertices labeled $0, 1, \ldots, n$. Given any triangulation $\mathcal{T}$ of P_{n+1} *with the same vertices as* P_{n+1}, we can think of the labeling of the vertices of P_{n+1} as orienting the edges of $\mathcal{T}$, as pictured below.



This means that we can identify each triangle of the triangulation as a 2-simplex, and each edge of the triangulation as a 1-simplex. Put another way, we can think of the subsets of $\{0, \ldots, n\}$ defining the 1- and 2-simplices of $\mathcal{T}$ as elements of Δ via the canonical order on $\{0, \ldots, n\}$. As such, we obtain a diagram in the simplex category consisting of these subsets, and can apply X to obtain a diagram in Set. In the case of the example above, these diagrams are

[9] With some additional structure on the inputs, the construction we give here can be formalized into a state-sum construction for punctured surfaces. See [7, §V] and [26, §3.3] for more details.



Let us denote the indexing category of this resulting diagram $I_{\mathcal{T}}$, and the functor $X_{\mathcal{T}}$. We can then define our proposed state-sum construction to be the limit of this diagram of sets, i.e.

$$(\mathcal{T}, X) := \lim_{I_{\mathcal{T}}} X_{\mathcal{T}}.$$

However, in general, this construction will depend explicitly on the choice of triangulation, which runs counter to the very idea of a state sum. To rectify this, we notice that, since our original diagram was defined in terms of ordered subsets of $[n] = \{0, \ldots, n\}$, the set X_n automatically fits into a cone over $X_{\mathcal{T}}$, the components of which are induced by the inclusions of the subsets into $[n]$. Thus, there is a unique map

$$f_{\mathcal{T}} : X_n \longrightarrow (\mathcal{T}, X)$$

induced by this cone. Following [8], we will call $f_{\mathcal{T}}$ the *2-Segal map corresponding to* $\mathcal{T}$. We can then define precisely the condition which will make our state-sum invariant under triangulation.

Definition 3.1. We call a simplicial set $X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ *geometrically 2-Segal*[10] if for any $n \geq 2$ and any triangulation $\mathcal{T}$ of the labeled $(n+1)$-gon P_{n+1} with the same vertex set as P_{n+1}, the 2-Segal map

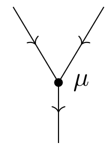
$$f_{\mathcal{T}} : X_n \longrightarrow (\mathcal{T}, X)$$

is an isomorphism. Equivalently, this is the requirement that the canonical cone with tip X_n over $X_{\mathcal{T}}$ is a limit cone.

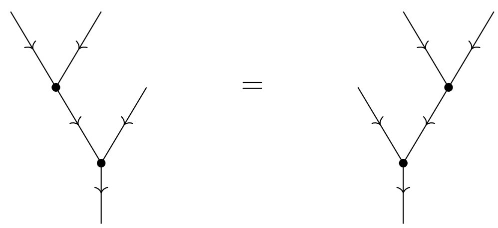
This definition is all well and good, and may even seem reasonable in isolation, but our choice of terminology should raise an important question: what does this state-sum invariance condition have to do with associativity?

[10] Warning: this terminology, unlike most of our terminology and notation, is non-standard.

One answer to this question lies in an operadic view of associativity. Schematically, suppose that an object Y of some monoidal category $\mathcal{C}$ has a multiplication operation $\mu : Y \otimes Y \longrightarrow Y$. We can draw this multiplication as the planar rooted tree

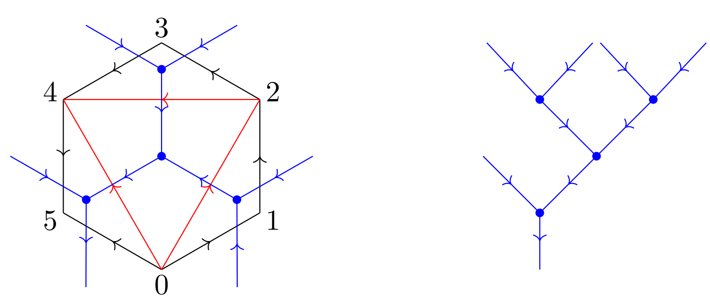


Associativity then says that any planar rooted binary tree with n inputs — interpreted as a composition of $n-1$ copies of μ with itself — must be equal. So, for instance, the associativity condition of (1) is expressed by the equality of planar rooted binary trees



Reinterpreting this in terms of a 2-Segal simplicial set $X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$, this requires that the limits of the corresponding diagrams be equal, precisely as above.

Given a triangulation $\mathcal{T}$ of P_{n+1} as above, there is a canonical way to extract a planar, rooted, binary tree. To each triangle of $\mathcal{T}$ one assigns a vertex of the tree, and to each edge $\{i, j\}$ of $\mathcal{T}$ one assigns an edge $e_{i,j}$ of the tree. The orientations are chosen so that the edges $\{i, j\}$ and $e_{i,j}$ form an oriented basis. For example:



We can thus see that the limits arising from our triangulations are precisely iterated versions of the pullbacks we used to explore our associativity conditions. As such, we should expect that the 2-Segal conditions defining associativity would be equivalent to the geometric 2-Segal conditions.

However, this association — from triangulations, to binary trees, to limits, can also be generalized. By allowing $\mathcal{T}$ to divide P_{n+1} not into triangles, but into $(k+1)$-gons, we obtain planar rooted trees which are not necessarily binary, and thereby obtain associativity conditions explicitly involving the n-fold composition operations defined by X. We will call such a subdivision $\mathcal{T}$ a *polygonal subdivision*, and note that for any polygonal subdivision $\mathcal{T}$, the construction of $I_{\mathcal{T}}$, $X_{\mathcal{T}}$, $(\mathcal{T}, X)$, and $f_{\mathcal{T}}$ works precisely it does for triangulations.

With this in hand, the intuition we have developed connecting the algebraic and geometric 2-Segal conditions can be made precise.

Theorem 3.2 ([8, Proposition 2.3.2]). *For a simplicial set* $X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ *, the following are equivalent.*

1. *The simplicial set* X *is 2-Segal.*
2. *The simplicial set* X *is geometrically 2-Segal.*
3. *For every polygonal subdivision* $\mathcal{T}$ *of* P_{n+1} *for* $n \geq 2$*, the canonical map*
$$f_{\mathcal{T}} : X_n \longrightarrow (\mathcal{T}, X)$$
is an isomorphism.

We will provide a proof of this theorem in the next section, as we formalize our geometric intuition.

4. Formalizing the geometric perspective

Now that we have walked through two intuitions behind 2-Segal simplicial sets, let's try to formalize them. To make the geometric intuition formal, we introduce a tool from [8, §2.2]: *membrane sets.*[11]

Definition 4.1. Let D and X be simplicial sets. We define the *set of D-membranes in X* is the set
$$(D, X) := \mathrm{Hom}_{\mathsf{Set}_\Delta}(D, X).$$
Applying the standard result that D is a colimit over its category of simplices, we can alternately express the membrane set as
$$(D, X) = \mathrm{Hom}_{\mathsf{Set}_\Delta}(D, X) \cong \mathrm{Hom}_{\mathsf{Set}_\Delta}(\mathrm{colim}_{\Delta/D}\, \Delta^n, X) \cong \lim_{(\Delta/D)^{\mathrm{op}}} X_n.$$

Remark 4.2. It is the latter formula which is of use when X is a simplicial object in a category other than sets, and which allows us to generalize the work in this section beyond simplicial sets.

The notion of membrane sets will be most useful when $D \subset \Delta^n$ is a subset of a standard n-simplex. In this case, the limit expressing the membrane set can be rewritten using the category of *non-degenerate* simplices of X. However, there is another important feature in this case which is key to the 2-Segal conditions.

Definition 4.3. Let X be a simplicial set, and $D \subset \Delta^n$ a simplicial subset. The inclusion $D \hookrightarrow \Delta^n$ induces a map
$$f_D : X_n \longrightarrow (D, X)$$

[11] In [8], the authors focus on the higher categorical (space-enriched) case, and thus always speak of membrane *spaces*, rather than membrane sets.

by the contravariant functoriality of $\mathrm{Hom}_{\mathsf{Set}_\Delta}(-, X)$. This map can alternately be viewed as the universal map induced by the cone with tip X_n over the functor

$$(\Delta/D)^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}} \xrightarrow{X} \mathsf{Set}.$$

Following [8], we will call f_D the *generalized Segal map* associated to $D \subset \Delta^n$.

Examples 4.4.

(1) The motivation for the terminology *generalized Segal map* comes from the case where $D = \mathrm{Sp}(\Delta^n) \subset \Delta^n$. The generalized Segal map associated to this inclusion is simply the n^{th} 1-Segal map

$$f_D : X_n \longrightarrow X_{\{0,1\}} \times_{X_{\{1\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}.$$

(2) Let $D \subset \Delta^n$ be a 1-dimensional simplicial set which is a tree and contains every vertex of Δ^n. We will call such a subset a *spanning tree for* Δ^n. Notice that each spine inclusion is, in particular, a spanning tree. If X is a simplicial set such that the generalized Segal maps associated to spanning trees are always isomorphisms, then X is isomorphic to the nerve of a groupoid (this statement is implicit in the discussion of [16, §3], but I am unaware of a more direct reference).

For our purposes, the most important examples of generalized Segal maps are those arising from triangulations and polygonal subdivisions. Let $\mathcal{T}$ be a polygonal subdivision of P_{n+1} which has the same vertex set as P_{n+1}. Then each polygon of $\mathcal{T}$ can be uniquely identified with a subset $I \subset [n]$ via the labeling of the vertices. We then define a simplicial subset $\Delta^{\mathcal{T}} \subset \Delta^n$ associated to the polygonal subdivision $\mathcal{T}$ by the formula

$$\Delta^{\mathcal{T}} := \bigcup_{I \in \mathcal{T}} \Delta^I.$$

That is, we define $\Delta^{\mathcal{T}}$ to be the full simplicial subset of Δ^n on the simplices representing polygons of the decomposition $\mathcal{T}$. With this terminology in place, we can reformulate our geometric definition of the 2-Segal conditions.

To make it clear how this connects to the dual graph pictures we drew above, let us fix some further definitions.

Definition 4.5. Let $\mathcal{T}$ be a polygonal subdivision of P_{n+1} with the same vertex set as P_{n+1}. Write $(\Delta/\Delta^{\mathcal{T}})_{\mathrm{nd}}$ for the category of non-degenerate simplices of $\Delta^{\mathcal{T}}$. Let $I_{\mathcal{T}} \subset (\Delta/\Delta^{\mathcal{T}})_{\mathrm{nd}}$ be the full subcategory on the objects of form Δ^I for I a polygon of $\mathcal{T}$ and the objects $\Delta^{\{i,j\}}$ for $\{i,j\}$ an edge of $\mathcal{T}$. We will call $I_{\mathcal{T}}$ the *dual graph category* of $\mathcal{T}$.

Lemma 4.6. *The geometric realization of $N(I_{\mathcal{T}})$ is isomorphic to the dual graph of $\mathcal{T}$*

Proof. Left to the reader. □

Proposition 4.7. *The inclusion*

$$\iota : I_{\mathcal{T}} \longrightarrow (\Delta/\Delta^{\mathcal{T}})_{\mathrm{nd}}$$

induces an isomorphism

$$\lim_{I_{\mathcal{T}}} X_m \cong \lim_{(\Delta/\Delta^{\mathcal{T}})^{\mathrm{op}}_{\mathrm{nd}}} X_m$$

Proof. This is immediate from the fact that the induced map

$$\mathrm{colim}_{I_{\mathcal{T}}} \Delta^m \longrightarrow \Delta^{\mathcal{T}}$$

is an isomorphism. □

We are now in a position to prove Theorem 3.2.

Proof (of Theorem 3.2). Since every triangulation is a polygonal subdivision, (3) $\Longrightarrow$ (2). To see that (3) $\Longrightarrow$ (1), let $n \geq 3$, and $0 \leq i < j \leq n$, and consider the square

$$\begin{array}{ccc} X_n & \longrightarrow & X_{\{i,\ldots,j\}} \\ \downarrow & & \downarrow \\ X_{\{0,\ldots,i,j,\ldots,n\}} & \longrightarrow & X_{\{i,j\}} \end{array}$$

This is pullback if and only if the induced map

$$X_n \longrightarrow X_{\{i,\ldots,j\}} \times_{X_{\{i,j\}}} X_{\{0,\ldots,i,j,\ldots,n\}}$$

is an isomorphism. However, this is simply the generalized Segal map corresponding to the polygonal subdivision which divides P_{n+1} into two polygons along the edge from i to j.

To see that (2) $\Longrightarrow$ (3), choose a polygonal subdivision $\mathcal{T}$ of P_{n+1}, and a refinement to a triangulation $\mathcal{S}$. For each polygon Δ^J of $\mathcal{T}$, there is a corresponding generalized Segal map g_J for the restriction $\mathcal{S} \cap J$ of triangulation $\mathcal{S}$ to J. The generalized Segal map $f_{\mathcal{S}}$ can be identified with the composite

$$X_n \xrightarrow{f_{\mathcal{T}}} \lim_{I_{\mathcal{T}}} X_J \xrightarrow{\{g_J\}} \lim_{I_{\mathcal{T}}} \lim_{I_{\mathcal{S}\cap J}} X_L$$

However, since (2) holds, the g_J's and $f_{\mathcal{S}}$ are isomorphisms. Thus, by 2-out-of-3, $f_{\mathcal{T}}$ is an isomorphism.

A similar 2-out-of-3 argument, building arbitrary subdivisions out of subdivisions into two polygons, shows that (1) $\Longrightarrow$ (3), completing the proof. □

4.1. **Digression: the 4-truncation.** It is well known that every 1-Segal simplicial set X is completely determined by the underlying 2-dimensional data of X, and thus is 2-coskeletal. Rather less well-known is that, if a 2-coskeletal simplical set X satisfies the 1-Segal conditions for 2-dimensional and 3-dimensional spines, then it is 1-Segal (see [22, Lemma 5.2] for a closely-related proof). In this section, we will establish analogues of these results for 2-Segal simplicial sets.

The geometric discussion above could be taken to suggest that 2-Segal simplicial sets are determined by their 2-dimensional data — i.e., their 2-truncations. However, as we will see, this is not quite the case. The reason for this, loosely speaking, is that the *choice* of isomorphism between the membrane sets associated to the two triangulations of the square is a fundamental part of the 2-Segal data. Since this isomorphism is determined by the set X_3 of 3-simplices and the concomitant maps to X_2, we might adjust our expectations to predict that 2-Segal simplicial sets are determined by their 3-truncations. This is, in fact, the case.

Proposition 4.8. *Every 2-Segal simplicial set X is 3-coskeletal.*

Proof. This is [1, Corollary 1.7] □

Perhaps surprisingly, it is not true that every 3-coskeletal simplicial set for which the 2-Segal maps

$$X_2 \times_{X_1} X_2 \longleftarrow X_3 \longrightarrow X_2 \times_{X_1} X_2$$

are isomorphisms is 2-Segal. The reason for this is easier to interpret from the algebraic perspective. These 3-dimensional 2-Segal conditions encode the existence of *associators*: isomorphisms encoding the associativity of the 2-fold composition/multiplication. However, true higher-categorical associativity requires that these associators satisfy coherence conditions, which it will turn out are encoded in X_4. As such, the proper converse to Proposition 4.8 is

Proposition 4.9. *Let X be a 3-coskeletal simplicial set. If X satisfies the 2-Segal conditions coresponding to subdivisions of the square and pentagon, then X is 2-Segal.*

Proof. Let $n > 4$, and let $\mathcal{T}$ be a triangulation of P_{n+1}. We must show that any map $\Delta^{\mathcal{T}} \to X$ extends uniquely to a map $\Delta^n \to X$. Since $\mathcal{T}$ is a triangulation, $\Delta^{\mathcal{T}} \to \Delta^n$ factors through the 3-skeleton of Δ^n, yielding

$$\Delta^{\mathcal{T}} \longrightarrow \mathrm{sk}_3(\Delta^n) \longrightarrow \Delta^n.$$

Since X is 3-coskeletal, it will suffice to show that we can uniquely extend to a map $\Delta^{\mathcal{T}} \to X$ to $\mathrm{sk}_3(\Delta^n)$. To see this, we define a poset U whose elements are the simplicial subsets $\Delta^{\mathcal{R}}$ where $\mathcal{R}$ is one of the following types of subdivision.

- A triangulation $\mathcal{R}$.
- A subdivision $\mathcal{R}$ consisting of triangles and a single square.
- A subdivision $\mathcal{R}$ consisting of triangles and two squares.
- A subdivision $\mathcal{R}$ consisting of triangles and a single pentagon.

By [11, Theorem 6.32], the realization of this poset is the 2-skeleton of n^{th} associahedron. Since the associahedra are contractible, by cellular approximation, we see that the realization of U is simply-connected.

Since X satisfies the 2-Segal conditions corresponding to subdivisions of the square and pentagon, the functor

$$\begin{aligned} M : U^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ \Delta^{\mathcal{R}} &\longmapsto (\mathcal{R}, X) \end{aligned}$$

sends every morphism to an equivalence. Thus, since U is simply connected, the limit of this diagram (or, equivalently, the set of maps $\operatorname{colim}_{\Delta^{\mathcal{R}} \in U} \Delta^{\mathcal{R}} \to X$) is isomorphic to $(\mathcal{T}, X)$ for each fixed triangulation $\mathcal{T}$. We thus obtain a commutative diagram

$$\begin{array}{ccccc} & & (\operatorname{colim}_{\Delta^{\mathcal{T}}\in U}\Delta^{\mathcal{T}}, X) & & \\ & \swarrow & \downarrow & \nwarrow{\scriptstyle\cong} & \\ (\Delta^{\mathcal{T}},X) & \underset{\cong}{\longleftarrow} & (\mathrm{sk}_3(\Delta^n),X) & \longleftarrow & (\Delta^n,X)\end{array}$$

Where the dashed arrow has not yet been constructed.

To prove the proposition, it will thus suffice to show that the 3-skeleton of $\operatorname{colim}_{\Delta^{\mathcal{T}}\in U}\Delta^{\mathcal{T}}$ is isomorphic to $\mathrm{sk}_3(\Delta^n)$, thereby producing the dashed arrow and showing it is an isomorphism. Since every 3-simplex of Δ^n is represented by a square in at least one triangulation of P_{n+1}, it is clear that the map $\operatorname{colim}_{\Delta^{\mathcal{T}}\in U}\Delta^{\mathcal{T}} \to \Delta^n$ defined by universal property surjects onto the 3-skeleton. However, any two subdivisions containing the same square are related by a sequence of moves which fix that square and act on the rest of the triangulation by replacing a pair of triangles forming a square by the other pair of triangles which could form that square (see, for instance, the main theorem of [18]). This sequence of moves corresponds to a zig-zag in U consisting of subdivisions containing that same square. Thus, any two 3-simplices which correspond to the same 3-simplex in Δ^n are identified in the colimit, completing the proof. □

This result and its proof, may seem very technical, and it certainly relies on a great many combinatorial facts and constructions — associahedra and cellular approximation, for example — which we do not have the space to fully explore here. However, this is to be expected. As we will see when we formalize the algebraic perspective, Proposition 4.9 can be seen as a simplicial set analogue of MacLane's celebrated COHERENCE THEOREM FOR MONOIDAL CATEGORIES [21, Theorem 5.2], and as such we should expect its proof to be similarly challenging.

The proof of [1, Proposition 1.6] can be adapted to show that a 4-coskeletal simplicial set that satisfies the 2-Segal conditions necessary for Proposition 4.9 to hold is, in fact, 3-coskeletal. This allows us to rephrase Proposition 4.9 into a statement about 4-coskeletal simplicial sets, in which one condition involves only statements we can verify on the the 4-truncation.

Proposition 4.10. *Let X be a 4-coskeletal simplicial set. The following are equivalent.*

(1) *X is 2-Segal.*
(2) *X satisfies the 2-Segal conditions corresponding to subdivisions of the square and pentagon.*

Proposition 4.10 will be of use in formalizing the algebraic perspective, since it allows us to identify the category of 2-Segal simplicial sets with the category of 4-truncated simplicial sets satisfying condition (2) of the proposition.

5. Formalizing the algebraic perspective

Now that we have a rigorous geometric perspective on the 2-Segal conditions, we can return to the algebraic implications. We will formalize our previous discussion of the 2-Segal conditions in terms of algebraic structures in span bicategories, as in [3, 4, 12, 25, 26].

It is worth pointing out that the formalization presented here will differ from what one might expect from the intuitive discussion above. Using the discussion in section 2 as a guide, one would be led to define a notion of categories (weakly) enriched in spans, a precise formalization of our notion of categories whose composition and unit morphisms are spans. This is the perspective taken in [8, §3.3], where such categories are called μ-categories.

To simplify our formalization, and avoid presenting the somewhat esoteric definition of μ-categories and their concomitant functors, we will need modify our intuitions slightly. If, for a moment, we stop thinking of the elements of X_0 as *objects*, we can reinterpret the spans defining our n-fold compositions as spans

$$X_1 \times \cdots \times X_1 \longleftarrow X_n \longrightarrow X_1$$

in which the left-hand leg simply *happens* to take values in $X_1 \times_{X_0} \cdots \times_{X_0} X_1$. That is, we can think of the span as defining a span from $X_1^{\times n}$ to X_1. Similarly, we can think of the degeneracy map $s_0 : X_0 \to X_1$ as defining a span

$$* \longleftarrow X_0 \longrightarrow X_1$$

from the singleton $*$ to X_1.

From this perspective, our discussion in section 2 yields the following structure associated to a 2-Segal simplicial set X:

- A set X_1 equipped with *n-fold multiplication spans* from $X_1^{\times n}$ to X_1.
- A *unit span* from $*$ to X_0.
- Associativity and unitality data for the multiplication and unit spans.

Conveniently, these data seem to specify a (coherently) associative algebra in the category of spans. We will make this observation precise in the remainder of this section. The reader preferring the multivalued category perspective is encouraged to supplement the treatment here with [8, §3.3].

As our formalization will rely on the bicategory of spans of sets, we take a moment to define this bicategory before continuing.

Definition 5.1. Let X and Y be sets. A *span* from X to Y is a diagram

$$X \xleftarrow{f_1} F \xrightarrow{f_2} Y$$

in Set. The *category of spans from X to Y* is the slice category $\mathsf{Set}_{/X\times Y}$, interpreted as the category whose objects are spans from X to Y, and whose morphisms are commutative diagrams

$$\begin{array}{ccccc} & & F & & \\ & \swarrow^{f_1} & & \searrow^{f_2} & \\ X & & \downarrow & & Y \\ & \nwarrow_{g_1} & & \nearrow_{g_2} & \\ & & G & & \end{array}$$

For X, Y, and Z sets, the *composition functor* of spans

$$\circ : \mathsf{Set}_{/Y\times Z} \times \mathsf{Set}_{/X\times Y} \longrightarrow \mathsf{Set}_{/X\times Z}$$

is given on objects by sending

$$\begin{array}{ccccccc} & & F & & & G & \\ & \swarrow^{f_1} & & \searrow^{f_2} & \swarrow^{g_1} & & \searrow^{g_2} \\ X & & & Y & & & Z \end{array}$$

to the pullback span

$$\begin{array}{ccc} & F\ {}^{f_2}\!\times_Y^{g_1} G & \\ \swarrow^{f_1\circ\pi_1} & & \searrow^{g_2\circ\pi_2} \\ X & & Y. \end{array}$$

On morphisms, the composition functor is uniquely determined up to unique natural isomorphism by universal property.

Definition 5.2. We define the *bicategory of spans of sets* $\mathsf{Span}(\mathsf{Set})$ — which we will typically denote simply by Span — to consist of the following data.

(1) The objects of Span are sets.
(2) The hom-categories $\mathsf{Span}(X, Y)$ are the categories $\mathsf{Set}_{/X\times Y}$ of spans of sets.
(3) The composition is the composition functor

$$\circ : \mathsf{Set}_{/Y\times Z} \times \mathsf{Set}_{/X\times Y} \longrightarrow \mathsf{Set}_{/X\times Z}$$

(4) The *identity* on X is the span

$$X \xleftarrow{\text{id}} X \xrightarrow{\text{id}} X$$

(5) The associators and unitors are the unique natural transformations defined by universal property.

It is a straightforward, if tedious, process, to verify that these data form a bicategory. A proof, in a more general setting, can be found in [2]. The bicategory Span is, in fact, a symmetric monoidal bicategory under the Cartesian product of sets, a fact which is also verified in [2].

5.1. **Algebras in spans.** The aim of this section is to provide a proof that 2-Segal simplicial sets correspond to coherently associative algebras in Span. To this end, we first define such coherently associative algebras and their isomorphisms, following [6].[12]

Definition 5.3. An *(associative) algebra* in Span consists of the following data:

(1) A set A_1.

[12] In op. cit., Day and Street use the term *pseudomonoids* for what we here call *algebras*. The reason for the terminology change is two-fold: on the one hand to better accord with the ∞-categorical terminology used in [25, 12], and on the other to reserve the term *monoid* for associative algebras in Cartesian monoidal structures.

(2) A *multiplication span*

$$A_1 \times A_1 \xleftarrow{(\mu_2,\mu_0)} A_2 \xrightarrow{\mu_1} A_1$$

which we will sometimes briefly denote as μ.

(3) A *unit span*

$$* \longleftarrow A_0 \xrightarrow{\nu_0} A_1$$

which we will sometimes briefly denote as ν.

(4) An isomorphism α in the category $\mathsf{Span}(A_1 \times A_1 \times A_1, A_1)$ from $\mu \circ (\mu \times \mathrm{id}_{A_1})$ to $\mu \circ (\mathrm{id}_{A_1} \times \mu)$, where we are identifying $(A_1 \times A_1) \times A_1$ with $A_1 \times (A_1 \times A_1)$ via the unique isomorphism determined by universal property of products.

(5) Isomorphisms in the category $\mathsf{Span}(A_1, A_1)$

$$\lambda : \mu \circ (\nu \times \mathrm{id}_{A_1}) \xRightarrow{\sim} \mathrm{id}_{\mathrm{A}_1}$$

and

$$\rho : \mu \circ (\mathrm{id}_{A_1} \times \nu) \xRightarrow{\sim} \mathrm{id}_{\mathrm{A}_1} .$$

Here we are identifying $A_1 \times * \cong A_1 \cong * \times A_1$, again by universal property.

These data must additionally satisfy the following *coherence conditions.*

(1) The *pentagon diagram* (Figure 1) commutes.
(2) The *triangle diagram* (Figure 2) commutes.

Remark 5.4. Notice that the isomorphisms α, λ, and ρ are equivalently isomorphisms of sets

$$\alpha : A_2{}^{\mu_2}\times^{\mu_1}_{A_1} A_2 \xrightarrow{\cong} A_2{}^{\mu_0}\times^{\mu_1}_{A_1} A_2$$

$$\lambda : A_2{}^{\mu_2}\times^{\nu_0}_{A_1} A_1 \xrightarrow{\cong} A_1$$

$$\rho : A_2{}^{\mu_0}\times^{\nu_0}_{A_1} A_1 \xrightarrow{\cong} A_1$$

which commute with the appropriate projections to A_1.

Definition 5.5. For two associative algebras A and B in Span, a *oplax morphism*[13] from A to B consists of maps of sets $f : A_i \to B_i$, for $0 \leq i \leq 2$ which commute with the defining maps of the spans μ and ν, as well as with the morphisms α, λ, and ρ in Remark 5.4. We will denote the 1-category of algebras in spans simply by Alg.

Construction 5.6. Let X be a 2-Segal simplicial set. Define an algebra $A(X)$ in Span to have underlying set X_1, multiplication span

$$X_1 \times X_1 \xleftarrow{(d_2,d_0)} X_2 \xrightarrow{d_1} X_1,$$

[13]As shown in [12], pseudonatural transformations between such algebras whose components are spans correspond to spans of morphisms of simplicial sets satisfying very specific properties. The morphisms defined here correspond to oplax transformations whose components are morphisms of sets, hence the terminology we have chosen.

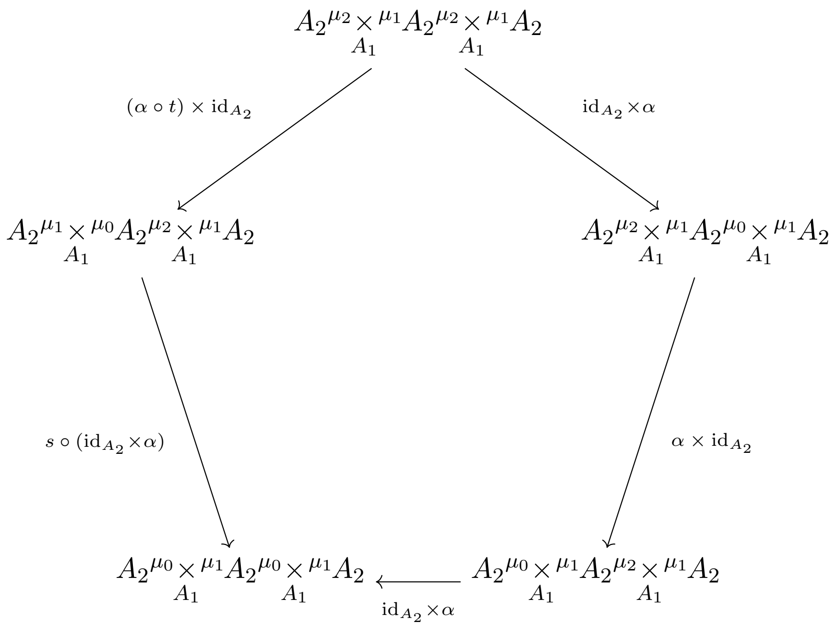


FIGURE 1. The associativity pentagon for algebras in Span, where permutations have been inserted to ease the writing of pullbacks. Here t denotes the unique non-identity permutation, and s denotes the permutation $(2, 1, 3)$. Each corner of the pentagon corresponds to a composite of three copies of μ, and thus corresponds to a binary tree and a triangulation of the pentagon P_5.

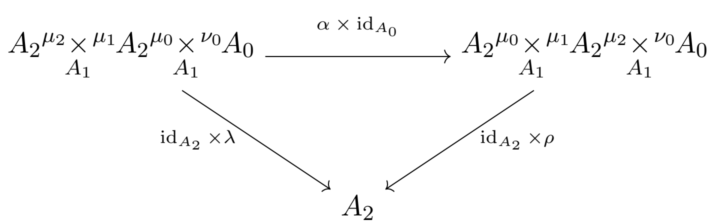


FIGURE 2. The triangle diagram for an algebra in Span. Here, we are implicitly identifying $A_2 \times_{A_1} A_1 \cong A_2$ at the bottom of the diagram

unit span

$$* \longleftarrow X_0 \xrightarrow{s_0} X_1$$

associator given by the (invertible) span

$$X_2 \times_{X_1} X_2 \xleftarrow{(d_0,d_2)} X_3 \xrightarrow{(d_3,d_1)} X_2 \times_{X_1} X_2$$

consisting of the 2-Segal maps, and unitors λ and ρ given by the inverses of the 2-Segal maps

$$\lambda^{-1} : X_1 \xrightarrow{s_0} X_2 \times_{X_1} X_0$$

$$\rho^{-1} : X_1 \xrightarrow{s_1} X_2 \times_{X_1} X_0$$

The pentagon condition follows by placing a copy of X_4 in the center of the pentagon of figure 1, with the 2-Segal maps for X_4 directed radially outward. This yields a commutative diagram in which every morphism is an equivalence by the 2-Segal conditions. The triangle condition follows similarly.

It is immediate that given a morphism $f : X \to Y$, the components f_0, f_1, and f_2 define an oplax morphism $A(f) : A(X) \to A(Y)$ in Alg. As a result we obtain a functor

$$\mathrm{Al} : \mathsf{2Seg} \longrightarrow \mathsf{Alg}.$$

The formalization of the algebraic condition then has the following form:

Theorem 5.7. *The functor*

$$\mathrm{Al} : \mathsf{2Seg} \longrightarrow \mathsf{Alg}.$$

is an equivalence of categories.

Before we prove this theorem it is worth noting that [15, Example 3.6] is a special case which is particularly approachable. A partially-defined map of sets can be viewed as a span whose left leg is injective, and so every partial monoid can be viewed as an algebra in Span.

5.2. **Essential surjectivity.** The construction of the inverse to A is a rather subtle business. In principal, it amounts to proving a kind of coherence result. However, we will make use of Proposition 4.9 to substantially simplify the matter, implicitly shifting the issue of coherence into that result. In particular, we need only construct a 4-truncated, 3-coskeletal simplicial set from an algebra. Even with this simplification, length constraints mean that we only sketch the construction which demonstrates essential surjectivity.

Let $A \in \mathsf{Alg}$ be an algebra in Span, and define a 4-truncated simplicial set X_A as follows. The sets of 0-, 1-, and 2-simplices are A_0, A_1, and A_2, respectively. We define A_3 to be the limit of the diagram

$$\alpha : A_2{}^{\mu_2}\times^{\mu_1}_{A_1} A_2 \xrightarrow{\cong} A_2{}^{\mu_0}\times^{\mu_1}_{A_1} A_2$$

of sets. Since the associator is an isomorphism, this is canonically isomorphic to both pullbacks. The set A_4 is defined to be the limit of the associativity pentagon. The face and degeneracy maps are defined as follows.

Face maps:

- The face maps $A_1 \to A_0$ are given by $d_1 = p \circ \lambda^{-1}$ and $d_0 = q \circ \rho^{-1}$, where p and q are the projections from the pullback to A_0.
- The face maps $A_2 \to A_1$ are the maps of the multiplication span, i.e., $d_i = \mu_i$.
- The face maps $A_3 \to A_2$ are the composites of the defining maps of the limit $A_3 \to A_2{}^{\mu_2}\times^{\mu_1}_{A_1} A_2$ and $A_3 \to A_2{}^{\mu_2}\times^{\mu_1}_{A_1} A_2$ with the projections to the four copies of A_2.

- The face maps $A_4 \to A_3$ are the composites of the defining maps of the limit from A_4 to the corners of the pentagon (1) with the projections of the pullbacks to copies of ${A_2}^{\mu_2}\times^{\mu_1}_{A_1} A_2$ and ${A_2}^{\mu_0}\times^{\mu_1}_{A_1} A_2$. These latter are canonically identified with A_3.

Degeneracy maps:

- The degeneracy $s_0 : A_0 \to A_1$ is identified with ν_0.
- The degeneracy maps s_0 and s_1 from A_1 to A_2 are identified with the composites of λ^{-1} and ρ^{-1}, respectively, with the projections to A_2.
- The definition of A_3 and A_4 yield isomorphisms

$$ {A_n}^{\phi_i}\times^{\nu_0}_{A_1} A_0 \xrightarrow{\cong} A_{n-1} $$

for $n = 3, 4$ and $0 \leq i < n$, where ϕ_i is dual to the inclusion of $\{i, i+1\}$ into $[n]$. The isomorphisms are obtained by decomposing A_n into a pullback of A_2's and then applying λ or ρ. The triangle diagram (2) implies that these are, in fact, unique. We can thus define the degeneracy map s_i to be the composite

$$ A_{n-1} \xrightarrow{\cong} {A_n}^{\phi_i}\times^{\nu_0}_{A_1} A_0 \xrightarrow{\mathrm{pr}_1} A_n $$

of the inverse isomorphisms with the projection.

It is time-consuming and notationally heavy to verify that these definitions yield a 4-truncated simplicial set, and we omit such verifications for the sake of brevity.

By construction, this 4-truncated simplicial set satisfies the 2-Segal conditions arising from triangulations of the square and pentagon. Thus, by Proposition 4.10, it uniquely determines a 2-Segal set. By construction, $\mathrm{Al}(A_X) \cong A$, and so the functor Al is essentially surjective.

5.3. **Fully faithfulness.** Again, we leverage coskeletalness to provide a short proof that the functor Al is fully faithful. In this case, since every 2-Segal simplicial set is 3-coskeletal by Proposition 4.8, we may consider our morphisms of 2-Segal simplicial sets to be morphisms between 3-truncations. This immediately implies that Al is faithful. To see that it is full, we need only note that, given an oplax morphism $(f_0, f_1, f_2) : A \to B$ of algebras, the fact that the f_i commute with α, λ and ρ allow us to uniquely extend them to a morphism of 3-truncated simplicial sets, completing the proof.

The one subtlety in the above argument is that, by the lowest-dimensional 2-Segal conditions, f_2 and f_1 uniquely determine maps $A_3 \to B_3$. However, there are two maps so determined, one corresponding to each subdivision of the square. Remembering f_3 amounts to remembering that these maps commute with the associativity isomorphism.

5.4. **Coherence, truncation, and coskeletalness.** We now arrive at a cluster of ideas which bear profound and interesting interrelations. Though we have defined associative algebras in Span by means of finite data satisfying relations, the most natural way of defining associative algebras in a bicategory is via infinite coherent data. More precisely, one can think of an algebra in a monoidal bicategory $(\mathbb{B}, \otimes, I)$ as consisting of an object A, together with morphisms

$$ \mu_n : A^{\otimes n} \longrightarrow A $$

for $n \geq 0$, together with 2-isomorphisms

$$\alpha_{m,n} : \mu_m \circ \mu_n \Longrightarrow \mu_{m+n-1}$$

and

$$\eta : \mu_1 \Longrightarrow \mathrm{id}_A$$

which are *coherent* in the sense that any two parallel 2-morphisms built from these data using tensor product and compositions are equal.[14]

The fact that algebras in the sense of our Definition 5.3 are equivalent to this kind of coherent data is a highly non-trivial theorem — a generalization of MacLane's famed COHERENCE THEOREM for monoidal categories.

Intuitively, however, algebras in Span presented by the infinite data above correspond very naturally to simplicial sets. The roof A_n of the span μ_n from $A_1^{\times n}$ to A_1 is the set of n-simplices, and the associativity conditions amount precisely to the 2-Segal conditions. As such, one can think of Proposition 4.10 as a kind of coherence theorem for algebras in Span — the 4-truncated data which correspond to algebras in the sense of Definition 5.3 can be uniquely extended to the infinite data of an algebra in the sense introduced in this section.

Our proof of Proposition 4.9 is very topological, and the essentially topological character of coherence theorems has long been recognized. In ∞-category theory, one can rephrase MacLane's coherence theorem to state that the space of n-ary multiplications on a monoidal category is contractible, i.e., is a trivial groupoid when equiped with (products and composites of) the associativity and unitality natural isomorphisms. For a treatment of coherence making more explicit use of topological methods see, e.g., [5].

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