Summer 0 Notes

Walker H. Stern

the following notes were created for the Math 5305 course taught during the Summer of 2023. These notes have been cobbled together from my other writings, and may not always hang together well. If you spot any errors or issues, please let me know. Exercises, theorems, etc. which are marked with a star ($\star$) are possible topics for a presentation. Let me know if you are interested in presenting one of these topics.

Contents

1 *Continuity and Topology* 5

1.1 *Continuity & Topology* 5

1.2 *Building spaces* 9

1.3 *Hausdorff Spaces* 13

1.4 *Connectedness and path-connectedness* 16

2 *The fundamental group* 19

2.1 *Homotopy or a first look at the fundamental group* 20

2.2 *Coverings and the circle* 27

2.3 *The Seifert-van Kampen Theorem* 29

2.4 *Topological manifolds and surfaces* 35

3 *Smooth manifolds* 39

3.1 *Differentiation on* $\mathbb{R}^n$ 39

3.2 *Smooth manifolds, smooth functions, and derivatives* 44

3.3 *Paracompactness and partitions of unity* 45

3.4 *Tangent spaces and differentials* 48

4 *Integration* 55

4.1 *Rewriting integration* 55

4.2 *Multilinear Algebra* 60

4.3 *Forms, orientations, and integration* 65

4.4 *The exterior derivative* 71

4.5 *The Mayer-Vietoris sequence* 80

1

Continuity and Topology

1.1 Continuity & Topology

The fundamental observation of topology is that virtually no continuity-related notions really require a metric. We can have satisfactory definitions of continuous, compact, etc. if we forget about the metric (the distance between points) and only remember which sets were open. This insight leads to the notion of a *topology*, which abstracts continuity from metric spaces to “spaces without distances”.[1]

The idea of a topological space is to forget the metric, and just remember the open sets:

Definition 1.1. Let X be a set. A *topology on* X is a subset $\tau_X \subset \mathbb{P}(X)$ of the power set such that

1. $X, \varnothing \in \tau_X$.[2]
2. Let $\{U_i\}_{i\in I}$ be a (possibly infinite) collection of sets in τ_X. Then
$$\bigcup_{i\in I} U_i \in \tau_X$$
3. Let $\{U_i\}_{i\in 1}^n$ be a finite collection of sets in τ_X. Then
$$\bigcap_{i=1}^n U_i \in \tau_X.$$

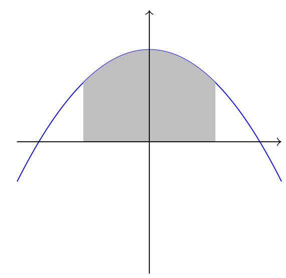
We call the elements $U \in \tau_X$ the *open sets* of the topology on X. We refer to a pair (X, τ_X), where τ_X is a topology on X, as a topological space.[3] We call a subset $C \subset X$ of a topological space *closed* if

$$C^c := X \setminus C$$

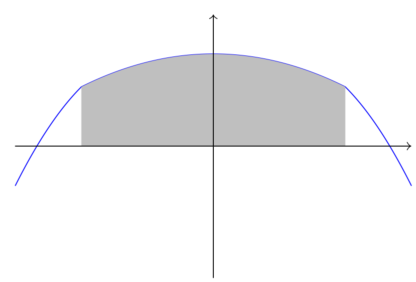
is an open set, i.e. is in τ_X.

Proposition 1.2. *Let (X, d) be a metric space. Denote by τ_d the collection of d-open subsets of X. Then τ_d is a topology on X.*[4]

[1] As one way to illustrate that continuity doesn’t depend on the specific metric, but rather on the open and closed sets, let’s consider the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}$.



We’ve highlighted a section of the x-axis, and we now stretch that section out, while leaving the other distances unchanged.



The function remains continuous, even though we’ve clearly changed the metric we’re using on $\mathbb{R}$.

[2] Applying general set-theoretic conventions, this assertion actually follows from the following two. Since the empty union is empty, and the empty intersection of subsets of X is all of X, condition (2) implies that $X \in \tau_X$, and condition (3) implies that $\varnothing \in \tau_X$

[3] We will often abuse notation and write X for a topological space in cases where the choice of topology is clear from context.

[4] This, together with Proposition 1.3 effectively tells us that we can study continuous functions between metric spaces in terms of the associated topological spaces.

Proof. Exercise. □

To begin our exploration of topology, let's make the claim that continuity doesn't depend on the metric more rigorous.

Exercise 1.3. Let (X, d) and (Y, s) be metric spaces. Show that a function $s : X \to Y$ is continuous if and only if, for every open subset $U \subset Y$, the subset $f^{-1}(U) \subset X$ is open.

This tells us that if we know the collection of open sets of (X, d), we can check whether a map is continuous regardless of whether we know the metric.

Having now established our general definitions, we can now proceed to the concepts necessary to study continuous maps

Definition 1.4. A map $f : X \to Y$ between topological spaces (X, τ_X) and (Y, τ_Y) is called a *continuous map* if, for every open subset $U \in \tau_Y$, the preimage $f^{-1}(U)$ is an element of τ_X. We say f is *open* if, for every open $U \subset X$, $f(U) \subset Y$ is open. We say f is *closed* if, for every closed $C \subset X$, $f(C) \subset Y$ is closed. We say that f is a *homeomorphism* if it is bijective and both f and f^{-1} are continuous.[5]

[5] This is simply formalizing what we discovered about metric spaces into a definition. We can now talk about continuity of maps between general topological spaces an know that this subsumes the case of metric spaces.

Exercise 1.5. Define $B^n := \{x \in \mathbb{R}^n \mid |x| < 1\}$ to be the open unit ball in $\mathbb{R}^n$ (equipped with the subspace topology). Define maps

$$\begin{aligned} f : B^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \tfrac{x}{1-|x|} \end{aligned}$$

and

$$\begin{aligned} g : \mathbb{R}^n &\longrightarrow B^n \\ x &\longmapsto \tfrac{x}{1+|x|} \end{aligned}$$

Show f is a homeomorphism with inverse g.

Lemma 1.6. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a map between topological spaces. The following are equivalent:*

1. *The map f is continuous.*
2. *For every closed set C of Y, $f^{-1}(C)$ is closed in X.*

Proof. First suppose that f is continuous, and let $C \subset Y$ be closed. Then $f^{-1}(C^c)$ is open in X. We can then compute

$$f^{-1}(C)^c = X \setminus f^{-1}(C) = f^{-1}(Y) \setminus f^{-1}(C) = f^{-1}(Y \setminus C) = f^{-1}(C^c)$$

and thus, $f^{-1}(C)^c$ is open, so $f^{-1}(C)$ is closed.

Now suppose that f satisfies condition 2. The same computation as above shows that for $U \in \tau_Y$, $f^{-1}(U)$ is open. □

Lemma 1.7. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a map between topological spaces. Then f is a homeomorphism if and only if f is continuous, bijective, and, for all $U \in \tau_X$, $f(U) \in \tau_Y$.*

Proof. Left as an exercise to the reader. □

So far, we have only seen topologies that arise from metrics. Let us briefly sketch two examples which are less familiar.

Example 1.8. Suppose that $(P, \leq)$ is a partially ordered set (poset). We can define a topology $\tau_{\leq}$ on P as follows. We define a set $U \subset P$ to be *downwards closed* if, for every $x \in U$ and $y \in P$, if $y \leq x$, then $y \in U$. We claim that the downwards closed sets form a topology τ_P on P. To see this, we check axioms (2) and (3) in the definition of topological spaces

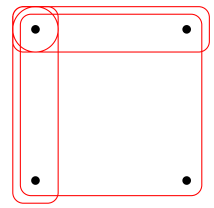
2. Suppose that $\{U_i\}_{i \in I}$ is a collection of downwards-closed sets. Let $x \in \bigcup_{i \in I} U_i$ and $y \in P$ such that $y \leq x$. Then there is some $j \in I$ such that $x \in U_j$. Thus $y \in U_j$, and so $y \in \bigcup_{i \in I} U_i$. Thus, $\bigcup_{i \in I} U_i$ is downwards-closed.
3. Suppose $\{U_i\}_{i \in I}$ is a collection of downwards-closed sets. Let $x \in \bigcap_{i \in I} U_i$ and $y \in P$ such that $y \leq x$. Then for each $i \in I$, $x \in U_i$, and so $y \in U_i$. Thus $y \in \bigcap_{i \in I} U_i$, and so $\bigcap_{i \in I} U_i$ is a downwards-closed set.

Notice that this topology has a curious feature: an *arbitrary* intersection of open sets is still open. This is not true, for instance, in the metric topology on $\mathbb{R}^n$.

Let us consider a specific poset — the power set of $\{1, 2\}$ ordered by inclusion. We can draw the order relation as arrows:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr] & \{1\} \arrow[d] \\ \{2\} \arrow[r] & \{1,2\} \end{tikzcd}$$

Since this poset is finite, we can also draw all of the corresponding non-empty open sets.



This topology is not trivial in any way — it encodes information about the original order relation. Enough, in fact, that we can recover the poset structure from the topology.

Exercise 1.9. Let $(P, \leq)$ and $(Q, \prec)$ be posets. Show that a map $f : P \to Q$ is monotone if and only if it is continuous with respect to the associated topologies.

Example 1.10. Let us describe a different topology on $\mathbb{R}^n$. We can consider the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ in n-variables. Let $I \subset \mathbb{R}[x_1, \ldots, x_n]$ be an ideal of this ring. We can define a corresponding subset of $\mathbb{R}^n$, the *vanishing set* of I, to be

$$V(I) := \{a \in \mathbb{R}^n \mid p(a) = 0 \ \ \forall p \in I\}.$$

Let us explore the behavior of the $V(I)$ under intersections and unions.

- Given I, J ideals, we can compute that

$$V(I) \cap V(J) = V(I + J)$$

  where $I + J$ is the ideal consisting of elements of the form $p + q$ for $p \in I$ and $q \in J$. More generally, for an arbitrary collection $\{I_s\}_{s \in S}$ of ideals, we have

$$\bigcap_{s \in S} V(I_s) = V\left(\left\langle \bigcup_{s \in S} I_s \right\rangle\right)$$

  where $\langle \bigcup_{s \in S} I_s \rangle$ denotes the ideal generated by all of the elements in the I_s.

- For a finite set S and a collection of ideals $\{I_s\}_{s \in S}$, we have[6]

$$\bigcup_{s \in S} V(I) = V\left(\prod_{s \in S} I_s\right).$$

[6] **Exercise.** Say what goes wrong here if we consider an infinite set of ideals.

This tells us that the collection of zero-sets of ideals is closed under arbitrary intersection and finite union — the opposite of what we want for a topology. However, this is a simple fix for this issue. We define the *Zariski topology* on $\mathbb{R}^n$ to be the topology whose open sets are of the form $\mathbb{R}^n \setminus V(I)$ for some ideal $I \subset \mathbb{R}[x_1, \ldots, x_n]$.

Definition 1.11. Let τ_1 and τ_2 be two topologies on a set X. If $\mathrm{id}_X : (X, \tau_2) \to (X, \tau_1)$ is continuous, we say that τ_1 is *coarser* than τ_2 or that τ_2 is *finer* than τ_1.

Examples. Some interesting examples of coarseness/fineness are the most extreme. Let X be a set

1. Define a topology τ_{dis} on X by declaring *every* subset of X to be an element of τ_{dis}. We call this the *discrete topology* on X. This is the finest possible topology on X, and it has a very interesting property. Let (Y, τ_Y) be any topological space, and let $f : X \to Y$ be any map of underlying sets. Then $f : (X, \tau_{dis}) \to (Y, \tau_Y)$ is continuous.
2. Define a topology τ_{ind} on X by $\tau_{ind} := \{\emptyset, X\}$. We call this the *indiscrete topology* on X – the coarsest possible topology on X. For any topological space (Y, τ_Y) and any map of sets $f : Y \to X$, the map $f : (Y, \tau_Y) \to (X, \tau_{ind})$ is continuous.

These two examples are *dual* to one another, in a sense that can be made explicit using category theory.

Exercise 1.12. Show that the following statements are equivalent:

1. $\tau_1 \subset \tau_2$
2. τ_1 is coarser than τ_2.

We now want a way to uniquely specify a topology on X by giving a simpler collection of sets. We will introduce two such notions, one stronger than the other.

Exercise 1.13. Let X be a set and let I be a set of topologies on X. Then

$$\gamma := \bigcap_{\tau \in I} \tau$$

is a topology on X.

Definition 1.14. Let X be a set, and $\mathcal{B} \subset \mathbb{P}(X)$ be a subset of the power set. Set

$$I := \{\tau \subset \mathbb{P}(X) \mid \mathcal{B} \subset \tau \text{ and } \tau \text{ is a topology on } X\}.$$

We define

$$\tau_{\mathcal{B}} := \bigcap_{\tau \in I} \tau$$

to be the *topology generated by* $\mathcal{B}$.

Definition 1.15. Let X be a set, and let τ_X be a topology on X. We call a subset $\mathcal{B} \subset \tau_X$ a *basis* of τ_X if every element of τ_X is a (possibly empty) union of elements of $\mathcal{B}$.

Proposition 1.16. *Let (X, τ_X) be a topological space, and let $\mathcal{B} \subset \tau_X$. Then $\mathcal{B}$ is a basis for τ_X if and only if, for every $U \in \tau_X$ and every $x \in U$, there is a $V \in \mathcal{B}$ such that $x \in V$ and $V \subset U$.*

Proof. First suppose that $\mathcal{B}$ is a basis for τ. Let $U \in \tau_X$ and $x \in U$. Then, in particular, there is a set $\{V_i\}_{i \in I}$ of elements in $\mathcal{B}$ such that

$$\bigcup_{i \in I} V_i = U$$

So, for at least one $i \in I$, $x \in V_i$, and every $V_i \subset U$. Therefore, our criterion is fulfilled.

Now suppose our criterion is fulfilled. Let $U \in \tau_X$. For each $x \in U$, let $V_x \in \mathcal{B}$ be a set such that $x \in V_x \subset U$. It is then immediate from the definition that

$$U = \bigcup_{x \in U} V_x$$

so $\mathcal{B}$ is a basis for τ_X. $\square$

Examples 1.17.

1. Let (X, d) be a metric space. The set of open balls in X forms a basis of the topology induced by d.
2. $\mathbb{R}^n$ actually has an even smaller basis: the set of open balls with rational radii about points with rational coordinates.

We can also make use of bases to more efficiently check when maps are continuous:

Exercise 1.18. Suppose (X, τ_X) and (Y, τ_Y) are topological spaces, $\mathcal{B}$ is a basis of τ_Y, and $f : X \to Y$ is a map of sets. Then f is continuous if and only if, for every $U \in \mathcal{B}$ $f^{-1}(U)$ is open.

We conclude with a criterion for determining when a collection of sets $\mathcal{B}$ is a basis for *some* topology:

Exercise 1.19. Let X be a set, and $\mathcal{B} \subset \mathbb{P}(X)$. Suppose X can be written as a union of elements of $\mathcal{B}$ (we say $\mathcal{B}$ *covers* X) and that, for every $U, V \in \mathcal{B}$ and $x \in U \cap V$, there is a set $W \in \mathcal{B}$ with $W \subset U \cap V$ such that $x \in W$. Then $\mathcal{B}$ is the basis of a topology.

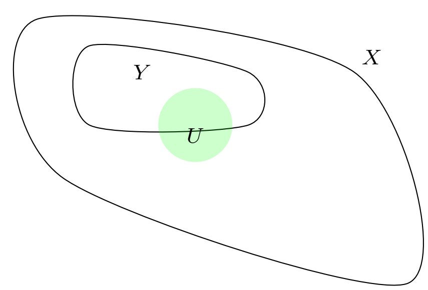
Given a topological space (X, τ_X), and a subset $A \subset X$, there are three further constructions we will make use of.

Definition 1.20. Define a subset $\overline{A} \subset X$ to be the intersection of all closed subsets containing A. We call $\overline{A}$ the *closure of A in X*. This is an intersection of closed sets, and thus is closed. By construction $A \subseteq \overline{A}$.

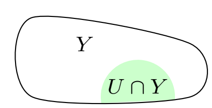
Definition 1.21. We define a subset $\mathring{A} \subset A$, the *interior* of A, to be $X \setminus \overline{(X \setminus A)}$. This is the complement of a closed set, and thus is open.

Definition 1.22. We define the *boundary* of A to be the intersection $\partial A := \overline{A} \cap \overline{(X \setminus A)}$.

A schematic depiction of the open sets in the supspace topology is as follows. In the first drawing we have a space X, a subset Y, and an open set U of X.



In the second, we have the corresponding open subset $Y \cap U$ of Y in the subspace topology



1.2 *Building spaces*

Having established our basic definitions, we now make a brief interlude to discuss some ways of constructing new topological spaces from old ones. We already have quite a large class of topological spaces — those which arise as metric spaces — however, for more general applications, we will want to construct topological spaces directly from other topological spaces.

Construction 1.23. Let (X, τ_X) be a topological space and let $Y \subset X$ be a subset. The topology τ_X on X induces a topology τ_Y on Y called the *subspace topology* as follows.

We define

$$\tau_Y := \{U \cap Y \mid U \in \tau_X\}.$$

To see that (Y, τ_Y) is a topological space, we first note that $Y = X \cap Y \in \tau_Y$ and $\emptyset = \emptyset \cap Y \in \tau_Y$. Suppose we have a set of open sets $\{V_i\}_{i \in I}$ in τ_Y. For each i, choose[7] a $U_i \in \tau_X$ such that $U_i \cap Y = V_i$. Then $U := \bigcup_{i \in I} U_i$ is in τ_X, and thus

$$\bigcup_{i \in I} V_i = \bigcup_{i \in I} Y \cap U_i = Y \cap U \in \tau_Y.$$

[7] This requires the axiom of choice.

Finally, for a finite collection $\{V_i\}_{i=1}^n$ of sets in τ_Y, we again choose[8] $U_i \in \tau_X$ with $U_i \cap Y = V_i$, and note that

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (Y \cap U_i) = Y \cap \bigcap_{i=1}^n U_i \in \tau_Y.$$

Thus, τ_Y is a topology on Y.

[8] This doesn't.

The subspace topology is, in a sense, the coarsest topology on X such that $Y \to X$ is continuous, as the next lemma makes clear.

Lemma 1.24. *Let (X, τ_X) be a topological space, let $Y \subset X$, and let τ_Y denote the subspace topology on Y. Then for any topology γ on Y such that the inclusion $\iota : (Y, \gamma) \to (X, \tau_X)$ is continuous, the identity map $\mathrm{id}_Y : (Y, \gamma) \to (Y, \tau_Y)$ is continuous.*

Proof. Let $V \in \tau_Y$. Then there is a $U \in \tau_X$ with $U \cap Y = V$. However, $\iota^{-1}(U) = Y \cap U$. Thus, since $\iota : (Y, \gamma) \to (X, \tau_X)$ is continuous, $V \in \gamma$. Therefore $\tau_X \subset \gamma$. □

Lemma 1.25. *Let (X, τ_X) be a topological space, $Y \subset X$, and τ_Y the subspace topology on Y. Denote the inclusion $\iota : Y \hookrightarrow X$. Let (Z, τ_Z) be a topological space, and $f : Z \to Y$ a map of sets. Then f is continuous if and only if $\iota \circ f$ is continuous.*

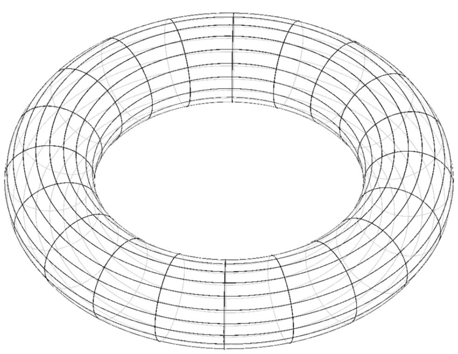
Proof. Left as an exercise to the reader. □

Example 1.26. Consider the unit circle $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$. The Euclidean metric on $\mathbb{R}^2$ induces a topology on $\mathbb{R}^2$ (called the *standard topology* on $\mathbb{R}^2$), and we can equip S^1 with the subspace topology.

1.2.1 The product topology

We now come to a slightly more subtle construction. We want to define topologies on the cartesian products of topological spaces $\prod_{i \in I} X_i$. However, in the case where the product has an infinite number of factors, care must be taken to get a sensible definition.

A nice example of the product topology is the *torus*. This is the product $S^1 \times S^1$, equipped with the product topology. One can also view the torus as a subspace of $\mathbb{R}^3$, and the subspace topology agrees with the product topology in this case. When drawn, the torus is a surface which looks a bit like a doughnut:



Construction 1.27. Let I be a set, and $\{(X_i, \tau_i)\}_{i \in I}$ be a collection of topological spaces indexed by I. We define a topology on the set

$$X := \prod_{i \in I} X_i$$

as follows. Define a set

$$\mathcal{B} := \left\{ \prod_{i \in I} U_i \;\middle|\; \substack{U_i \in \tau_i \text{ and} \\ U_i = X \text{ for all but a finite} \\ \text{number of } i \in I} \right\}$$

One can easily check that $\mathcal{B}$ satisfies the criteria from Proposition 1.19, and thus, defines a topology τ_X on X. We call this topology the *product topology*.

Example 1.28. Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be equipped with the standard topologies, and denote the product topology on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ by $\tau_{n\times m}$. Denote the standard topology on $\mathbb{R}^{n+m}$ by γ. It is immediate from the definitions that the identity map defines a homeomorphism $(\mathbb{R}^{n+m}, \tau_{n\times m}) \xrightarrow{\cong} (\mathbb{R}^{n+m}, \gamma)$, so $\tau_{n\times m} = \gamma$.

Exercise 1.29. Formulate and prove the statement that the product topology is the coarsest topology on $\prod_{i\in I} X_i$ such that the projections

$$\pi_j : \prod_{i\in I} X_i \to X_j$$

are all continuous. Prove the universal property of the product topology.

Proposition 1.30 (Universal Property of the Product)**.** *Let $\{X_i\}_{i\in I}$ be a set of topological spaces. Given a topological space Y and a set of continuous maps $f_i : Y \to X_i$, there is a unique continuous map*

$$f : Y \longrightarrow \prod_{i\in I} X_i$$

such that $\pi_i \circ f = f_i$ for all $i \in I$.

Construction 1.31. Let $\{(X_i, \tau_i)\}_{i\in I}$ be a collection of topological spaces, we define a topology τ on $X := \coprod_{i\in I} X_i$ called the *coproduct topology* or *disjoint union topology* by setting

$$\tau = \{U \subset X \mid U \cap X_i \in \tau_i \forall i \in I\}.$$

The verification that this is indeed a topology is left as an exercise to the reader.

Exercise 1.32. Formulate and prove the statement that the coproduct topology is the finest topology on $\coprod_{i\in I} X_i$ such that all of the inclusions

$$\iota_j : X_j \to \coprod_{i\in I} X_i$$

are continuous. Formulate an prove a universal property for the coproduct topology.[9]

[9] Hint: this will be *dual* to the universal property of the product in the sense that it looks the same, but all the maps go in the opposite direction.

Definition 1.33. Let X, Y, and Z be topological spaces, and suppose we are given continuous maps $f : X \to Z$ and $g : Y \to Z$. We can define a new space, $X \times ZY$, called the *pullback* or *fibre product* to be

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subset X \times Y$$

equipped with the subspace topology. The pullback comes equipped with continuous *projection maps* $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$.

Exercise 1.34. Prove the universal property of the pullback. If W is a topological space, and $u_X : W \to X$, $u_Y : W \to Y$ are continuous maps such that $f \circ u_X = g \circ u_Y$, then there is a unique map $u : W \to X \times_Z Y$ such that $p_X \circ u = u_X$ and $p_Y \circ u = u_Y$.[10]

[10] This would more conventionally be written

1.2.2 The Quotient topology

Our final construction is perhaps the most important to the study of algebraic topology. It is a way for us to *glue* two spaces to obtain a new space. As in the overture, we will try to view our spaces, wherever possible, as being glued together out of simple, well-understood pieces.

Construction 1.35. Let (X, τ_X) be a topological space, and $\sim$ an equivalence relation on X. There is a canonical map of sets $\pi : X \to X_{/\sim}$ from X to the quotient set. We now define the *quotient topology*

$$\tau_\sim := \left\{ U \subset X_{/\sim} \mid \pi^{-1}(U) \in \tau_X \right\}.$$

We claim that $(X_{/\sim}, \tau_\sim)$ is a topological space. We leave the verification that this is, in fact a topology to the reader

Exercise 1.36. Rigorously formulate and prove the statement that '$\tau_\sim$ is the finest topology on $X_{/\sim}$ such that $\pi : X \to X_{/\sim}$ is continuous'.

Proposition 1.37 (The universal property of the quotient topology)**.** *Let X be a topological space, and $\sim$ an equivalence relation on X. Let $f : X \to Y$ be a continuous map such that for $x_1, x_2 \in X$, if $x_1 \sim x_2$, then $f(x_1) = f(x_2)$. Then there is a unique continuous map $\overline{f} : X_{/\sim} \to Y$ such that $\overline{f} \circ \pi = f$.*

Proof. We first show that the underlying map of sets exists and is uniquely determined. First uniqueness: suppose that $\overline{f} : X_{/\sim} \to Y$ is such a map. Then, for any equivalence class $[x] \in X_{/\sim}$, the condition that $\overline{f} \circ \pi = f$ requires that

$$\overline{f}([x]) = \overline{f}(\pi(x)) = f(x)$$

so that f is uniquely determined. To see that $\overline{f}$ is well-defined, we need only note that if $[x] = [y]$, then $x \sim y$, and thus $f(x) = f(y)$ by hypothesis. Thus, there is a unique map of underlying sets satisfying the desired property.

To see continuity, let $U \subset Y$ be open. Then since $\overline{f} \circ \pi = f$, we have

$$f^{-1}(U) = \pi^{-1}(\overline{f}^{-1}(U)).$$

Since f is continuous by hypothesis, this means that $\pi^{-1}(\overline{f}^{-1}(U))$ is open, and so $\overline{f}^{-1}(U)$ is open. Thus $\overline{f}$ is continuous, completing the proof. $\square$

Examples 1.38.

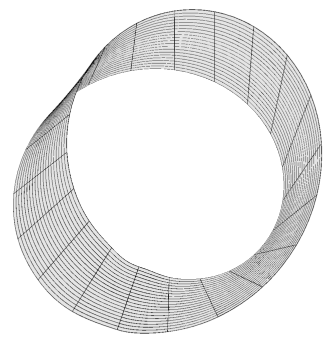
1. Consider $[0,1] \subset \mathbb{R}$ equipped with the subspace topology, and define an equivalence relation on $[0,1]$ by setting $0 \sim 1$. We then get a quotient topological space $S = [0,1]_{/\sim}$ with topology $\tau_\sim$. Consider the map

$$\begin{aligned} f : [0,1] &\to \mathbb{C} = \mathbb{R}^2 \\ t &\mapsto \exp(2\pi i t) \end{aligned}$$

   This map is well-defined, continuous, and has image $S^1 \subset \mathbb{R}^2$. Moreover, f is a bijection onto its image, and one can check (using a basis for the standard topology) that it is a homeomorphism. Therefore, $(S, \tau_\sim)$ is homeomorphic to S^1.

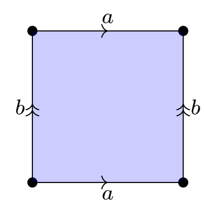
2. Let $X = [0,1] \times [-1,1] \subset \mathbb{R}^2$, equipped with the subspace topology. Define an equivalence relation on X by setting $(0, x) \sim (1, -x)$. The quotient space $(X_{/\sim}, \tau_\sim)$ is called the *Möbius band*.[11]

[11] The Möbius band is a rare example of a *non-orientable* surface which is easy to visualize:



Indeed, one can easily construct a Möbius band from paper or fabric.

We will often draw pictures meant to convey such equivalence relations. For example, the gluing described in the exercise might be drawn as



The labels tell us which intervals should be identified homeomorphically, and the arrows tell us whether to reverse the direction when we glue or not.

A more involved example of quotient spaces is that of the *real projective spaces*.

Construction 1.39. Consider $\mathbb{R}^{n+1} \setminus \{0\}$ with the subspace topology coming from $\mathbb{R}^{n+1}$. Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by $x \sim \lambda x$ for every $\lambda \in \mathbb{R} \setminus \{0\}$. We define the *$n$-dimensional real projective space* $\mathbb{R}P^n$ to be the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})_{/\sim}$.

Note that we can also consider $\mathbb{R}P^n$ as the quotient of the unit sphere S^n by the equivalence relation $x = -x$.

Lemma 1.40. *The spaces S^1 and $\mathbb{R}P^1$ are homeomorphic.*

Proof. Let $p : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}P^1$ be the quotient map, and let $q : [0,1] \to S^1 = [0,1]_{/0\sim 1}$ be the quotient map.

We first define a map $f : [0,1] \to \mathbb{R}P^1$ by $x \mapsto p(\exp(\pi i x))$. It is clear that, as a composite of continuous maps, f is continuous. Note that, if $x \in (0,1)$, then there is no $y \in [0,1]$ such that $-\exp(\pi i x) = \exp(\pi i y)$, so f is injective on $[0,1]$. Moreover, $f(0) = f(1)$. Consequently, f descends to an injective continuous map $f : S^1 \to \mathbb{R}P^1$. Since every element of $\mathbb{R}P^n$ has a representative in the upper half-circle, this map is surjective, and thus is a bijection.

Denote the inverse of f by f^{-1}. The map f^{-1} sends an element in the upper half-circle $x \in C$ to $q(\frac{\ln(x)}{\pi i})$. The assignment $x \mapsto \frac{\ln(x)}{\pi i}$ is a continuous assignment from the upper half-circle to $[0,1]$, and therefore, f^{-1} is continuous. $\square$

Exercise 1.41. Consider the unit rectangle $I \times I$, and define an equivalence relation on $I \times I$ by $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, t)$. Show that the quotient space $(I \times I)_{/\sim}$ is homeomorphic to $S^1 \times S^1$.

1.3 Hausdorff Spaces

Now that we have the basic tools necessary to construct topological spaces, we can begin exploring their properties. Of particular interest is the degree to which our concepts and intuitions from analysis carry over to topologies.

Definition 1.42. Let (X, τ_X) be a topological space. An *open cover* of X is a collection $\mathcal{U} := \{U_i\}_{I \in I} \subset \tau_X$ of open subsets of X such that $\bigcup_{i \in I} U_i = X$. We say that a cover $\mathcal{V}$ is a *subcover* of a cover $\mathcal{U}$ if $\mathcal{V} \subset \mathcal{U}$

Definition 1.43. Let (X, τ_X) be a topological space. We say that X is *compact* if every cover $\mathcal{U}$ of X admits a finite subcover $\mathcal{V}$.

Intuitively, compact sets should be thought of as playing the role of 'closed and sufficiently small' in topology. However, this intuition is significantly complicated by some pathological counterexamples. We do, however, have the following nice property.

Lemma 1.44. *Let (X, τ_X) be a compact topological space, and let $Y \subset X$ be a closed set. Then Y is compact.*

Proof. Let $\mathcal{U} := \{U_i\}_{i \in I}$ be an open cover of Y. Since Y is closed, the collection $\mathcal{V} := \mathcal{U} \cup \{X \setminus Y\}$ is an open cover of X, and therefore admits a finite subcover $U_1, \dots, U_k, X \setminus Y$. Since $(X \setminus Y) \cap Y = \emptyset$, this means that $U_1, \dots, U_k$ is a finite subcover of Y. □

It is a classical theorem of analysis that, space (X, d), every compact subset is closed and bounded. In the case of $\mathbb{R}^n$ with the Euclidean metric, this can be strengthened to an 'if and only if' statement (the *Heine-Borel Theorem*). However, in topological spaces, things become rather stranger.

Example 1.45. Let X be a set, and $\tau_X := \{\emptyset, X\}$ be the indiscrete topology on X. Let $x \in X$. Then $\{x\}$ is clearly a compact subset of X (the only open covers are finite), however, if X has more than one point, then $\{x\}$ is not the complement of either X or $\emptyset$, and therefore cannot be closed.

To avoid this particular pathology, we need to impose some condition on our topological spaces to make them better match our intuition.

Definition 1.46. A topological space (X, τ_X) is called a *Hausdorff space* (or just *Hausdorff*) if, for every two distinct points $x, y \in X$, there exist open sets $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$. We say that a Hausdorff space *separates points.*

Lemma 1.47. *Let (X, τ_X) be a Hausdorff space. Then every compact subset of X is closed.*

Proof. Let $Y \subset X$ be a closed subset of X. We show the equivalent statement that $X \setminus Y$ is open. Fix a point $x \in X \setminus Y$; for each point $y \in Y$ choose open sets $y \in V_y$ and $x \in U_y$ such that $V_y \cap U_y = \emptyset$. The collection $\{V_y\}_{y \in Y}$ is an open cover of Y, and therefore admits a finite subcover $V_{y_1}, \dots, V_{y_n}$. By construction the intersection $U_x := \bigcap_{i=1}^n U_{y_i}$ has empty intersection with $\bigcup_{i=1}^n (V_{y_i})$ and thus has empty intersection with Y. But, as a finite intersection of open sets, U_x is an open set, and since $x \in U_{y_i}$ for all $1 \leq i \leq n$, $x \in U_x$. Therefore, U_x is an open neighborhood of x in X, and $U_x \subset X \setminus Y$.

Construct such a U_x for every $x \in X \setminus$. Then $X \setminus Y = \bigcup_{x \in X \setminus Y} U_x$ is open. □

Proposition 1.48. *Let (X, τ_X) be a Hausdorff space, and $Y \subset X$ a subspace. Then Y is Hausdorff.*

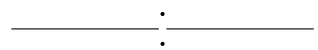
Proof. Exercise. □

Proposition 1.49. *Let $f : X \to Y$ be a continuous map of topological spaces such that X is compact. Then $f(X)$ is compact.*

Proof. Let $\{U_\alpha\}$ be a cover of $f(X)$ by open sets of Y. Then by continuity, $\{f^{-1}(U_\alpha)\}$ is a cover of X by open sets of X. Since X is compact, there is a finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_k)$. Since $f(f^{-1}(U_i)) \subseteq U_i$, this means that the sets $U_1, \dots, U_k$ cover $f(X)$, and thus, the cover $\{U_\alpha\}$ admits a finite subcover. □

Example. Every metric space is Hausdorff.

Example (Non-example)**.** Let (Y, τ_Y) be the topological space $\mathbb{R} \times \{0,1\}$, where $\{0,1\}$ is equipped with the discrete topology. Note that Y can also be identified with $\mathbb{R} \amalg \mathbb{R}$. Define an equivalence relation on Y by $(x, 0) \sim (x, 1)$ for all $x \neq 0$. The quotient space $(Y_{/\sim}, \tau_\sim)$ is a standard counterexample in topology, called the *line with two origins.* Schematically, it looks like



If we label the copies of the origin 0_1 and 0_2, it is not hard to see that every open ball $B_r(0_1)$ of 0_1 must intersect every open ball $B_R(0_2)$, and thus that any open sets $0_1 \in V$ and $0_2 \in U$ with have non-empty intersection.

There are two important theorems whose proofs we omit here, but which have an outsize impact on the study of topology. The first pertains to the topology of $\mathbb{R}^n$.

($\star$) **Theorem 1.50** (Heine-Borel). *For a subset $C \subset \mathbb{R}^n$ the following are equivalent.*

1. *The set C is closed and bounded.*
2. *The set C is compact.*

The second theorem explicates the relation of products to compactness.[12]

($\star$) **Theorem 1.51** (Tychonoff). *Let $\{X_i\}_{i\in I}$ be a set of compact spaces. Then $\prod_{i\in I} X_i$ is compact.*

[12] The Tychonoff Theorem is, in fact, equivalent to the *axiom of choice* (at least in the usual Zermelo-Fraenkel axioms).

Axiom of Choice. *Let I be a set whose elements are nonempty sets. Then there exists a function*

$$f : \longrightarrow \bigcup_{i\in I} i$$

such that for all $i \in I$, $f(i) \in i$.

Exercise 1.52. Let (X, d_X) be a metric space. Show that X endowed with the metric topology is Hausdorff.

As in analysis, we can consider limits of sequences, and limit points of sets in topological spaces. In sufficiently nice topological spaces, limits behave much as we might expect, but in general, limits in topological spaces can be a lot odder.

Definition 1.53. Let X be a topological space, and $Y \subset X$ a subset of X. A *limit point of Y in X* is an element $x \in X$ such that, for every open neighborhood U of x, $U \cap Y \neq \varnothing$. A *sequence in X* is a function $x_{(-)} : \mathbb{N} \to X$, written as $\{x_i\}_{i\in\mathbb{N}}$. A *limit* of a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X is an element $x \in X$ such that, for any open neighborhood U of x in X, there exists an $n \in \mathbb{N}$ such that, for all $m > n$, $x_i \in U$. We call a sequence *convergent* if it has a limit.

Exercise 1.54. Show that a limit of a sequence $x_{(-)} : \mathbb{N} \to X$ is a limit point of the set $\{x_i\}_{i\in\mathbb{N}}$. Is the converse true?

Lemma 1.55. *Let X be a Hausdorff space, and $\{x_i\}_{i\in\mathbb{N}}$ a sequence in X. If x and y are limits of $\{x_i\}_{i\in\mathbb{N}}$, then $x = y$.*

Proof. Since x and y are limits of $\{x_i\}_{i\in\mathbb{N}}$, for any neighborhoods U and V of x and y, we have that $U \cap V$ contains infinitely many elements of $\{x_i\}_{i\in\mathbb{N}}$. As such, every two open neighborhoods of x and y intersect non-trivially, and so since X is Hausdorff, $x = y$. $\square$

Exercise 1.56. Give an example of a non-Hausdorff space in which limits of sequences are not unique.

Definition 1.57. We call a space X *sequentially compact* if every sequence $\{x_i\}_{i\in\mathbb{N}}$ in X has a convergent subsequence.

Exercise 1.58. Show that a metric space is sequentially compact if and only if it is compact.

The Hausdorff property is one of what are sometimes called *separation axioms*. Though there are many such, we will only list one more, and then describe two important consequences of this axiom.

Definition 1.59. A topological space X is called *normal* if, given two closed sets $C, D \subset X$ such that $C \cap D = \varnothing$, there are open sets $U, V \subset X$ such that $C \subset U$, $D \subset V$, and $U \cap V = \varnothing$. It is sometimes said that in normal spaces, *closed sets can be separated by neighborhoods.*

($\star$) **Theorem 1.60** (Urysohn's Lemma)**.** *Let X be a topological space. The following are equivalent.*

1. *The space X is normal.*

2. *For every pair $C, D \subset X$ of disjoint closed subsets of X, there exists a continuous function*
$$f : X \longrightarrow [0, 1]$$
such that $f|_C = 0$ and $f|_D = 1$.

Urysohn's Lemma can be used to prove our final important theorem of this section.

($\star$) **Theorem 1.61** (Tietze Extension Theorem)**.** *Let X be a normal space and let $A \subset X$ be a closed subspace. Given a continuous function $f : A \to \mathbb{R}$, there is a continuous function $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f}|_A = f$.*

1.4 Connectedness and path-connectedness

Above, we constructed the *disjoint union* of topological spaces, $X \amalg Y$, which can be viewed as consisting of two separate 'parts': X and Y. However, given a topological space (X, τ_X), we do not yet have any way of testing whether it has been built in this way. Such a criterion is provided by notions of *connectedness*.

Definition 1.62. Let (X, τ_X) be a topological space. If, for every pair of non-empty open sets $U, V \in \tau_X$ such that $U \cup V = X$, the intersection $U \cap V \neq \emptyset$, we say that X is connected. A *connected component* of X is a maximal connected subspace $Y \subset X$.

In a sense made precise by the following proposition, connectedness measures 'discreteness of maps out of X'.

Proposition 1.63. *Let (X, τ_X) be a topological space. The following are equivalent*

1. *(X, τ_X) is connected.*

2. *Every continuous map $f : X \to Y$ to a discrete space is constant.*

Proof. We first show 2. $\Rightarrow$ 1. Suppose that (X, τ_X) is not connected. Then there are two non-empty sets $U, V \in \tau_X$ with $U \cup V = X$ and $U \cap V = \emptyset$. Define a map to $f : X \to \{0, 1\}$ by sending every element of U to 0, and every element of V to 1. It is immediate from the definition that f is continuous with respect to the discrete topology on $\{0, 1\}$ and non-constant.

We now show 1. $\Rightarrow$ 2. Suppose that there is a continuous, non-constant map $f : X \to Y$, where Y is equipped with the discrete topology. In particular, choose two distinct elements y_0 and y_1 in Y such that both are in the image of f. Choose any map of sets $p : Y \to \{0,1\}$ such that $p(y_0) = 0$ and $p(y_1) = 1$. Since this is continuous with respect to the discrete topologies, we get a non-constant continuous map $p \circ f : X \to \{0,1\}$. Since this is continuous, the sets $U := (p \circ f)^{-1}(0)$ and $V := (p \circ f)^{-1}(1)$ are open. Since $p \circ f$ is non-constant, both U and V are non-empty. By definition $U \cup V = X$ and $U \cap V \neq X$. □

Definition 1.64. Let (X, τ_X) be a topological space, and let $A \subset X$. We define the *closure* of A to be the subset $\overline{A} \subset X$ which is the intersection of all closed subsets of X which contain A.

Lemma 1.65. *Let $f : X \to Y$ be a continuous map, and let X be connected. Then $f(X) \subset Y$ is connected.*

Proof. Exercise. □

Lemma 1.66. *Let (X, τ_X) be a topological space, and $A \subset X$. For every element $x \in \overline{A}$ and every open $x \in U$, $U \cap A \neq \emptyset$.*

Proof. Left as an exercise to the reader. □

Proposition 1.67. *Let (X, τ_X) be a topological space, and A a connected subset. If $B \subset X$ such that $A \subset B \subset \overline{A}$, then B is connected.*

Proof. Suppose that there were two open sets $U, V \subset X$ such that $U \cup V = B$ and $U \cap V \cap B = \emptyset$. Since A is connected, we must then have that $A \subset U$ or $A \subset V$. WLOG, assume $A \subset U$. But then, for $b \in B$, we have that $b \in \overline{A}$ and V is an open subset containing b. Therefore by lemma 1.66, $V \cap A \neq \emptyset$, and thus, $V \cap U \cap B \neq \emptyset$, which is a contradiction. □

So if connectedness measures the discreteness of maps *out* of X, can we also measure the discreteness of maps *into* X?

Definition 1.68. A *path* in a topological space (X, τ_X) is a continuous map $p : [0,1] \to X$, where $[0,1]$ is equipped with the subspace topology inherited from $\mathbb{R}$. We say that p is a *path from x to y* if $p(0) = x$ and $p(1) = y$.

We define an equivalence relation on X by $x \sim y$ if and only if there exists a path in X from x to y. A *path component of x* is an equivalence class $[x] \in X_{/\sim}$, viewed as a subspace $[x] \subset X$. We say that X is *path connected* if $X_{/\sim}$ is the one-point space.

Exercise 1.69. Show that $\sim$ is indeed an equivalence relation on X.

Exercise 1.70. Show that the interval $[0,1]$ is connected.

Proposition 1.71. *Let (X, τ_X) be a path-connected topological space. Then X is connected.*

Proof. Suppose X is not connected. Then there exists a continous map $f : X \to \{0,1\}$ (where $\{0,1\}$ is equipped with the discrete topology) such that f is non-constant. Let $x \in f^{-1}(0)$ and $y \in f^{-1}(1)$. A path $p : [0,1] \to X$ from x to y would then yield a continuous, non-constant map $f \circ p : [0,1] \to \{0,1\}$. Since $[0,1]$ is connected, this cannot occur, and thus, X is not path connected. □

Warning 1.72. The converse of Proposition 1.71 is *not* true. There are connected spaces which are not path-connected. We need to make additional assumptions about our space X if we want connectedness and path-connectedness to be equivalent.

A standard counterexample in topology is the *topologist's sine curve.* Let $X \subset \mathbb{R}^2$ be the collection of all points $(x, \sin(\frac{1}{x}))$ for $x > 0$, together with the point $(0,0)$. This inherits a topology from $\mathbb{R}^2$ (indeed, this topology is even Hausdorff).

Lemma. *The topologist's sine curve is connected.*

Proof. We simply need note that the subspace $A := \{(x, \sin(\frac{1}{x})) \mid x > 0\}$ is path-connected, and thus connected. Moreover $A \subset X \subset \overline{A}$. Therefore, by Proposition 1.67, X is connected. □

Lemma. *The topologist's sine curve is not path connected.*

Proof. Suppose we have a path $p : [0,1] \to X$ going from $(1, \sin(1))$ to $(0,0)$. Consider the component functions $p_x, p_y : [0,1] \to \mathbb{R}$. Since p_x is continuous, its image is connected, and therefore is the interval $[0,1]$. But then, p is the map $t \mapsto (t, \sin(\frac{1}{t}))$. But for every $\delta > 0$, there is a $0 < t < \delta$ such that $\sin(\frac{1}{t}) = 1$, i.e $|p(t) - (0,0)| > 1$. Therefore, p cannot be continuous. □

Definition 1.73. We call a topological space (X, τ_X) *locally path-connected* if, for every $x \in X$ and every open U containing x, there is an open V with $x \in V \subset U$ such that V is path connected.

Proposition 1.74. *If X is a connected, locally path-connected topological space, then X is path-connected.*

Proof. Exercise. □

Exercise 1.75. This exercise will serve to prove the *Intermediate Value Theorem*, familiar from single-variable calculus courses.

1. Prove that a subset $X \subset \mathbb{R}$ is connected if and only if it is an interval of the form $[a,b]$, (a,b), $[a,b)$, or $(a,b]$ for $a \leq b$.
2. Prove the Intermediate Value Theorem: Let $a, b \in \mathbb{R}$ with $a \leq b$, and let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then for any $c \in \mathbb{R}$ such that c lies between $f(a)$ and $f(b)$, there exists a $d \in [a,b]$ such that $f(d) = c$.
3. Prove the 1-dimensional case of the *Brouwer Fixed-point Theorem.* Suppose that $f : [a,b] \to [a,b]$ is continuous. Then f has a *fixed point*, i.e., a point $x \in [a,b]$ such that $f(x) = x$.

2

The fundamental group

In this section, we will seek to understand the fundamental group, covering spaces, and the Seifert Van Kampen Theorem. While there are many questions that could lead to the fundamental group, the one we will focus on here is a deceptively simple one:

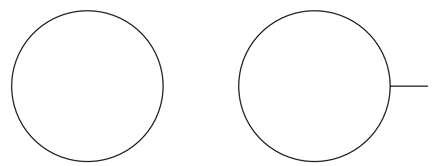
Question: How can we tell when two topological spaces are *not* the same?

Rather counter-intuitively, this question is often substantially *harder* than ascertaining when two topological spaces *are* the same. If two spaces are homeomorphic, one can often play around with maps for a little while until one finds a homeomorphism between them, and then prove it is, indeed, a homeomorphism.

Exercise 2.1. Provide an explicit homeomorphism between the unit square centered on the origin in $\mathbb{R}^2$ and the unit ball centered on the origin in $\mathbb{R}^2$.

On the other hand, showing that two spaces are *not* homeomorphic often requires us to stretch beyond the most direct approach. In most cases, we cannot simply test every possible map to show that it is not a homeomorphism, and so we must appeal to *invariants* — properties, numbers, or mathematical objects we can assign to spaces which do not change under homeomorphism.

The spaces in Example 2.2 look like



Example 2.2. Consider the space S^1 — the unit circle in $\mathbb{R}^2$ — and the space $S^{\vdash}$, which we define to be the union of S^1 and the segment from $(1,0)$ to $(3/2,0)$. I claim that these spaces are not homeomorphic.

To see this, suppose that there were a homeomorphism $f : S^{\vdash} \to S^1$. Then f sends the point $x = (1,0)$ to some point $y = f(x) \in S^1$. Then f would induce a homeomorphism $S^{\vdash} \setminus \{x\} \cong S^1 \setminus \{y\}$. However, $S^{\vdash} \setminus \{x\}$ is not connected, whereas $S^1 \setminus \{y\}$ is connected. Since the number of connected components is invariant under homeomorphism, we see that the two original spaces are not homeomorphic.

This example relies on a number of statements which I will leave as exercises.

Exercise 2.3. Complete the example by proving the following statements.

1. Given a homeomorphism $f : X \to Y$ of topological spaces, and a subspace $Z \subset X$, f induces a homeomorphism $Z \cong f(Z)$.

2. For any point $x \in S^1$, the space $S^1 \setminus \{x\}$ is connected.

3. The number of connected components of a space is a homeomorphism invariant.[1]

[1] This means that if two spaces are homeomorphic they have the same number of connected components.

The invariant we will define and study in this section is rather more powerful, and has applications substantially beyond identifying when spaces are homeomorphic.

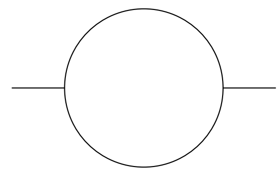
2.1 Homotopy or a first look at the fundamental group

As we saw with $\mathbb{R}P^1$ and S^1, it is often not too hard to explicitly write down a homeomorphism between two spaces. However, to be able to make meaningful statements about topological spaces, it is necessary for us to be able to say when two spaces are *not* the same (i.e. homeomorphic). At first blush this may seem easy. After all, it should be obvious that two spaces are different. Once one starts looking at an example, however, it is not at all clear how one should go about distinguishing two spaces.

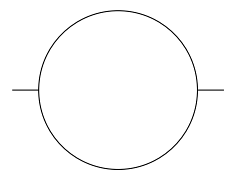
As an example, consider $\mathbb{R}^2$, equipped with the topology induced by the Euclidean metric. and $\mathbb{R}^2 \setminus \{0\}$, equipped with the subspace topology. By inspection, it should be fairly intuitive that these are not homeomorphic spaces, but how do we prove it? The two underlying sets have the same cardinality, and it's not possible to write down every possible continuous map between $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0\}$. So we seem to be stuck.

Paradoxically, the answer comes by considering an even weaker notion of equivalence: *homotopy equivalence*. In loose, intuitive terms, two spaces are homotopy equivalent if one can be 'stretched' or 'shrunk' into another.

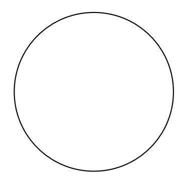
A heuristic depiction of homotopy equivalence might be the following. We consider the space:



This sort of looks like the circle, but they are not homeomorphic. If we remove one of the intersections of the line and the circle, then we get a space with three different path components, but if we remove any single point from the circle, we get a space with only one path component. However, If we are allowed to shrink without tearing our space, we can shrink it to



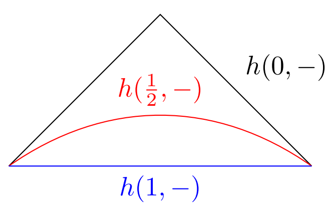
and then to



yielding the circle. The aim of this section will be to prove rigorous results about this kind of a process.

Definition 2.4. Let X and Y be topological spaces, and $f, g : X \to Y$ continuous maps. A *homotopy* from f to g is a continuous map $h : [0,1] \times X \to Y$ (where $[0,1]$ is equipped with the subspace topology inherited from the Euclidean topology on $\mathbb{R}$) such that $h(0,-) : X \to Y$ is the map f and $h(1,-) : X \to Y$ is the map g. If there is a homotopy from f to g, we say that f and g are *homotopic*.

Example 2.5. The image below shows a homotopy $h : [0,1] \times [0,1] \to \mathbb{R}^2$ between paths $[0,1] \to \mathbb{R}^2$.



The idea is that we continuously morph one path into another.

Lemma 2.6. *Let $[a,b] \subset \mathbb{R}$ be an interval equipped with the subspace topology, and let $f, g : X \to Y$ be continuous maps of topological spaces. The following are equivalent:*

1. *There is a homotopy h from f to g.*
2. *There is a map $H : [a,b] \times X \to Y$ such that $H(a,-) = f$ and $H(b,-) = g$.*

Proof. It is immediate that 1. ⇒ 2. Suppose that 2. holds, and we have such a map H. Define a map

$$q : [0,1] \times X \to [a,b] \times X; \quad (t,x) \mapsto (\rho(t),x)$$

where

$$\rho(t) := a + t(b-a)$$

It is easy to verify that q is continuous, and therefore $H \circ q : [0,1] \times X \to Y$ is a continous map. However, by definition $(H \circ q)(0,-) = H(a,-) = f$ and $(H \circ q)(1,-) = H(b,-) = g$. Thus $H \circ q$ is a homotopy from f to g. □

Lemma 2.7. *Denote by* $\mathbf{Top}(X,Y)$ *the set of continuous maps between two topological spaces X and Y. Then the relation*

$$f \sim g \Leftrightarrow f \text{ is homotopic to } g$$

is an equivalence relation.

Proof. First, we show reflexivity. Let $f : X \to Y$ be a continuous map. Since the projection $p_2 : [0,1] \times X \to X$ is continuous, the composite $f \circ p_2$ is as well, and provides a homotopy from f to f.[2]

[2] This is sometimes referred to as the *constant homotopy*.

Second, we show symmetry. Let $h : [0,1] \times X \to Y$ be a homotopy. Define $p : [0,1] \to [0,1]$ to send $t \mapsto 1-t$. This is a homeomorphism (as one can easily verify), and exchanges 0 and 1. Therefore the map $\tilde{h} : [0,1] \times X \to Y$ given by $\tilde{h}(t,x) = h(p(t),x)$ is a homotopy from g to f.

Finally, supose that h is a homotopy from f to g, and k is a homotopy from g to ℓ. We define a map

$$k * h : [0,2] \times X \to Y$$

via

$$(k*h)(t,x) = \begin{cases} h(t,x) & 0 \le t \le 1 \\ k(t-1,x) & 1 \le t \le 2 \end{cases}$$

It is straightforward to verify that this is well-defined and continuous, and therefore, by Lemma 2.6, f is homotopic to ℓ. □

This now allows us to define our notion of homotopy equivalence:

Definition 2.8. Two continous maps $f : X \to Y$ and $g : Y \to X$ are said to be *homotopy inverses* if $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. In this situation, we call f (or g) a *homotopy equivalence*, and say that X and Y are *homotopy equivalent*.

Remark 2.9. It is immediate from the definitions that every homeomorphism is a homotopy equivalence.

Example 2.10. Consider $X := \mathbb{R}^2 \setminus \{0\}$ and S^1, both with the subspace topology inherited from $\mathbb{R}^2$. There is a canonical inclusion $\iota : S^1 \to X$. We now claim that ι is a homotopy equivalence. Define a map

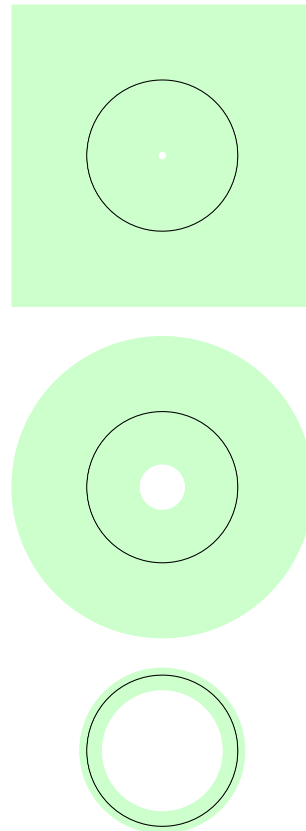
$$r : X \to S^1; \quad x \mapsto \frac{x}{|x|}$$

Since $x \neq 0$, this is well-defined, and it is easy to check that it is continuous. We note that $r \circ \iota : S^1 \to S^1$ is *equal to* the identity on S^1.

In the other direction, we wish to define a homotopy between $\iota \circ r$ and id_X. Define a continuous map

$$H : [0,1] \times X \to X; \quad (t,x) \mapsto \frac{x}{|x|^t}$$

Note $H(0,x) = \frac{x}{|x|^0} = x$, so $H(0,-) = \mathrm{id}_X$, and that $H(1,x) = \frac{x}{|x|^1} = \frac{x}{|x|} = (\iota \circ r)(x)$. Thus, H is a homotopy from id_X to $\iota \circ R$, and ι is a homotopy equivalence.



Examples 2.11. The following examples are quite straightforward, and you should attempt to verify for yourself that they hold:

1. $\mathbb{R}^n$ is homotopy equivalent to the one-point topological space $*$.
2. The Möbius band is homotopy equivalent to S^1.

Definition 2.12. If, as in Example 2.11 (1), a space X is homotopy equivalent to the one-point topological space $*$, we will call X *contractible*.

Lemma 2.13. *Let $h : [0,1] \times X \to Y$ be a homotopy from f to g, and let $p : Y \to Z$ be a continuous map. Then $p \circ h$ is a homotopy from $p \circ f$ to $p \circ g$.*

Proof. Immediate from the definitions. □

So, how can we use homotopies and homotopy equivalences to show that $\mathbb{R}^2 \setminus \{0\}$ is not homeomorphic to $\mathbb{R}^2$? The answer lies in a speciallized invariant: the fundamental group.

Definition 2.14. We call a continuous map $f : [a,b] \to X$ a *loop in* X with *basepoint* $x \in X$ if $f(a) = f(b) = x$. Denote the set of loops in X with basepoint x by $L(X,x)$.

We say that two loops $f, g : [a,b] \to X$ with basepoint x are *based-homotopic* if there is a homotopy $h : [0,1] \times [a,b] \to X$ from f to g such that $h(s,a) = h(s,b) = x$ for all $s \in [0,1]$.

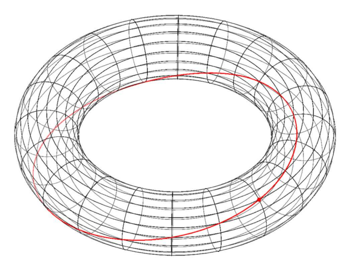
We say that two loops $f : [a,b] \to X$ and $g : [c,d] \to X$ are *equivalent* if there is a homeomorphism $p : [a,b] \to [c,d]$ with $p(a) = c$ and $p(b) = d$ such that f is based-homotopic to $g \circ p$. We write $f \simeq g$ is f and g are equivalent loops.

Exercise 2.15. Show that equivalence of loops is an equivalence relation.

A loop in X is pretty easy to visualize, since it matches our intuition precisely. Let's consider the example of the torus $S^1 \times S^1$. We can define a loop in $S^1 \times S^1$ by

$$\begin{aligned} [0,1] &\longrightarrow S^1 \times S^1 \\ t &\longmapsto (e^{2\pi i t}, e^{2\pi i t}) \end{aligned}$$

If we draw this, we get something like:



Where the path is drawn in red, and the basepoint is represented by a red dot.

Definition 2.16. Let X be a topological space, and $x \in X$ a basepoint. We define a subset $L^1(X, x) \subset L(X, x)$, consisting of those loops in X with basepoint x which are defined on the unit interval $[0, 1]$. We call such loops *unit loops* in X with basepoint x. We will write $f \sim g$ if the unit loops f and g are based-homotopic.

Proposition 2.17. *For any topological space X with basepoint x, the inclusion $L^1(X, x) \hookrightarrow L(X, x)$ induces a bijection*

$$L(X, x)_{/\simeq} \cong L^1(X, x)_{/\sim}.$$

Before we can prove this result, we need the following lemma.

Lemma 2.18. *Let $f : [0, 1] \to [0, 1]$ be a homeomorphism preserving 0 and 1. Then there is a homotopy h from f to id_x such that $h(t, 0) = 0$ and $h(t, 1) = 1$ for all $t \in [0, 1]$*

Proof. We define a continuous map

$$h : [0, 1] \times [0, 1] \to [0, 1], \quad (s, t) \mapsto sf(t) + (1 - s)g(t).$$

This is well defined since, for all $(s, t) \in [0, 1]^2$, we have

$$sf(t) + (1 - s)g(t) \geq 0 \cdot 0 + 0 \cdot 0 = 0$$

and

$$sf(t) + (1 - s)g(t) \leq 1 \cdot 1 + 1 \cdot 1 = 1.$$

It is immediate from the definitions that $h(0, -) = g$ and $h(1, -) = f$. Moreover,

$$h(s, 0) = sf(0) + (1 - s)g(0) = 0 + 0 = 0$$

and

$$h(s, 1) = sf(0) + (1 - s)g(0) = s + (1 - s) = 1$$

proving the lemma. □

Proof of Proposition 2.17. The proof of Lemma 2.6 can be used to show that every loop in X with basepoint x is equivalent to a unit loop in X with basepoint x. Therefore, it suffices to show that two unit loops are homotopic if and only if they are equivalent. By definition, if $f \sim g$ is a homotopy, then $f \simeq g$, so it suffices to show that any two equivalent unit loops are homotopic.

Suppose $f \simeq g$. By definition this means that there is a homeomorphism $p : [0, 1] \to [0, 1]$ which preserves 0 and 1, such that $f \circ p \sim g$. However, by Lemma 2.18, we have a homotopy h from $\mathrm{id}_{[0,1]}$ to p which respects endpoints. Composing h with f thus yields a based homotopy $f = f \circ \mathrm{id}_{[0,1]}$ to $f \circ p$. Thus, there is a based homotopy between f and g. □

We can define additional structure on $L^1(X, x)_{/\sim}$. In fact, by tracing through loops one after another, we can define a group structure on $L^1(X, x)_{/\sim}$:

Construction 2.19. Given two paths $\alpha : [a,b] \to X$ and $\beta : [c,d] \to X$ with $\alpha(b) = x = \beta(c)$ we define the *concatenation* of α and β to be the path

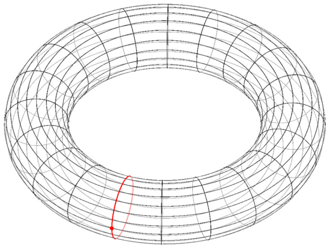
$$\beta * \alpha : [a, b + (d - c)] \to X$$

given by

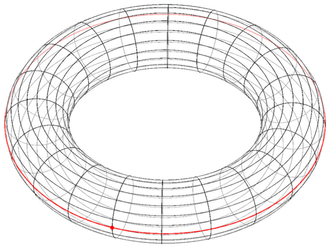
$$(\beta * \alpha)(t) = \begin{cases} \alpha(t) & t \in [a,b] \\ \beta(t - b + c) & t \in [b, b + (d-c)]. \end{cases}$$

Note that when α and β are loops, so is $\beta \star \alpha$.

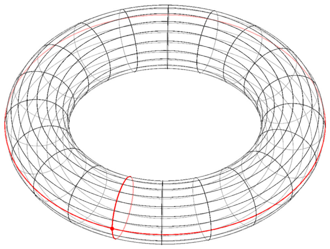
The concatenation of paths is straightforward, but can be a bit tricky to draw. Lets visualize two paths in the torus $S^1 \times S^1$. The first we call α:



and the second, β:



The concatenation $\beta * \alpha$ is a path that looks like



where we first trace through α, and then β.

Exercise 2.20. Show that $\alpha * \beta$ yields a well-defined map on equivalence classes

$$L(X,x)_{/\simeq} \times L(X,x)_{/\simeq} \to L(X,x)_{/\simeq}.$$

Given two unit loops α and β in X, find a unit loop representing the equivalence class of $\beta * \alpha$.

Proposition 2.21. *The binary operation*

$$* : L(X,x)_{/\simeq} \times L(X,x)_{/\simeq} \to L(X,x)_{/\simeq}$$

defines a group structure on $L(X,x)_{/\simeq}$.

Proof. It is immediate from the definitions that $*$ is associative, so we need only define a unit element and inverses.

Let $e_x : [0,1] \to X$ be the constant loop at the basepoint x, i.e. $e_x(t) = x$ for all $t \in [0,1]$. Let $\alpha : [a,b] \to X$ be a loop with basepoint x. We claim that $e_x * \alpha : [a, b+1] \to X$ is equivalent to α.

Define a endpoint-preserving homeomorphism $p : [a, b+1] \to [a,b]$ via

$$p(t) = \frac{b-a}{b+1-a} t + \frac{a}{b+1-a}.$$

And a continuous map $q : [a, b+1] \to [a,b]$ via

$$q(t) = \begin{cases} t & t \in [a,b] \\ b & t \in [b, b+1] \end{cases}$$

Note that $\alpha \circ q = e_x * \alpha$. The construction of Lemma 2.18 can be used to define an endpoint-preserving homotopy $q \sim p$, which then gives rise to a based homotopy $\alpha \circ q \sim \alpha \circ p$. This shows $\alpha \simeq e_x * \alpha$, so e_x is a left unit for $*$. The proof that e_x is a left unit is totally analogous.

We now need only show that every path α has an inverse up to homotopy. Since every equivalence class can be represented by a unit loop, we may assume without loss of generality that $\alpha : [0,1] \to X$ is a based unit loop. Define $p : [0,1] \to [0,1]$ by $p(t) = 1 - t$, and set $\alpha^{-1} = \alpha \circ p$. We define a based homotopy $h : [0,1] \times [0,2] \to X$ from $\alpha * \alpha^{-1}$ to the constant loop $e_x^2 : [0,2] \to X$ as follows:

$$h(s,t) = \begin{cases} (\alpha * \alpha^{-1})(t - st) & 0 \le t \le 1 \\ (\alpha * \alpha^{-1})((2-t)s + t) & 1 \le t \le 2 \end{cases}$$

We leave it to the reader to verify that this provides the desired homotopy, and to check the analogous case of $\alpha^{-1} * \alpha$. □

Proposition 2.22. *Let $f : X \to Y$ be a continuous map of topological spaces such that $f(x) = y$. Then composing with f defines a group homomorphism*

$$f_* : L(X,x)_{/\simeq} \to L(Y,y)_{/\simeq}$$

Proof. There are two things to check: First, that composition defines a well-defined map on equivalence classes, and second, that this map preserves the group structure.

The map in question sends a loop $\ell : [a,b] \to X$ based at x to the loop $f \circ \ell : [a,b] \to X$. Since reparameterization by a homeomorphism operates on the interval $[a,b]$, it will suffice to show that this map sends based homotopy class to based homotopy classes. This follows (with some extra care paid to the basepoint) from Lemma 2.13.

To see that the map preserves the group structure, we first note that $f \circ e_x$ is clearly e_y. For two loops $\beta : [a,b] \to X$ and $\alpha : [c,d] \to X$ based at x, we have

$$f \circ (\beta * \alpha)(t) = \begin{cases} f \circ \alpha(t) & t \in [a,b] \\ f \circ \beta(t-b+c) & t \in [b, b+(d-c)] \end{cases} = ((f \circ \beta) * (f \circ \alpha))(t)$$

completing the proof. □

Remark 2.23. Note that, while we have worked with $L(X,x)$ the propositions above hold true for $L^1(X,x)$ via the canonical isomorphism $L^1(X,x)_{/\sim} \cong L(X,x)_{/\simeq}$.

Definition 2.24. The set $L(X,x)_{/\simeq}$ together with the group structure constructed above is denoted by $\pi_1(X,x)$, and is called *the fundamental group* of X at x.

We will now explain how the fundamental group can be used to distinguish topological spaces. We will not compute any fundamental groups in this section, instead deferring such proofs to after we have developed some technology for computation.

Claim 2.25. Suppose that $f : X \to Y$ is a homotopy equivalence with $f(x) = y$ then $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ is an isomorphism of groups.

Claim 2.26. For $x, x' \in X$ in the same path component, there is an isomorphism $\pi_1(X,x) \cong \pi_1(X,x')$.

This tells us something very important, namely that if two spaces have different fundamental groups, they cannot be homotopy equivalent, and thus cannot be homeomorphic. We now have almost everything we need to distinguish $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0\}$.

Claim 2.27. We have

$$\pi_1(\mathbb{R}^2, 1) = \{e\}$$

and

$$\pi_1(\mathbb{R}^2 \setminus \{0,\}, 1) = \pi_1(S^1, 1) = \mathbb{Z}$$

Consequently, $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0\}$ cannot be homeomorphic.

Remark 2.28. In general, if a space X is contractible (i.e. homotopy equivalent to the one-point space), then X is path-connected, and so Claim 2.25 implies that $\pi_1(X, x) \cong \pi_1(*, *) \cong \{e\}$ is a trivial group for any basepoint $x \in X$.

2.2 *Coverings and the circle*

We will now aim to compute the fundamental group of the circle $S^1 \subset \mathbb{C}$. Throughout this section, we will be considering the map

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ x &\longmapsto \exp(2\pi i x). \end{aligned}$$

We will use the basepoint $1 \in S^1 \subset \mathbb{C}$.

Definition 2.29. Let $q : Y \to Z$ be a continuous map. Let $f : X \to Z$ be another continuous map. A *lift* $\widetilde{f}$ of f is a continuous map $\widetilde{f} : X \to Y$ such that $q \circ \widetilde{f} = f$.

Diagrammatically, a lift is a continuous map which makes the diagram

$$\begin{tikzcd} & Y \arrow[d, "q"] \\ X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

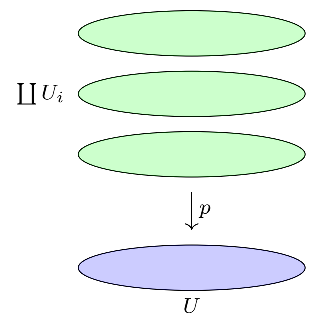
The key property which the map p has is the following

Homotopy Lifting Property A map $q : Y \to Z$ is said to have the *homotopy lifting property* if, for any space X any homotopy

$$H : [0,1] \times X \longrightarrow Z$$

from f to g, and any lift $\widetilde{f}$ of f, there is a unique lift $\widetilde{H}$ of H such that $\widetilde{H}|_{\{0\}\times X} = \widetilde{f}$.

Schematically, a covering space should locally look like



Exercise 2.30. Consider the map $p : \mathbb{R} \to S^1$.

1. Show that, for any $x \in S^1$, there is an open subset $U \subset S^1$ with $x \in U$ such that $p^{-1}(U) \cong \coprod_{i\in\mathbb{Z}} U_i$, and such that $p|_{U_i} : U_i \to U$ is a homeomorphism. (We say that $p : \mathbb{R} \to S^1$ is a *covering space*)

2. Prove the *Tube Lemma*: Let X and Y be topological spaces and suppose Y is compact. Given an open subset $N \subset X \times Y$ which contains $\{x\} \times Y$ for some $x \in X$, there is an open neighborhood $U \subset X$ with $x \in U$ such that $U \times Y \subset N$.

3. Prove the *Lebesgue Number Lemma*: For any compact metric space (X, d) and any open cover $\{U_i\}_{i\in I}$ of X, there is a $\delta > 0$ such that, for any subset $V \subset X$ with
$$\operatorname{diam}(V) := \sup\{d(x,y) \mid x, y \in V\} < d$$
then V is contained in at least one of the U_i.

4. Prove that p has the homotopy lifting property.

Our goal will be to show, using the homotopy lifting property, that $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Construction 2.31. We define a map

$$\Phi : \pi_1(S^1, 1) \longrightarrow \mathbb{Z} =: p^{-1}(1) \subset \mathbb{R}$$

as follows. Given $[\alpha] \in \pi_1(S^1, 1)$, let $\widetilde{\alpha}$ denote the unique lift of α starting from 0 guaranteed by the homotopy lifting property. Set $\Phi([\alpha]) = \widetilde{\alpha}(1)$.

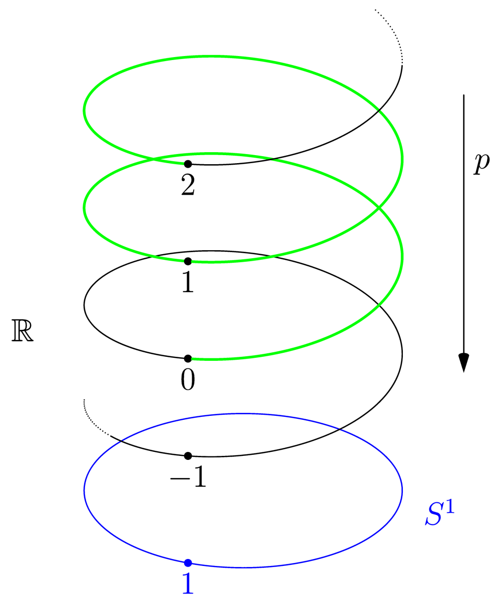
To see that this is well-defined, suppose $H : [0,1]\times[0,1] \to S^1$ is a path homotopy from α to β. Then there is a unique lift $\widetilde{H}$ starting from $\widetilde{\alpha}$. However, this means that $\widetilde{H}|_{[0,1]\times\{0\}}$ and $\widetilde{H}|_{[0,1]\times\{1\}}$ are lifts of the constant path e_1. The uniqueness guaranteed by the homotopy lifting property then implies that both are constant paths.

This, in turn, means that $\widetilde{H}|_{\{1\}\times[0,1]}$ is the unique lift $\widetilde{\beta}$ of β starting from $0 \in \mathbb{R}$. Thus, we see that

$$\Phi([\alpha]) = \widetilde{\alpha}(1) = \widetilde{\beta}(1) = \Phi([\beta])$$

and so Φ is well-defined.

To illustrate the homotopy lifting property: if we lift a loop α that winds twice around the circle counterclockwise to a path in $\mathbb{R}$ starting at 0, we end up with a path $\tilde{\alpha}$, as pictured in green below.



Claim. The map Φ is a group homomorphism.

Proof. Let $[\alpha], [\beta] \in \pi_1(S^1, 1)$, and let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be the unique lifts of α and β starting from 0 in $\mathbb{R}$.

Define $\beta : [0,1] \to \mathbb{R}$ by

$$\overline{\beta}(t) := \widetilde{\beta}(t) + \widetilde{\alpha}(1).$$

Since $\widetilde{\alpha}(1)$ is an integer, we have

$$\begin{aligned} \exp(2\pi i\overline{\beta}(t)) &= \exp(2\pi i(\widetilde{\beta}(t) + \widetilde{\alpha}(1))) \\ &= \exp(2\pi i\widetilde{\beta}(t)) \exp(2\pi i\widetilde{\alpha}(1)) \\ &= \exp(2\pi i\widetilde{\beta}(t)) \\ &= \beta(t). \end{aligned}$$

So that $\overline{\beta}$ is the unique lift of β starting from $\widetilde{\alpha}(1)$. Thus $\overline{\beta} * \widetilde{\alpha}$ is the unique lift of $\beta \star \alpha$ starting from 0. Finally, we can then compute:

$$\begin{aligned} \Phi([\beta \star \alpha]) &= (\overline{\beta} * \widetilde{\alpha})(1) \\ &= \widetilde{\beta}(1) + \widetilde{\alpha}(1) \\ &= \Phi([\beta]) + \Phi([\alpha]) \end{aligned}$$

completing the proof. □

Lemma 2.32. *Let $f, g : [a,b] \to \mathbb{R}$ be two maps such that $f(a) = g(a)$ and $f(b) = g(b)$. Then there is a homotopy of paths between f and g.*

Proof. The homotopy in question is the straight-line homotopy

$$\begin{aligned} H : \quad [0,1] \times [0,1] &\longrightarrow \mathbb{R} \\ (t, s) &\longmapsto tg(s) + (1-t)f(s). \end{aligned}$$

We leave the checks that this is indeed a homotopy of paths to the reader. □

Claim. The homomorphism Φ is injective.

Proof. Suppose that $\Phi([\alpha]) = 0$, so $\widetilde{\alpha}(1) = 0$. Then by the preceding lemma, there is a homotopy $\widetilde{H}$ from $\widetilde{\alpha}$ to to the constant path e_0 on $0 \in \mathbb{R}$. This then implies that $H := p \circ \widetilde{H}$ is a homotopy from α to e_1. Thus $[\alpha] = [e_1]$. $\square$

Definition 2.33. For each $n \in \mathbb{Z}$, we define a path

$$\widetilde{\alpha}_n : [0,1] \longrightarrow \mathbb{R}$$

$$t \longmapsto nt$$

from 0 to n in $\mathbb{R}$. We write $\alpha_n := p \circ \widetilde{\alpha}_n$.

Claim 2.34. The homomorphism Φ is surjective.

Proof. We need only note that $\widetilde{\alpha}_n$ is the unique lift starting from 0 of the loop α_n, so

$$\Phi([\alpha_n]) = \widetilde{\alpha}_n(1) = n.$$

$\square$

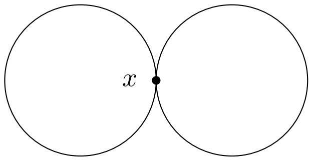
Proposition 2.35. *The group $\pi_1(S^1, 1)$ is freely generated by $[\alpha_1]$, yielding an isomorphism*

$$\pi_1(S^1, 1) = \langle [\alpha_1] \rangle \cong \mathbb{Z}.$$

2.3 The Seifert-van Kampen Theorem

Before we develop the complex technology which allows us to compute some fundamental groups directly, we want a statement that allows us to build up the fundamental group of a complicated space X if we can break X apart into pieces whose fundamental groups we know.

Example 2.36. Consider the *wedge of two circles* $S^1 \vee S^1$, which is the space



Take the point x as a basepoint. We note that we get a non-contractible loop α around the left-hand circle, and a non-contractible loop β around the right-hand circle. Each of these would generate the corresponding fundamental group of a circle, but how do they interact.

It is pretty easy to convince ourselves that the elements of the resulting group will have the form

$$\alpha^{k_1} * \beta^{k_2} * \alpha^{k_3} * \cdots * \alpha^{k_{r-1}} * \alpha^{k_r} \qquad k_i \in \mathbb{Z}$$

so that $\pi_1(S^1 \vee S^1, x)$ is the free group on $\{\alpha, \beta\}$.

The **wedge sum** of two path-connected topological spaces X and Y is a way of gluing X and Y. Formally, given pointed spaces (X, x) and (Y, y) we define

$$X \vee Y := (X \coprod Y)_{\sim}$$

where we define $x \sim y$. What this means intuitively is that we glue X to Y at a single point.

This definition depends on the choice of x and y, but when we are just gluing circles together, the choice doesn't matter as much because of the symmetries of the circle.

This isn't quite a proof, but we'd like to prove this in a much more general context. To do this, we'll need two definitions:

Definition 2.37. Let G and H be groups. We want to define a group $G * H$ which is sort of like a disjoint union of G and H, but such that elements of G do not commute with elements of H. The idea is to let the elements be *formal* products

$$a_1 \cdot a_2 \cdot a_3 \cdots a_k$$

of elements in G and H. This, however, forgets the original group structures of G and H. To add them back in, we identify

$$a_1 \cdot a_2 \cdots a_i \cdot a_{i+1} \cdots a_k = a_1 \cdot a_2 \cdots (a_i a_{i+1}) \cdots a_k$$

whenever a_i and a_{i+1} are both in G, or both in H. We similarly identify

$$a_1 \cdots a_{i-1} \cdot a_i \cdot a_{i+1} \cdots a_k = a_1 \cdots a_{i-1} \cdot a_{i+1} \cdots a_k$$

whenever a_i is the identity element in either G or H.[3]

We call the resulting group the **free product** of G and H, and denote it by $G * H$.[4]

[3] Note that this formally identifies e_G and e_H with the empty product, which is the identity of the free group.

[4] To see why this is relevant, notice that we can rephrase our first example as saying

$$\pi_1(S^1 \vee S^1, x) \cong \pi_1(S^1, x) * \pi_1(S^1, x) \cong \mathbb{Z} * \mathbb{Z}.$$

Exercise 2.38. Show that the free product is coproduct in the category **Grp**.

To this definition, we add a little bit of extra structure.

Definition 2.39. Suppose we are given homomorphisms of groups

$$G \xleftarrow{\phi} K \xrightarrow{\psi} H$$

We define the **pushout** of these groups to be

$$G *_K H := (G * H)_{/\sim}$$

where we declare $\phi(k) \sim \psi(k)$ for any $k \in K$.[5]

[5] Technically, we are taking the smallest normal subgroup of $G * H$ containing the elements $\phi(k) * \phi(k^{-1})$, and quotienting by that.

Remark 2.40. It is worth noting that, while the notation $G *_K H$ does not include ϕ and ψ, the nature of the maps ϕ and ψ is *very* important. For example, consider two diagrams

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and

$$\mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}$$

The pushout of the first diagram is simply the free product $\mathbb{Z} * \mathbb{Z}$, since ψ and ϕ do not give us any new relations.

In the second case, however, we get that the pushout is $(\mathbb{Z} * \mathbb{Z})_{/\sim}$, where $\sim$ is a relation which identifies elements from the second copy of $\mathbb{Z}$ with elements from the first copy of $\mathbb{Z}$. Consequently, we see that the pushout is simply $\mathbb{Z}$ again.

Exercise 2.41. Show that the pushout of groups described in Definition 2.39 is the pushout in the category **Grp**.

With this new terminology, we can state our main theorem:

Theorem 2.42 (Seifert-van Kampen)**.** *Let X be a topological space, and let U_1 and U_2 be two open, path-connect subsets of X such that $X = U_1 \cup U_2$ and such that $U_1 \cap U_2$ is path-connected and non-empty. Let $x \in U_1 \cap U_2$. Then there is an isomorphism*

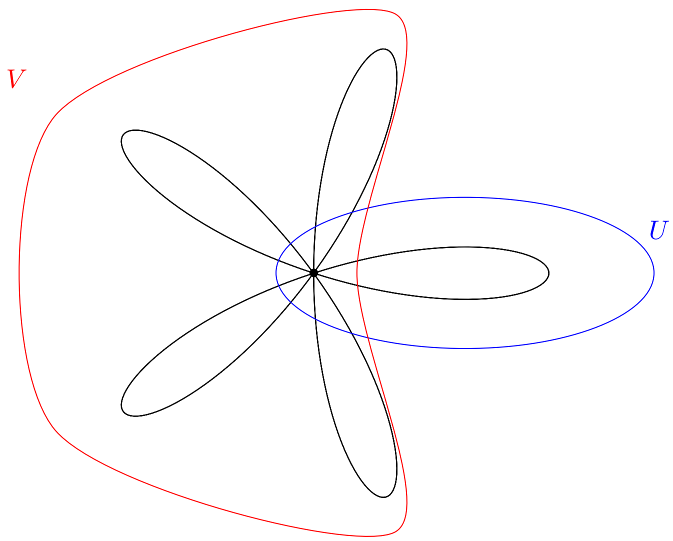
$$\pi_1(X, x) \cong \pi_1(U_1, x) *_{\pi_1(U_1 \cap U_2, x)} \pi_1(U_2, x)$$

induced by the inclusions.

Before we prove this theorem, lets see some applications.

Example 2.43. Let $B_n := S^1 \vee S^1 \vee \cdots \vee S^1$ be a **bouquet of n circles**[6], i.e. n circles joined together at a single point x.

To compute $\pi_1(B_n, x)$, we can apply an induction. Suppose that we know $\pi_(B_{k-1}, x) := \mathbb{Z}^{*(k-1)}$. We can choose open sets U and V in B_k as pictured below:



We can see that V is homotopy equivalent to B_{k-1}, U is homotopy equivalent to S^1, and $U \cap V$ is homotopy equivalent to the 1-point space. Consequently, by SvK, we get

$$\pi_1(B_k, x) \cong \pi_1(B_{k-1}, x) * \pi_1(S^1, 1) \cong \mathbb{Z}^{*(k-1)} * \mathbb{Z} \cong \mathbb{Z}^{*k}.$$

Since the base case (wedge of one circle) is taken care of by the fact that $\pi_1(S^1, 1) \cong \mathbb{Z}$, we can conclude that $\pi_1(B_n, x) \cong \mathbb{Z}^{*n}$.

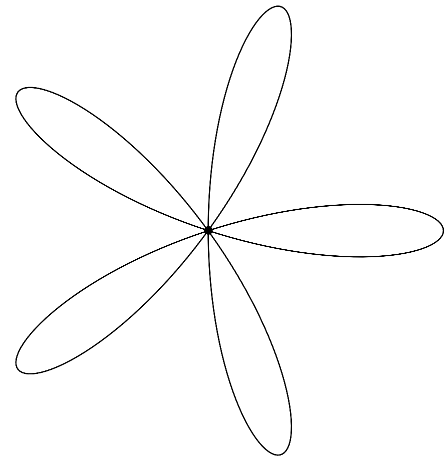
Claim 2.44. Let S^n be the unit sphere with $n > 1$, and let $x \in S^n$ be any point. Then

$$\pi_1(S^n, x) \cong \{1\}.$$

Proof. Write coordinates on $\mathbb{R}^{n+1}$ as $x = (x_1, x_2, \ldots, x_{n+1})$, and view S^n as the unit sphere

$$S^n := \{x \in \mathbb{R}^n \mid |x| = 1\} \subset \mathbb{R}^n$$

[6] We tend to draw the bouquet of n circles as, e.g.,



We can now define two open sets in S^n

$$U_+ := \{x \in S^n \mid x_{n+1} > -\frac{1}{5}\}$$

and

$$U_- := \{x \in S^n \mid x_{n+1} < \frac{1}{5}\}.$$

Using the exercise below, we see that U_+ and U_- are contractible, $U_+ \cup U_- = S^n$, and $U_+ \cap U_- \simeq S^{n-1}$. Thus, since $n > 1$, we have that $U_+ \cap U_-$ is path-connected and non-empty, so that SvK applies. We thus see that

$$\pi_1(S^n, x) \cong \{1\} *_{\pi_1(S^{n-1},x)} \{1\} \cong \{1\}$$

is the trivial group, as desired. $\square$

Exercise 2.45. Consider the sets U_+ and U_- from the proof above. Prove that U_+ and U_- are contractible, and that $U_+ \cap U_- \simeq S^{n-1}$.

Exercise 2.46. Define a space X by gluing the north pole of S^2 to the south pole of S^2. Compute the fundamental group of X.

Exercise 2.47. Define a space H to be

$$\mathsf{H} := \bigcup_{n=1}^{\infty} \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\} \subset \mathbb{R}^2.$$

Can you apply the Seifert-van Kampen Theorem to compute the fundamental group of H in terms of the fundamental group of S^1? If so, compute it.

Exercise 2.48. Let $T^2 = S^1 \times S^1$ denote the torus.

1. Argue that if we delete a single point from T^2, the result is homotopy equivalent to $S^1 \wedge S^1$.
2. Argue that if we delete two points from T^2, the result is homotopy equivalent to $S^1 \wedge S^1 \wedge S^1$.
3. Let $x, y \in T^2$ be two distinct points. Apply SvK to $U_1 = T^2 \setminus \{x\}$ and $U_2 = T^2 \setminus \{y\}$ to give an alternate computation of the fundamental group of the torus.

Proof of Theorem 2.42. We start by defining a map. By Exercise 2.41, we obtain a homomorphism

$$\begin{aligned} \Psi : \pi_1(U_1, x) *_{\pi_1(U_1 \cap U_2, x)} \pi_1(U_2, x) &\longrightarrow \pi_1(X, x) \\ [\alpha_1] \cdot [\beta_1] \cdot [\alpha_2] \cdots [\alpha_k] \cdot [\beta_k] &\longmapsto [\alpha_1 * \beta_1 * \alpha_2 * \cdots * \alpha_k * \beta_k] \end{aligned}$$

It thus remains only for us to to show that this map is surjective and injective. For ease of notation, we will write

$$G := \pi_1(U_1, x) *_{\pi_1(U_1 \cap U_2, x)} \pi_1(U_2, x)$$

To see surjectivity, let α be a loop in X based at x. Divide I into intervals $[0,t_1],[t_1,t_2],\ldots,[t_{k-1},1]$ so that either $\alpha([t_{i-1},t_i]) \subset U_1$ or $\alpha([t_{i-1},t_i]) \subset U_2$ for every $0 < i \leq k$.

We fix some notation: $x_i = \alpha(t_i)$, and α_i is the path from x_{i-1} to x_i obtained by restricting α to the interval $[t_{i-1},t_i]$. Since $U_1 \cap U_2$ is path-connected, we can define a path γ_i from x to x_i in $U_1 \cap U_2$, with γ_0 and γ_k the constant loop on x. We can then write

$$[\alpha] = [\gamma_0 * \alpha_1 * \gamma_1^{-1} * \gamma_1 * \alpha_2 * \gamma_2^{-1} * \cdots * \gamma_{k-1} * \alpha_k * \gamma_k^{-1}]$$

Breaking this up, we see that $\gamma_{i-1} * \alpha_i * \gamma_i^{-1}$ is a loop contained entirely in either U_1 or U_2. Thus, we can write an element

$$A := ([\gamma_0 * \alpha_1 * \gamma_1^{-1}]) \cdot ([\gamma_1 * \alpha_2 * \gamma_2^{-1}]) \cdots ([\gamma_{k-1} * \alpha_k * \gamma_k^{-1}]) \in G$$

Such that $\Psi(A) = [\alpha]$. Therefore, the map is surjective.

We now turn to injectivity, the most involved part of the proof. Suppose that

$$A := [\alpha_k] \cdots [\alpha_1] \cdot \in G$$

is an element such that $\Psi(A) = e_x$. This means that there is a homotopy $H : I \times I \to X$ of loops from the concatenation

$$\alpha_k * \alpha_{k-1} * \cdots \alpha_1$$

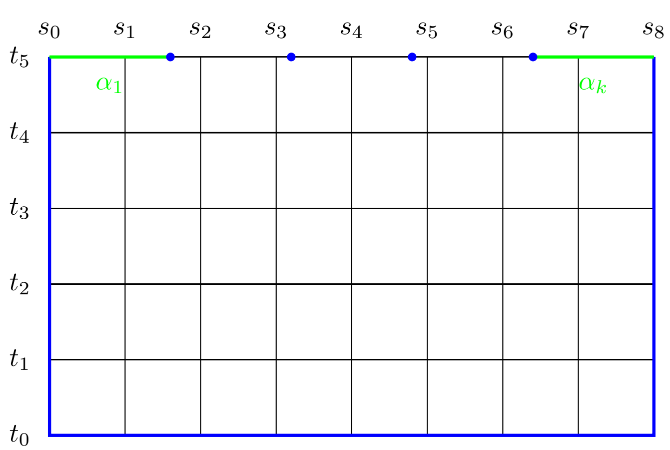
to e_x.

We now choose subdivisions $[0,t_1],[t_1,t_2],\ldots,[t_{r-1},1]$ and $[0,s_1],[s_1,s_2],\ldots,[s_{q-1},1]$ of $[0,1]$ such that, for every i,j, the set

$$H\left([t_{i-1},t_i] \times [s_{j-1},s_j]\right)$$

is contained in either U_1 or U_2 as sketched below.[7]

[7] If you are not convinced we can do this, try to prove it yourself or look up the Lebesgue number lemma.



In this image, the lines in blue are sent to the basepoint x, and the composite $\alpha_k * \cdots * \alpha_1 =: \alpha$ lies along the top edge, read from left to right.

We now wish to simplify the formal word

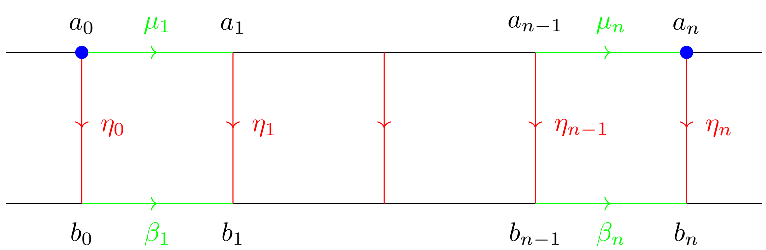
$$[\alpha_1][\alpha_2]\cdots[\alpha_k]$$

to the identity using relations which are valid on words in G. Since we will work with specific loops representing the letters of our word, there are three such relations we can use: (1) Homotopies on one of the letters, so long the homotopy is contained entirely in U_1 or entirely in U_2. (2) Multiplication (concatenation) in $\pi_1(U_1, x)$ **or** $\pi_2(U_2, x)$ can be identified with the corresponding formal product in G. (3) A letter $[\alpha_\ell]$ which is in the image of $\pi_1(U_1 \cap U_2, x)$ can be considered as either a letter in $\pi_1(U_1, x)$ or in $\pi_1(U_2, x)$.

Using these rules, we will attempt to simplify the word indicated by the upper horizontal line of our homotopy H to a word corresponding to a subdivision of the horizontal line second from the top. Iterating this process will show that A is equivalent to the identity in G, since any subdivision of the bottommost horizontal line of the homotopy will represent the identity in G.

Without loss of generality, we may assume that our subdivision $0 < s_1 < \cdots < s_{q-1} < 1$ also contains the endpoints of the α_i, since once we have established a subdivision with the desired property, we may refine it.

Let us now consider just a section of the top two lines, containing the path α_ℓ. We label the vertical paths defined by our subdivision as η_i, and the paths in the induced subdivision of α_ℓ by $\mu_1, \ldots, \mu_n$. We similarly label the paths induced by the subdivision of the bottom line by $\beta_1, \ldots, \beta_n$. Finally, we label the images of the vertices along the top under H as $a_0, \ldots, a_n$, and of the vertices along the bottom by $b_0, \ldots, b_n$.



We will call each of the small rectangles in our subdivision a *cell*, and we will call the (not necessarily unique) U_i which contains the image of a cell under H the *U_i associated to the cell.* For each $0 \leq i \leq n$, the point $a_i \in X$ lies in the U_i's associated to the two neighboring cells. Moreover, it lies in the U_i which contains α_ℓ. Thus by our path-connectedness hypotheses, we may choose a path γ_i from x to a_i which lies entirely in the intersection of these U_i's.[8] Without loss of generality, we may take $\gamma_0 = \gamma_n = e_x$.

[8] By this, we mean the intersection $U_1 \cap U_2$ if the U_i associated to either neighboring cell differs from the U_i containing α_ℓ, and the U_i which contains α_ℓ otherwise.

Let $j \in \{0, 1\}$ be an index such that α_ℓ lies in U_j. Then we may form loops in U_j

$$\lambda_i := \gamma_i^{-1} \star \mu_i \star \gamma_{i-1}$$

and note that that, in $\pi_1(U_j, x)$, we have

$$[\alpha_\ell] = [\lambda_n \star \cdots \star \lambda_1]$$

As such, in G, we have that the word $[\alpha_\ell]$ is equivalent to the word $[\lambda_n][\lambda_{n-1}]\cdots[\lambda_1]$.

Similarly, $\eta_i \star \gamma_i$ is a path from x to b_i which lies in the U_i associated to the cells which neighbor η_i. We can thus form loops

$$\delta_i := \gamma_i^{-1} \star \eta_i^{-1} \star \beta_i \star \eta_{i-1} \star \gamma_{i-1}.$$

which lie entirely in U_j associated to the cell above β_i. Moreover, each cell provides a homotopy, entirely in the U_j associated to that cell, from μ_i to $\eta_i^{-1} \star \beta_i \star \eta_{i-1}$. Thus, we see that, in G,

$$\lambda_i \sim \delta_i.$$

Thus, as elements of G,

$$[\alpha_\ell] = [\delta_n]\cdots[\delta_1].$$

We then attempt to simplify the product

$$[\delta_{i+1}][\delta_i]$$

in G. There are two cases to consider:

1. If the U_j associated to the cells above β_i and β_{i+1} are the same, then δ_i and δ_{i+1} can be viewed as loops in $\pi_1(U_j, x)$, and so

$$[\delta_{i+1}][\delta_i] = [\delta_{i+1} * \delta_i]$$

   in G. This means that, for any path ζ_i from x to b_i in U_j, we can replace δ_i with

$$\zeta_i^{-1} * \beta_i * \eta_{i-1} * \gamma_{i-1}$$

   and δ_{i+1} with

$$\gamma_{i+1}^{-1} * \eta_{i+1}^{-1} * \beta_i * \zeta_i.$$

2. If, on the other hand these two U_j's are different, then $\eta_i * \gamma_i$ lies entirely in $U_1 \cap U_2$.

We then iterate this argument. If the cells above and below β_i correspond to different U_j's, the two numbered points above tell us that we may assume the that "conjugating paths" $\eta_i * \gamma_i$ lie entirely within the intersection $U_1 \cap U_2$, and so does δ_i. We may then consider $[\delta_i]$ as a class in either $\pi_1(U_1, x)$ or $\pi_1(U_2, x)$, allowing us to iterate the argument.

Finally, we iterate this process to find an element in G which is equivalent to A, and consists of constant loops on x conjugated by paths in $U_1 \cap U_2$ from x to x. As a result, we see that this element is simply the identity element, and the proof is complete. □

2.4 Topological manifolds and surfaces

A number of the spaces we have met in examples — the spheres S^n, the torus $T^2 = S^1 \times S^1$, the real projective space $\mathbb{R}P^n$, and the Möbius band, for example

— have a striking property: if you zoom in close enough to any point, they look like $\mathbb{R}^n$ for some n. Such spaces form the backbone of many modern disciplines of mathematics: Differential topology, differential geometry, etc.[9] To make this idea rigorous, we make the following definition.

[9] Indeed, if we all more general spaces than $\mathbb{R}^n$ as our "model space", the idea of a space "locally modeled on our chosen space suffuses much of modern mathematics.

Definition 2.49. A *dimension k (topological) manifold* (or just a k-manifold, for short) is a topological space M with the following properties.

1. For every $x \in M$, there is an open $U \subset M$ containing x and a homeomorphism
$$\phi : U \xrightarrow{\cong} V$$
where V is an open subset of $\mathbb{R}^k$.
2. There is a countable basis for the topology on M.[10]
3. The space M is a Hausdorff space.

[10] This property is often referred to as *second countability.*

We call a 2-dimensional manifold a *surface.*

The first condition is the one that captures our essential intuition for a space that is "locally like $\mathbb{R}^k$". The other two are mild conditions to exclude difficult-to-work with cases. Many sources also assume that manifolds are *paracompact* — an assumption which we will see follows from the others when discussing the basics of differential topology. To understand the reasons behind and implications of our three properties, let us consider three non-examples.

Non-examples 2.50.

1. Recall the *line with two origins*: Let (Y, τ_Y) be the topological space $\mathbb{R} \times \{0, 1\}$, where $\{0, 1\}$ is equipped with the discrete topology. Note that Y can also be identified with $\mathbb{R} \amalg \mathbb{R}$. Define an equivalence relation on Y by $(x, 0) \sim (x, 1)$ for all $x \neq 0$, and let L be the quotient space of Y by this equivalence relation.

   The space L satisfies property (1) of our definition. If $x \in L$ is not one of the two origins, then any open ball around x which has radius less than $|x - 0|$ is homeomorphic to the same ball in $\mathbb{R} \times \{0\}$ via the quotient map $q : Y \to L$. If we consider x to be one of the origins — WLOG $(0, 0)$, we can take an open ball $B_1(0) \subset \mathbb{R}$, and define U to be $q(B_1(0) \times \{0\})$. This open subset is homeomorphic to $B_1(0)$, virtually by definition.

   Similarly, this space satisfies condition (2) of our definition. We know $\mathbb{R} \times \{0, 1\}$ has a countable basis, and it is not hard to check that the quotient map q in this case is an open map. Thus , L admits a countable basis.

   However, L is not Hausdorff, as we have already seen, and so we exclude it from our considerations.
2. Consider the union
$$X := \{x \in \mathbb{R}^3 \mid x_3 = 0\} \cup \{x \in \mathbb{R}^3 \mid x_2 = 0\}$$

as a subspace of $\mathbb{R}^3$. This looks like two planes intersecting in a line. As a subspace of $\mathbb{R}^3$, X has properties (2) and (3), however, any open ball centered on the line of intersection will not be homeomorphic to an open subset of $\mathbb{R}^2$. Thus, the all-important property (1) is violated.

3. As a final example, we consider the *long line*. Let λ be an uncountable ordinal[11] and let 0_λ be the minimum of λ. Consider the set $X := ([0,1) \times \lambda) \setminus (0, 0_\lambda)$, equipped with the lexicographic order

$$(x,i) \leq (y,j) \Leftrightarrow \begin{cases} i < j & \text{or} \\ i = j \text{ and } x \leq y \end{cases}$$

We define an *interval* in X to be a set of the form

$$(a,b) := \{x \in X \mid a < x < b\}$$

for elements $a, b \in X$. Then the set of intervals forms the basis for a topology on X. We call the resulting space the *long line*. As the next exercise demonstrates, the long line satisfies all of the properties of a manifold except the existence of a countable basis.

[11] An *ordinal* is a totally ordered set μ such that, for any non-empty subset $S \subset \mu$, there is a minimum of S, i.e., $x \in S$ such that, for every $y \in S$, $x \leq y$. For a proof of the existence of uncountable ordinals, see Hartog's Theorem

Exercise 2.51. Show that the long line satisfies proeprties (1) and (3) from the definition of a manifold, but not (2).

To contrast with these non-examples, let's consider some examples.

Examples 2.52.

1. The n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$ is a Hausdorff space and has a countable basis — properties it inherits from $\mathbb{R}^{n+1}$. The *stereographic projection*

$$\begin{aligned} \phi : S^n \setminus \{(0,0,\ldots,0,1)\} &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \left(\tfrac{x_1}{1-x_n}, \ldots, \tfrac{x_{n-1}}{1-x_n}\right) \end{aligned}$$

is a homeomorphism. Since we can freely permute the deleted coordinate in the definition, variants of the stereographic projection cover all of S^n in neighborhoods homeomorphic to $\mathbb{R}^n$.

2. The torus $T^2 = S^1 \times S^1$ is a surface. Since S^1 has a countable basis and is a Hausdorff space, so is its product with itself. We can view T^2 as the quotient of $\mathbb{R}^2$ by the relations $(x,y) \sim (x+n, y+m)$ for $(n,m) \in \mathbb{Z} \times \mathbb{Z}$. The resulting quotient map $q : \mathbb{R}^2 \to T^2$ is an open map. Restricting q to open subsets small enough to pass unchanged through the quotient provides the desired local isomorphism.

Exercise 2.53. Prove that $\mathbb{R}P^2$ is a surface.

To increase our collection of surfaces, we will define an operation called the *connected sum*. A priori, this depends on a wide variety of different input data, but it turns out — a result beyond the scope of this lecture — that up to homeomorphism, the connected sum of path-connected manifolds is independent of our choices.

3

Smooth manifolds

3.1 *Differentiation on* $\mathbb{R}^n$

I'm not going to aim for a rigorous and complete exposition of differentiation and integration in these notes. I will, however, try to develop some computational tools, as well as some of the underlying intuitions.

The key point of a derivative is to approximate an arbitrary function by something linear. In single-variable calculus, when we differentiate a function $f : \mathbb{R} \to \mathbb{R}$ at a point x, the defining property is that

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

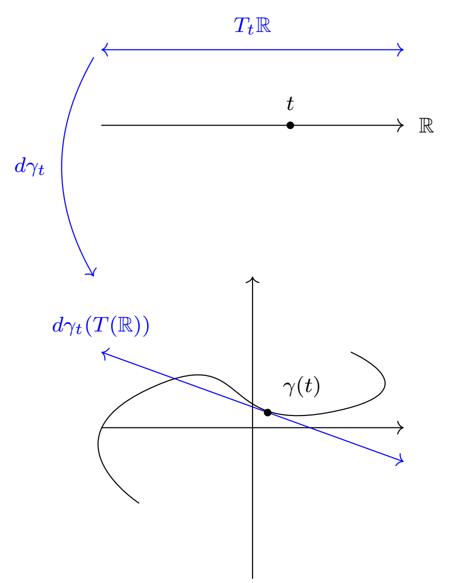
for x close to x_0. We make this formal by defining the derivative $f'(x_0)$ to be the real number (if one exists) such that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - f'(x_0 + h)}{h} = 0.$$

Equivalently, we can define

$$f'(x_0) = \lim_{h \to 0} .$$

Pictorially



Our first question to answer is *what does the number $f'(x_0)$ mean?* We can think of it as the slope of the tangent line to $f(x)$, or as a velocity. More useful, however, is to view $f'(x_0)$ as a *linear transformation.* We can think of $f'(x_0)$ as the linear transformation which turns possible velocities at the point x_0 to possible velocities at the point $f(x_0)$. We typically formalize this viewpoint by defining the *tangent space* to $\mathbb{R}$ at x_0 to be $T_{x_0}\mathbb{R} := \mathbb{R}$, which we view as the possible velocities of linear paths through x_0 in $\mathbb{R}$. We then

Even clearer is when we consider a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$. To each point $z \in \mathbb{R}^2$ we associate a tangent space $T_z\mathbb{R}^2 := \mathbb{R}^2$ — viewed as the possible velocities of linear paths through z — The derivative $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))$ can then be viewed as a transformation

$$d\gamma_t : T_t\mathbb{R} \longrightarrow T_{\gamma(t)}\mathbb{R}^2,$$

The image of this transformation is a linear subspace, which can be identified with the space of all possible tangent vectors to the image of γ at the point $\gamma(t)$ — The *tangent space of the curve* γ at $\gamma(t)$.

To make this intuition formal, we make the following definitions.

Definition 3.1. Let $U \subset \mathbb{R}^n$ be an open set. For any $x \in U$, the *tangent space of U at x* is the vector space
$$T_xU := \mathbb{R}^n.$$
The *tangent bundle* of U is the set TU consisting of pairs (x, v) where $x \in U$ and $v \in T_xU$. We call (x, v) (or $v \in T_xU$) a *tangent vector to U at x*. Notice that there is a canonical identification
$$TU \cong U \times \mathbb{R}^n,$$
and so we can view TU as a subset of $\mathbb{R}^{2n}$.

Definition 3.2. Let $U \subset \mathbb{R}^n$ be an open subset, and $f : U \to \mathbb{R}^m$ a function. The *(total) derivative Df_x* at a point $x \in U$ is the linear map
$$(Df)_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
such that
$$\lim_{y \to x} \frac{|f(y) - f(x) - Df_x(y - x)|}{|y - x|} = 0.$$
If the total derivative of f exists at every point $x \in U$, we call f *differentiable on U*, and we define a map
$$\begin{aligned} df :\ TU &\longrightarrow T\mathbb{R}^m \\ (x, v) &\longmapsto (f(x), Df_x(v)) \end{aligned}$$
called the *differential* of f.

Remark 3.3. Under the canonical identifications $T_xU \cong U \times \mathbb{R}^n$, we can identify df_x with Df_x. Moreover, we can equivalently view df as a map
$$\begin{aligned} U &\longrightarrow \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m) \\ x &\longmapsto Df_x \end{aligned}$$
from U to linear maps $\mathbb{R}^n \to \mathbb{R}^m$.

Remark 3.4. Notice that if the differential of $f : U \to \mathbb{R}^m$ exists at every point, then f is continuous.

One of the key questions we need to answer is ”how do we compute using total derivatives and differentials?” The solution is the following lemmata.

Lemma 3.5. *Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^m$. Denote by $f_i : U \to \mathbb{R}$ for $1 \leq i \leq m$ the i^{th} component function. If f_i is differentiable for every $1 \leq i \leq m$, then f is differentiable*

Proof. It suffices to show the lemma at a point $x \in U$. Suppose the differential of f_i at x exists, and represent it by a $1 \times n$ matrix A_i, so that, for any $1 \leq i \leq m$,
$$\lim_{y \to x} \frac{|f_i(y) - f_i(x) - A_i(y - x)|}{|y - x|} = 0.$$

For $\epsilon > 0$, choose $\delta > 0$ such that, for any $y \in U$ with $|y - x| < \delta$,

$$|f_i(y) - f_i(x) - A_i(y - x)| < \frac{\epsilon}{\sqrt{m}}|y - x|$$

for every $1 \leq i \leq m$.

Define a $m \times n$ matrix A by

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}.$$

We then see that for $y \in U$ such that $|y - x| < \delta$, we have

$$\begin{aligned} |f(y) - f(x) - Ay - x|^2 &= \sum_{j=1}^{m} (f_i(y) - f_i(x) - A_i(y - x))^2 \\ &< \sum_{i=1}^{m} \frac{\epsilon^2}{m}|y - x|^2 = \epsilon^2|y - x|^2. \end{aligned}$$

We thus see that

$$\lim_{y \to x} \frac{|f(y) - f(x) - A(y - x)|}{|y - x|} = 0,$$

as desired. □

Our second lemma is a generalization of the chain rule from single-variable calculus.

Exercise 3.6. Let B be an $m \times n$ matrix. Define

$$\|B\| := \sup_{v \in \mathbb{R}^n} \frac{|Bv|}{|v|}.$$

Show that $\|B\|$ is always finite. Note that for any $v \in \mathbb{R}^n$,

$$|Bv| \leq \|B\||v|.$$

Lemma 3.7. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ be differentiable. Then $g \circ f$ is differentiable, and*

$$D(g \circ f)_x = (Dg)_{f(x)} \circ Df_x$$

for any $x \in \mathbb{R}^n$

Proof. Fix $x \in \mathbb{R}^n$, and denote by A the $m \times n$ matrix representing Df_x, and by B the matrix representing $Dg_{f(x)}$. By the triangle inequality

$$\begin{aligned} |g(f(y)) - g(f(x)) - BA(y - x)| \leq &|g(f(y)) - g(f(x)) - B(f(y) - f(x))| \\ &+ |Bf(y) - B(f(x)) - BA(y - x)| \end{aligned}$$

Since f is differentiable at x, for any $\epsilon > 0$ we may choose $\delta_1 > 0$ such that for $|y - x| < \delta_1$, we have

$$\frac{|f(y) - f(x)|}{|y - x|} < \|A\| + \epsilon$$

and

$$|f(y) - f(x) - A(y-x)| < \epsilon |y-x|.$$

Similarly, since g is differentiable at $f(x)$ and f is continuous, we may choose $\delta_2 > 0$ such that when $|y-x| < \delta_2$,

$$|g(f(y)) - g(f(x)) - B(f(x) - f(y))| < \epsilon |f(y) - f(x)|.$$

Taking $\delta = \min(\delta_1, \delta_2)$ we see that for $|y-x| < \delta$, we have

$$\begin{aligned}|g(f(y)) - g(f(x)) - BA(y-x)| \leq& |g(f(y)) - g(f(x)) - B(f(y) - f(x))| \\ &+ |Bf(y) - B(f(x)) - BA(y-x)| \\ <& \epsilon |f(y) - f(x)| + \epsilon \|B\| |y-x|\end{aligned}$$

Dividing through by $|y-x|$, we obtain

$$\frac{|g(f(y)) - g(f(x)) - BA(y-x)|}{|y-x|} < \epsilon \left(\frac{|f(y) - f(x)|}{|y-x|} + \|B\| \right)$$

And apply the property of δ_1 to see that

$$\frac{|g(f(y)) - g(f(x)) - BA(y-x)|}{|y-x|} < \epsilon \left(\|A\| + \epsilon + \|B\| \right).$$

Since ϵ can be chosen arbitrarily small, this completes the proof. $\square$

Remark 3.8. When considering differentials, the statement of this lemma can be simplified even further. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^k$ be open subsets, and $g : U \to V$ and $f : V \to W$ be smooth maps. Then

$$d(f \circ g) = df \circ dg$$

as maps $TU \to TW$.

This lemma has an immediate corollary, allowing us to give a matrix representing Df_x.

Corollary 3.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function, and denote by $x_i : \mathbb{R} \to \mathbb{R}^n$ the i^{th} coordinate function*

$$\gamma_i(t) = (0, \ldots, 0, \underbrace{t}_{i^{th}}, 0 \ldots, 0).$$

Denote by A the matrix representing Df_0 with respect to the standard bases. Then

$$A_{i,j} = \frac{d}{dt}(f_i \circ \gamma_j)|_{t=0}.$$

The implication of this corollary is that we can compute the matrix representation of $A_{i,j}$ using only the techniques of 1-variable calculus.

Definition 3.10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Define a curve in $\mathbb{R}^n$ through a by

$$\gamma_i(t) = (a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n).$$

The i^{th} *partial derivative of* f[1] at a is

$$\frac{\partial f}{\partial x_i}(a) := \frac{d}{dt} f \circ \gamma_i|_{t=0}.$$

[1] The partial derivative function $\frac{\partial f}{\partial x^i}$ can be computed by applying the usual differentiation rules to an expression for f, treating variables other than x_i as constants.

Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$, and $x \in \mathbb{R}^n$ the matrix representing Df_x is called the *Jacobian* of f at x, and is denoted by Jf_x. By the corollary, we have

$$(Jf_x)_{i,j} = \frac{\partial f_i}{\partial x^j}.$$

We can rewrite Lemma 3.7 in terms of Jacobians.

$$(J(f \circ g))_{i,j} = \frac{\partial (f \circ g)_i}{\partial_x^j} = \sum_{k=1}^{m} \frac{\partial f_i}{\partial y^k} \frac{\partial g_k}{\partial x^j}.$$

Which is the usual chain rule for partial derivatives.

We will also make use of the notion of *smoothness*. This is more or less the same as the corresponding notion for single-variable functions.

Definition 3.11. Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^m$ a function.

- We call f a C^1 *function* if f is differentiable at every point $x \in U$, and the map

$$\begin{aligned} Df : U &\longrightarrow \mathbb{R}^{n \times m} \\ x &\longmapsto (Jf)_x \end{aligned}$$

  is continuous.

- We call f a C^2 *function* if f is C^1 and the map

$$\begin{aligned} Df : U &\longrightarrow \mathbb{R}^{n \times m} \\ x &\longmapsto (Jf)_x \end{aligned}$$

  is C^1.

- $\vdots$

We say that f is C^∞ (or *smooth*) if it is C^k for any k.

Remark 3.12. If $f : U \to V$ is smooth, then its differential, viewed as a map

$$df : U \times \mathbb{R}^n \longrightarrow V \times \mathbb{R}^m,$$

is smooth. Indeed, f is smooth if and only if f is C^1 and df is smooth.

We will not focus on proving smoothness here, but will rather note some facts which allow us to check when functions are smooth.

Fact 3.13. • Any composite of smooth functions is smooth.

- Any sum or difference of smooth functions is smooth.
- Any rational function of n variables is smooth on its domain.
- The function $\sqrt{x}$ is smooth on $(0, \infty)$. The functions exp, sin, cos, and ln are all smooth on their respective (open) domains.

As our final definition, we want a smooth notion of "sameness"

Definition 3.14. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets. We call a function $f : U \to V$ a *diffeomorphism* if f is a C^∞ bijection, and the function f^{-1} is also C^∞.

Exercise 3.15. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty open subsets. Show that if $f : U \to V$ is a diffeomorphism, then $n = m$.

3.2 Smooth manifolds, smooth functions, and derivatives

We can now revisit our definition of a manifold. Previously, we defined manifolds as topological objects — equipped with notions of continuity, but not much more. However, if we want to use calculus to study manifolds, we run into problems.

For instance, given a k-manifold M and a continuous function $f : M \to \mathbb{R}$, we ideally would be able to say whether f is differentiable at a point $x \in M$. Naïvely, this should be easy. Choose $x \in U \subset M$, and a homeomorphism $\phi : U \to V \subset \mathbb{R}^k$, and say that f is differentiable at x if $f \circ \phi^{-1}$ is differentiable at $\phi(x)$. However, this runs into a problem. If $x \in W \subset M$ is another such set, and $\psi : W \to Q \subset \mathbb{R}^k$ another homeomorphism, there is no guarantee that $\psi \circ \phi^{-1}$ is differentiable at $\phi(x)$, thus, depending on our choices, we might get different answers to whether f is smooth at x!

To solve this problem, we require additional compatibilities of our local homeomorphisms.

Definition 3.16. Let M be a topological space. A *chart on* M is a pair (U, ϕ) consisting of an open subset $U \subset M$ and a homeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^n$ onto an open subset of $\mathbb{R}^n$. We say that two charts (U, ϕ) and (V, ψ) on M are *(smoothly) compatible* if the *transition maps*

$$\phi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \phi(U \cap V)$$

and

$$\psi \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

are smooth, and hence mutually inverse diffeomorphisms.

A *smooth atlas on* M is a set $\mathcal{A} := \{(U_i, \phi_i)\}_{i \in I}$ of pairwise compatible charts on M such that the open sets $\{U_i\}_{i \in I}$ cover M. A smooth atlas $\mathcal{A}$ is called *maximal* if, for any other atlas $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} = \mathcal{B}$.

We are now ready to provide our definition of a smooth manifold:

Definition 3.17. A *smooth manifold* of dimension k is a topological manifold with a maximal smooth atlas $\mathcal{A}_M$ consisting of charts to subsets of $\mathbb{R}^k$.

Remark 3.18. One can prove, using Zorn's Lemma, that every smooth atlas is contained in a maximal smooth atlas. As such, to prove that a space is a manifold, it suffices to show that it has a smooth atlas.

Exercise 3.19. Show that S^n and $\mathbb{R}P^n$ are smooth manifolds.

With this in mind, we can now define smooth maps precisely as we expected.

Definition 3.20. Let M be an m-manifold and let N be an n-manifold. Let $f : M \to N$ be a function and let $p \in M$. We say f *is* C^k *at* p if there are charts (U, x) on M around p and (V, y) on N around $f(p)$ such that the composite

$$y \circ f \circ x^{-1}$$

is C^k at $x(p)$. We say that f is C^k if it is C^k at every point $p \in M$.

Exercise 3.21. Show that if f is C^k at p, then for *any* charts (U, x) and (V, y) as in the definition, $y \circ f \circ x^{-1}$ is smooth at $x(p)$.

Notice that, while we have a definition of a differentiable function, we do not yet have a notion of derivative! We will rectify this, however, to approach it cleanly, we must develop some additional technology.

3.3 Paracompactness and partitions of unity

We are now ready to discuss one of the most important properties that smooth manifolds possess: paracompactness.

Definition 3.22. A chart (U, x) on a k-manifold M is called a *coordinate ball* if $x(U) = B_r(a) \subset \mathbb{R}^k$ for some $r > 0$ and $a \in \mathbb{R}^k$.

Exercise 3.23. Show that any k-manifold M has a countable basis of coordinate balls (U, x) such that $\overline{U}$ is compact.

Definition 3.24. A cover $\{U_i\}_{i\in I}$ of a space X is called *locally finite* if, for any $x \in X$, there is an open U_x with $x \in U_x$ such that

$$|\{i \in I \mid U_i \cap U_x \neq \varnothing\}| < \infty.$$

We say that a cover $\{V_j\}_{j\in J}$ is a refinement of a cover $\{U_i\}_{i\in I}$ of X if, for every $j \in J$ there is an $i \in I$ with $V_j \subset U_i$.

A space X is called *paracompact* if every open cover has a locally finite refinement.

Proposition 3.25. *Let M be a k-manifold, and let $\{U_i\}_{i\in I}$ an open cover. Then there is a locally finite cover of M by coordinate balls (V_j, x_j) which refines $\{U_i\}_{i\in I}$ and such that $\overline{V}_j$ is compact. In particular, M is paracompact.*

Exercise 3.26. Prove the proposition in the following steps.

1. Prove that M has an *exhaustion by compact sets*: a set $\{K_i\}_{i=1}^{\infty}$ of compact subsets such that
   - $\bigcup_{i=1}^{\infty} = M$, and
   - $K_i \subset \mathrm{Int}(K_{i+1})$ for all $i \in \mathbb{N}$.
2. Define
$$W_i = \mathrm{Int}(J_{i+2}) \setminus K_{i-1}$$
$$Q_i = K_{i+1} \setminus \mathrm{Int}(K_i)$$
   Show that Q_i admits a finite cover by coordinate balls contained in W_i. Conclude that the proposition holds.

Definition 3.27. Let M be a k-manifold. A *bump function* on $V \subset M$ around p is a smooth map
$$f : M \longrightarrow \mathbb{R}$$
such that

1. the *support* of f
$$\mathrm{supp}(f) := \{q \in M \mid f(q) \neq 0\}$$
   is contained in V;
2. there is an open neighborhood W of p such that $f|_W \equiv 1$; and
3. For any $q \in M$, $0 \leq f(q) \leq 1$.

Lemma 3.28. *There is a bump function on* $(-1, 1) \subset \mathbb{R}$ *around* 0.

Proof. We begin with the smooth function
$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} \exp(-\frac{1}{x}) & x > 0 \\ 0 & x \leq 0 \end{cases} \end{aligned}$$
We can then define
$$g(x) = \frac{f(x+1)}{f(x+1) + f(\frac{1}{2} - x)}$$
and notice that

1. The function g is smooth, as the denominator is never zero.
2. For any $x \in \mathbb{R}$, we have $0 \leq g(x) \leq 1$.
3. We have that $x \leq -1$ if and only if $g(x) = 0$.
4. We have that $x \geq -\frac{1}{2}$ if and only if $g(x) = 1$.

We can then note that the function $g(-|x|)$ is smooth away from 0 as a composite of smooth functions, and is smooth at 0 since it is constant on 1 near 0. Thus $g(-|x|)$ is our desired bump function. $\square$

Exercise 3.29. Prove that f is smooth at 0.

Lemma 3.30. *For any open $U \subset \mathbb{R}^n$ and $p \in U$, there is a bump function on $U \subset \mathbb{R}^n$ around p.*

Proof. Let $\epsilon > 0$ such that $B_\epsilon(p) \subset U$. Then $g(-\frac{1}{\epsilon}|x-p|)$ is the desired bump function. $\square$

Definition 3.31. Let M be a smooth manifold, and let $\{U_i\}_{i\in I}$ be an open cover of M. A *partition of unity subordinate to* $\{U_i\}_{i\in I}$ is a set of functions

$$\left\{ \psi_j : M \longrightarrow \mathbb{R} \right\}_{j\in J}$$

such that

1. For all $j \in J$ and all $p \in M$, $0 \leq \psi_j(p) \leq 1$.
2. For all $j \in J$, $\mathrm{supp}(\psi_j) \subset U_i$ for some $i \in I$.
3. The cover $\{\mathrm{supp}(\psi_j)\}_j \in J$ is locally finite.
4. The sum[2]
$$\sum_{j\in J} \psi_j \equiv 1.$$

[2] This may seem like a strange sum to consider, but since only finitely many of the ψ_j are non-zero on a neighborhood of any given point, the sum is well-defined and smooth.

Proposition 3.32. *Let M be a smooth k-manifold and let $\{U_i\}_{i\in I}$ be an open cover of M. Then there is a partition of unity subordinate to M.*

Proof. Choose a countable cover of M by coordinate balls $\{(V_j, x_j)\}$ with the following properties:

1. $x_j(V_j) = B_1(0) \subset \mathbb{R}^k$ for all $j \in J$.
2. $\{V_j\}_{j\in J}$ is locally finite.
3. The set $\{x_j^{-1}(B_{\frac{1}{2}}(0))\}_{j\in J}$ is a cover of M.

Choose a bump function

$$\theta : B_1(0) \to \mathbb{R}$$

on $B_{\frac{1}{2}}(0) \subset B_1(0)$ about 0 such that $\theta_j(a) > 0$ if and only if $x \in B_{\frac{1}{2}}(0)$, and then define

$$\begin{aligned} \theta_j : M &\longrightarrow \mathbb{R} \\ p &\longrightarrow \begin{cases} \theta(x_j(p)) & p \in V_j \\ 0 & \text{else} \end{cases} \end{aligned}$$

Since the cover $\{V_j\}_{j\in J}$ is locally finite,

$$\mu = \sum_{j\in J} \theta_j$$

is smooth and well-defined Moreover, since the supports of the θ_j cover M, $\mu(p) > 0$ for $p \in M$. We can thus define a smooth function

$$\psi_j := \frac{\theta_j}{\mu}$$

which is the desired bump function. □

3.4 Tangent spaces and differentials

We now have the tools to define derivatives of a smooth map $f : M \to N$ between manifolds. Initially, we might have the following

IDEA: Given (U, x) a chart on M around $p \in M$ and (V, y) a chart on N around $f(p)$, we want to define the derivative of $f : M \to N$ at p to be the Jacobian $J(y \circ f \circ x)_{x(p)}$.

However, we run into a similar problem to the one we encountered in defining differentiability. For example, $\overline{y} = 2y$ is another chart on N about p, defined on the same domain V. However, by chain rule

$$J(\overline{y} \circ f \circ x^{-1})_{x(p)} = 2J(y \circ f \circ x^{-1})_{x(p)}.$$

What this means is that derivatives give us a *system* of numerical values which depend on the chart. The way to solve this problem is by appealing to our idea of the total derivative as a *linear map* between vector spaces. Within the framework we develop, the derivative of a smooth map between smooth manifolds will be a linear map between vector spaces. Each chart will give a basis of these vector spaces, and the Jacobians computed with respect to these charts will give the matrix representations with respect to these bases.

Our basic idea is that the derivative turns possible velocities (or *tangent vectors*) at $p \in M$ into possible velocities at $f(p) \in N$.

So we arrive at

QUESTION: How do we define *tangent vectors* or *velocity vectors* at $p \in M$?

3.4.1 First definition: Curves

Any curve through p — which we can interpret as the motion of a particle in time — should have an associated velocity vector at p. To make this formal, we must ascertain when two curves should have the same tangent vector.

However, this is not as hard as it seems. Two curves should have the same tangent vector when they *locally* have the same velocity. That is, when, *in the same chart* their derivatives agree.

Definition 3.33. Let M be a smooth k-manifold, and let $p \in M$. A *curve through* p is a continuous map

$$\gamma : \qquad (-\epsilon, \epsilon) \longrightarrow M$$

for some $\epsilon > 0$ such that $\gamma(0) = p$.[3]

We call a curve γ *differentiable* (or C^1) if there is a chart (U, x) such that $x \circ \gamma$ has a continuous total derivative on $(\gamma^{-1}(U))$.

[3] One can define curves in a more general fashion, but requiring that $\gamma(0) = p$ will be very convenient for our purposes here.

We now define an equivalence relation on curves

$$\gamma_1 : \qquad (-\epsilon_1, \epsilon_1) \longrightarrow M$$

and

$$\gamma_2 : \qquad (-\epsilon_2, \epsilon_2) \longrightarrow M$$

through p. We say that $\gamma_1 \sim_p \gamma_2$ if and only if there exists a chart (U, x) on M around p such that

$$D(x \circ \gamma_1)_0 = D(x \circ \gamma_2)_0$$

or, equivalently,

$$\frac{d(x \circ \gamma_1)}{dt}(0) = \frac{d(x \circ \gamma_2)}{dt}(0).$$

Exercise 3.34. If $\gamma_1 \sim_p \gamma_2$, then for *any* chart (V, y) on M around p,

$$D(y \circ \gamma_1)_0 = D(y \circ \gamma_2)_0.$$

Definition 3.35. A *tangent vector* at $p \in M$ is an equivalence class $[\gamma]$ of C^1 curves through p under $\sim_p$. The *tangent space* is

$$T_pM = \{C^1 \text{ curves through } p\}_{/\sim_p}$$

However, we have already given a definition of tangent space — for $U \subset \mathbb{R}^n$ an open subset and $a \in U$, we previously defined

$$T_aU := \mathbb{R}^n.$$

To connect these two definitions, we prove a short lemma.

Lemma 3.36. *Let U be an open subset of $\mathbb{R}^n$ and $a \in U$. The map*

$$\begin{aligned} \mathrm{d}_a : T_aU &\longrightarrow \mathbb{R}^n \\ &\longmapsto \tfrac{d\gamma}{dt}(0) \end{aligned}$$

is a bijection.

Proof. It is immediate that d_a is well-defined on equivalence classes. We define an inverse which sends $v \in \mathbb{R}^n$ to the equivalence class of the curve

$$\begin{aligned} \gamma_v : (-\epsilon, \epsilon) &\longrightarrow U \\ t &\longmapsto a + tv \end{aligned}$$

where $\epsilon > 0$ is some real number small enough that $B_\epsilon(a) \subset U$. Note that the equivalence class does not depend on the choice of ϵ.

We then note that

$$\mathrm{d}_a(\gamma_v) = \frac{d\gamma_v}{dt}(0) = v.$$

Moreover, given any curve γ with $\frac{d\gamma}{dt}(0) = v$, by definition, $\gamma \sim_a \gamma_v$. Thus d_a is a bijection. □

As a generalization of this lemma, we can define an induced map between tangent spaces associated to any smooth (indeed, even C^1) map between manifolds.

Definition 3.37. Let $f : M \to N$ be a C^1 map between manifolds. Define the *pushforward* or *differential* of the map f at $p \in M$ to be the map

$$\begin{aligned} df_p : T_pM &\longrightarrow T_{f(p)}N \\ [\gamma] &\longmapsto [f \circ \gamma]. \end{aligned}$$

To see that the differential is well-defined, we note that, choosing charts (U, x) on M around p and (V, y) on N around $f(p)$. We then can simply compute that

$$\begin{aligned} \frac{d}{dt}|_{t=0}\,(y \circ f \circ \gamma) =& \frac{d}{dt}|_{t=0}\left(y \circ f \circ x^{-1} \circ x \circ \gamma\right) \\ =& D(y \circ f \circ x^{-1})_{x(p)}\left(\frac{d}{dt}|_{t=0}(x \circ \gamma)\right) \end{aligned}$$

Thus, if $\gamma_1 \sim_p \gamma_2$, we have $f \circ \gamma_1 \sim_{f(p)} f \circ \gamma_2$.

Proposition 3.38. *Given $f : M \to N$ and $g : N \to L$ two C^1 maps of smooth manifolds, $d(g \circ f)_p = dg_{f(p)} \circ df_p$. In particular, if $f : M \to N$ is a diffeomorphism (i.e., has a smooth inverse) then df_p is a bijection for all $p \in M$.*

Proof. Exercise. □

Corollary 3.39. *If (U, x) is a chart on a smooth manifold M around p, then $dx_p : T_pM \to T_{x(p)}\mathbb{R}^m \cong \mathbb{R}^m$ is an isomorphism of vector spaces.*

Proof. The definition of T_pM is local (i.e., depends only on U) and

$$x : U \longrightarrow x(U)$$

is a diffeomorphism. □

Lemma 3.40. *Let $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ be a C^1 map between open subsets of Euclidean spaces. Let $p \in U$. Under the identifications $T_pU \cong \mathbb{R}^n$ and $T_{f(p)}V \cong \mathbb{R}^m$ of Lemma 3.36, the map df_p is identified with the differential Df_p. More formally,*

$$\frac{d}{dt}(f \circ \gamma_v)(0) = Df_p(\frac{d\gamma_v}{dt}).$$

Proof. This is the chain rule. □

We now can define a vector space structure on T_pM — choosing the vector space structure which makes dx_p for a given chart an isomorphism of vector spaces. As is now the norm, we make our definition with the aid of a chart (U, x) around p. Let $[\gamma_1], [\gamma_2], [\gamma_3] \in T_pM$, and let $\lambda \in \mathbb{R}$.

- We set
$$[\gamma_1] + [\gamma_2] = [\gamma_3]$$
if and only if
$$\frac{d(x \circ \gamma_1)}{dt}(0) + \frac{d(x \circ \gamma_2)}{dt}(0) = \frac{d(x \circ \gamma_3)}{dt}(0).$$
- We set
$$\lambda[\gamma_1] = [\gamma_2]$$
if and only if
$$\lambda \frac{d(x \circ \gamma_1)}{dt}(0) = \frac{d(x \circ \gamma_1)}{dt}(0).$$

Lemma 3.41. *For a smooth k-manifold M and $p \in M$ the definitions above are independent of the choice of chart.*

Proof. For notice that if (V, y) is another chart, we have that
$$\frac{d}{dt}(y \circ x^{-1} \circ x \circ \gamma) = J(y \circ x^{-1})_x(p)(\frac{d(x \circ \gamma)}{dt}(0))$$
so that the derivatives of $y \circ \gamma$ and $x \circ \gamma$ differ by a linear map. Thus, the definitions are independent of the choice of chart. □

Definition 3.42. Let $f : M \to N$ be a C^1 map between manifolds. Define the *pushforward* or *differential* of the map f at $p \in M$ to be the map
$$\begin{aligned} df_p : T_pM &\longrightarrow T_{f(p)}N \\ [\gamma] &\longmapsto [f \circ \gamma]. \end{aligned}$$

Proposition 3.43.

1. *For a C^1 map $f : M \to N$ between smooth manifolds, (U, x) a chart around $p \in M$, and (V, y) a chart around $f(p) \in N$ the diagram*
$$\begin{tikzcd} T_pM \arrow[r, "df_p"] \arrow[d, "dx_p"'] & T_{f(p)}N \arrow[d, "dy_{f(p)}"] \\ \mathbb{R}^{,} \arrow[r, "D(y \circ f \circ x^{-1})_{x(p)}"'] & \mathbb{R}^n \end{tikzcd}$$
commutes.
2. *For a C^1 map $f : M \to N$ between smooth manifolds, the differential $df_p : T_pM \to T_{f(p)}N$ is linear for any $p \in M$.*

Proof. For (1), suppose we are given charts (U, x) on M around p and (V, y) on N around $f(p)$. By restricting the chart (U, x), we may assume that $f(U) \subset V$. We thus have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "f|_U"] \arrow[d, "x"'] & V \arrow[d, "y"] \\ \mathbb{R}^m \arrow[r, "y \circ f \circ x^{-1}"'] & \mathbb{R}^m \end{tikzcd}$$

of C^1 maps among smooth manifolds. By Proposition 3.38, the diagram

$$\begin{tikzcd} T_pM \arrow[r, "df_p"] \arrow[d, "dx_p"'] & T_{f(p)}N \arrow[d, "dy_{f(p)}"] \\ \mathbb{R}^{,} \arrow[r, "D(y \circ f \circ x^{-1})_{x(p)}"'] & \mathbb{R}^n \end{tikzcd}$$

commutes.

This immediately implies that df_p is linear, since it shows that

$$df_p = (dy_{f(p)})^{-1} \circ \circ D(y \circ f \circ x^{-1})_{x(p)} \circ dx_p$$

is a composite of linear maps. $\square$

3.4.2 Second definition: derivatives

The other interpretation one can take of tangent directions in $\mathbb{R}^n$ is that they are "directions one can take a derivative in." To generalize this definition to manifolds, we need to define what we mean by a *derivative operator at a point* $p \in M$

Definition 3.44. Let M be a smooth m-manifold, and $p \in M$ a point. A *p-derivation* on M is a linear map

$$\delta : C^\infty(M) \longrightarrow \mathbb{R}$$

which satisfies the *Leibniz Rule* at p, namely, for any $f, g \in C^\infty(M)$, we have

$$\delta(f, g) = \delta(f)g(p) + f(p)\delta(g).$$

The set of p-derivations on M is a vector space under pointwise addition, which we denote by $\mathrm{Der}_p(M)$.

Construction 3.45. Let M be a smooth manifold, $p \in M$, and $v = [\gamma] \in T_pM$. For any $f \in C^\infty(M)$, we define the *derivative of f at p in the v-direction* to be

$$\delta_v(f) := \frac{d(f \circ \gamma)}{dt}(0)$$

Exercise 3.46. Show that δ_v is a p-derivation. Show that the map $v \mapsto \delta_v$ is linear.

Proposition 3.47. *The map*

$$\begin{aligned} T_pM &\longrightarrow \mathrm{Der}_p(M) \\ v &\longmapsto \delta_v \end{aligned}$$

is an isomorphism of vector spaces.

To prove this proposition, we will need a famous lemma.

Lemma 3.48 (Hadamard's Lemma)**.** *Let M be an m-manifold, $p \in M$, and (U, x) a chart around p. Then there is an open subset $V \subset U$ containing p such that, for any $f \in C^\infty(M)$, there exist smooth functions $f_i : V \to \mathbb{R}$ for $1 \leq i \leq m$ such that*

$$f|_V = f(p) + \sum_{i=1}^{m} (x^i - x^i(p)) f_i$$

and

$$f_i(p) = \frac{\partial f \circ x^{-1}}{\partial x^i}(x(p)).$$

Proof. We will prove the lemma when $M = \mathbb{R}^m$. The general case follows directly by passing through a chart. Let $U \subset \mathbb{R}^m$ be an open subset and $a \in U$, and set $V = B_r(a) \subset U$ be an open ball. Then for any $x \in V$ we can define a smooth function

$$g_x(t) = f(a - t(x - a))$$

for $t \in [0, 1]$. By the fundamental theorem of calculus, we find

$$f(x) - f(a) = \int_0^1 \frac{d}{dt} g_x(t) dt$$

so that

$$f(x) = f(a) + \int_0^1 \frac{d}{dt} g_x(t) dt.$$

We then note that by the chain rule,

$$\frac{d}{dt} g_x(t) = \sum_{i=1}^{m} (x^i - a^1) \frac{\partial f}{\partial x^i}(a - t(x - a)).$$

Thus, setting

$$f_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(a - t(x - a)) dt$$

we obtain

$$f(x) := f(a) + \sum_{i=1}^{m} (x^i - a^i) f_i(x)$$

with

$$f_i(a) = \int_0^1 \frac{\partial f}{\partial x^i}(a - t(x - a)) dt = \int_0^1 \frac{\partial f}{\partial x^i}(a) dt = \frac{\partial f}{\partial x^i}(a)$$

as desired. □

of Proposition 3.47. We begin with surjectivity. Let $\delta \in \mathrm{Der}_p(M)$. First, we note that for the constant function 1, we have

$$\delta(1) = \delta(1 \cdot 1) = \delta(1) + \delta(1) = 2\delta(1)$$

and so, by linearity $\delta(c) = 0$ for any constant function c. Given a function $f \in C^\infty(M)$ on V we can write

$$f = f(p) + \sum_{i=1}^{m} (x^i - x^i(p)) f_i.$$

Applying δ to this expression, we see that $\delta(f(p))$ vanishes, and so, by the Leibniz rule, we obtain

$$\delta(f) = \sum_{i=1}^{m} \delta(x^i - x^i(p)) \frac{\partial f}{\partial x^i}(p).$$

But then, taking $u \in \mathbb{R}^m$ to be the vector

$$u = \left(\delta(x^1 - x^1(p)), \ldots, \delta(x^m - x^m(p))\right).$$

Then we see that $\delta = \delta_v$ where v is the equivalence class of the path

$$\gamma_u(t) := x^{-1}(x(p) + ut)$$

in V. Thus the map is surjective.

To show injectivity, let (U, x) be a chart around p and suppose that $u \in \mathbb{R}^m$ such that $\delta_{[\gamma_u]} = 0$. Then define a function f_i on U by

$$f_i(q) = x^i(q)$$

and extend it to all of M by a bump function around p. Then

$$\delta_{[\gamma_u]}(f_i) = \frac{df_i \circ \gamma}{dt} = \frac{\partial f_i \circ x^{-1}}{\partial x^j} u^j = u^i$$

Thus, $u = 0$, and the map is injective. □

Definition 3.49. Given a chart (U, x) around $p \in M$, the identification

$$\mathbb{R}^m \cong T_p M$$

induced by x sends the standard basis $e_1, \ldots, e_m$ to a basis which we denote by

$$\left\{ \frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p \right\}.$$

Viewing these tangent vectors as derivative operators, we have

$$\frac{\partial}{\partial x^i}|_p f = \frac{\partial f \circ x^{-1}}{\partial x^i}(p).$$

justifying the notation we chose.

4

Integration

4.1 Rewriting integration

Differential forms are a key modern[1] innovation in the study of calculus on manifolds, and so it is now time for us to begin exploring them. The basic idea is that differential k-forms are "k-dimensional measuring tools". A 1-form is a way of measuring lengths (a yardstick, so to speak), a 2-form is a way of measuring areas, and so on. From a related perspective, a k-form can be seen as a "thing that can be integrated over a k-dimensional submanifold."

[1] Relatively speaking. The formal theory of differential forms is sometimes said to have begun with Élie Cartan's Sur certaines expressions différentielles et le problème de Pfaff in 1899.

Both of these heuristic ideas are realized in a common algebraic framework. Our first task in this section will be to build up this framework on $\mathbb{R}^n$. Once we have done this, we will proceed to defining differential forms on manifolds, and then to applications.

Before getting to the definition of forms, let's take a step back, and try to remember what we are doing when we are integrating something. The basic idea we will work with, familiar from high-school calculus, is that of a Riemann sum. Suppose first we have a function $f : \mathbb{R} \to \mathbb{R}$. When we integrate f on the interval $[a, b]$, we take some subdivisions of $[a, b]$ into N little pieces $[t_i, t_{i+1}]$, and then take the limit of Riemann sums

$$\int_a^b f dt = \lim_{N\to\infty} \sum_{i=1}^{N} f(t_i)\Delta_i t,$$

where $\Delta_i t = t_{i+1} - t_i$.

If we take this into n-dimensions, we can suppose that we have a curve $\gamma : [a, b] \to \mathbb{R}^n$, and want to integrate a tangent vector field X along γ. In this case, we can write our integral as a limit

$$\int \gamma f = \lim_{N\to\infty} \sum_{i=1}^{N} \left\langle X_{\gamma(t_i)}, \Delta_i \gamma \right\rangle$$

where $\Delta_i \gamma = \gamma(t_{i+1}) - \gamma(t_i)$.

How do we make sense of the expression $\left\langle X_{\gamma(t_i)}, \Delta_i \gamma \right\rangle$? Well, the first step is to reinterpret how we think of $\Delta_i \gamma$. *A priori*, this is simply a vector $\gamma(t_{i+1}) - \gamma(t_i)$. However, under the identification $T_{\gamma(t_i)}\mathbb{R}^n \cong \mathbb{R}^n$, we can think of this as a *tangent* *vector at* $\gamma(t_i)$, which, as the mesh of the limit we are taking gets finer, becomes close to being a tangent vector to γ. We can thus think of the expression

$$\langle X_p, - \rangle$$

as being a linear map $T_p\mathbb{R}^n \to \mathbb{R}$. That is, a linear map which takes a tangent vector at p and gives us a scalar.

On the other hand, suppose that, for each $p \in M$, we have a linear map $\omega_p : T_pM \to \mathbb{R}$, and that the map ω which sends $p \mapsto \omega_p$ is in some sense smooth.[2] Then along any curve γ, we can define an integral

[2] We'll get to what this means later.

$$\int_\gamma \omega = \lim_{N\to\infty} \sum_{i=1}^{N} \omega_{\gamma(t_i)}(\Delta_i \gamma) = \int_a^b \omega_{\gamma(t)}(\frac{\partial}{\partial t})dt.$$

The second of these equalities requires some justification, but it should not be hard to convince yourself that it holds.

The fact that this map is *linear* in the tangent vector is very useful. In particular, it should be easy to convince yourself using linearity that the following expected integral equalities hold:[3]

[3] Here we write $\overline{\gamma}$ for the curve which traces γ backwards.

$$\int_\gamma (\omega + \nu) = \int_\gamma \omega + \int_\gamma \nu$$
$$\int_{\overline{\gamma}} \omega = -\int_\gamma \omega = \int_\gamma (-\omega)$$
$$\int_\gamma (c\omega) = c \int_\gamma \omega$$

where γ is a curve, $-\gamma$ is the curve that traces γ backwards, ω and ν are our chosen "smooth assignments of linear maps $T_p\mathbb{R}^n \to \mathbb{R}$", and $c \in \mathbb{R}$ is a constant.

We can turn this reformulation into a definition.

Definition 4.1. Given a manifold M, a *covector* at $p \in M$ is a linear map

$$\omega_p : T_pM \longrightarrow \mathbb{R}.$$

The *cotangent space* to M at p is the $\mathbb{R}$-vector space

$$T_p^*M := \mathrm{Lin}(T_pM, \mathbb{R}).$$

Our next goal is to formalize what it means to assign a covector in a *smooth* way to every point in M. We will call such an assignment ω a *(smooth) 1-form*.

First, let's try to think about what the space $\mathrm{Lin}(T_pM, \mathbb{R})$ is in terms of coordinates. If (U, x) are our coordinates on M, we have a corresponding basis of tangent fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$ on MU. At each $p \in M$, we can take the *dual basis* of $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$. The dual basis consists of the linear maps

$$\begin{aligned} \mathsf{d}x_p^i : T_pM &\longrightarrow \mathbb{R} \\ \tfrac{\partial}{\partial x^j} &\longmapsto \delta_j^i \end{aligned}$$

where δ_j^i is the Krönecker delta. From linear algebra, this is a basis of T_p^*M. As such, we have an identification $T_p^*M \cong \mathbb{R}^n$.

Definition 4.2. We define the *cotangent bundle* of M to be the space

$$T^*M \cong \{(p, \nu_p) \mid \nu_p \in T_p^*M\}$$

It is at this point that we introduce the

EINSTEIN SUMMATION CONVENTION: If an index occurs twice in a product, once above, and once below, we sum over all possible values of that index. Thus, for example

$$u^i(1 + v_i^j) = \sum_{i=1}^{m} u^i(1 + v_i^j).$$

With this convention in place, we can simplify many of the written expressions in our computations. For instance, given $\omega \in T_p^*M$ we can find numbers $\omega_i \in \mathbb{R}$ such that

$$\omega = \omega_i dx_p^i = \sum_{i=1}^{m} \omega_i dx_p^i$$

where the last equality is simply expanding out the sum implicit in the Einstein summation convention.

From the smooth atlas $\mathcal{A}$ on M, we can define an atlas

$$\left\{ \begin{array}{rcl} T^*M|_U & \longrightarrow & x(U) \times \mathbb{R}^m \\ (p, \omega_i dx_p^i) & \longmapsto & (x(p), (\omega_1, \ldots, \omega_m)) \end{array} \right\}_{(U,x) \in \mathcal{A}}$$

To show that this is, indeed, a smooth atlas, we need to understand how the coefficients ω_i behave under a change of local coordinates.

Lemma 4.3. *Let (U, x) and (V, y) be two charts on M with $U \cap V \neq 0$. Then*

$$dx^i = \frac{\partial x^i}{\partial y^\ell} dy^\ell.$$

Proof. We compute the value of the left-hand side on the tangent vector

$$V = V^j \frac{\partial}{\partial y^j}.$$

Obtaining

$$\begin{aligned} dx^i \left(V^j \frac{\partial}{\partial y^j} \right) &= dx^i \left(V^j \frac{\partial x^k}{\partial y^j} \frac{\partial}{\partial x^k} \right) \\ &= V^j \frac{\partial x^k}{\partial y^j} dx^i\left(\frac{\partial}{\partial x^k}\right) \\ &= V^j \frac{\partial x^i}{\partial y^j} \\ &= V^j \frac{\partial x^i}{\partial y^\ell} \delta_j^\ell \\ &= V^j \frac{\partial x^i}{\partial y^\ell} dy^\ell\left(\frac{\partial}{\partial y^j}\right) \\ &= \frac{\partial x^i}{\partial y^\ell} dy^\ell \left(V^j \frac{\partial}{\partial y^j} \right) \end{aligned}$$

as desired. $\square$

Corollary 4.4. *For M a smooth manifold, the atlas above defines a smooth manifold structure on T^*M. The projection*

$$\begin{aligned} \pi : T^*M &\longrightarrow M \\ (p, v) &\longmapsto p \end{aligned}$$

is smooth.

Proof. By the lemma, the transition function for the two charts on $T^*M|_U$ defined by (U, x) and (V, y) is given by

$$(a, (\omega_i)) \mapsto (y^{-1}(x(a)), \left(\omega_i \frac{\partial x^i}{\partial y^j}\right))$$

which is clearly smooth.

In any such chart, the projection π is simply given by forgetting coordinates, and thus is smooth. $\square$

Definition 4.5. A *(smooth) 1-form on M* is a smooth map

$$\omega : M \longrightarrow T^*M$$

such that $\pi \circ \omega = \mathrm{id}_{\mathbb{R}^n}$. That is, such that $\omega(p) \in T_p^*M$.

Example 4.6. In particular, given a chart (U, x) we have the coordinate 1-forms $\mathsf{d}x^i$ on U which assign to every point $p \in U$ the covector $\mathsf{d}x^i_p$. By construction, for any smooth 1-form ω on U, there are unique smooth functions $\omega_i : U \to \mathbb{R}$ such that

$$\omega = \omega_i \mathsf{d}x^i = \omega_1 \mathsf{d}x^1 + \cdots + \omega_n \mathsf{d}x^n,$$

where we again use the Einstein summation convention.

Since the entire point of defining 1-forms was that they should be "things we can integrate along curves", let's think about what the integral of such a 1-form along a curve $\gamma : [a, b] \to \mathbb{R}^n$ should be. From our definition, we should have

$$\begin{aligned} \int_\gamma \omega &= \int_\gamma \omega_i \mathsf{d}x^i \\ &= \int_a^b \omega_i(\gamma(t)) \mathsf{d}x^i\,(\gamma')\,dt. \end{aligned}$$

But we can write $\gamma'(t) = \frac{d\gamma^i}{dt}\partial_{x^i}$, so by the definition of $\mathsf{d}x^i$ we have

$$\int_a^b \omega_i(\gamma(t)) \mathsf{d}x^i\,(\gamma')\,dt = \int_a^b \omega_i(\gamma(t)) \frac{d\gamma^i}{dt} dt$$

In particular, if we are simply integrating the form $\omega = f(x)\mathsf{d}x^1$, then we are, effectively, integrating f along γ in only the x^i-direction.

Returning to the general case, we now study the transformation behavior of 1-forms.

Definition 4.7. Let $f : M \to N$ be a smooth map between smooth manifolds, and let $\omega \in T^*_{f(p)}N$ be a covector at $f(p) \in N$. The *pullback* of ω along f is defined to be the 1-form $f_p^*\omega \in T_p^*M$ defined by

$$(f_p^*\omega)(v) := \omega(df_p(v))$$

for $v \in T_pM$. The pullback of a 1-form $\omega : M \to T^*M$ is given by

$$(f^*\omega)(p) = f_p^*(\omega(f(p))).$$

Lemma 4.8. *For smooth maps*

$$M \xrightarrow{g} N \xrightarrow{f} L$$

then

$$(f \circ g)^* = g^* \circ g^*.$$

Proof. Exercise. □

Proposition 4.9. *Let $f : M \to N$ be a smooth map, let $p \in M$, and let (U, x) and (V, y) be charts around p and $f(p)$, respectively. For a 1-form ω written with respect to x as*

$$\omega = \omega_i dx^i$$

then

$$f^*(\omega) = (\omega_i \circ f)\frac{\partial f^i}{\partial x^\ell} dx^\ell$$

Proof. It suffices to show that

$$f^*dy^i = \frac{\partial f^i}{\partial x^\ell}\partial x^\ell,$$

which follows by computing both sides' application to $V = V^i \frac{\partial}{\partial x^i}$. □

Definition 4.10. Let $\omega = f(t)dt$ be a 1-form on $\mathbb{R}$, and $[a, b] \subset \mathbb{R}$ an interval. We define

$$\int_{[a,b]} \omega := \int_a^b f(t)dt.$$

Let $\gamma : [a, b] \to M$ be a smooth curve, and ω a 1-form on M. The *line integral of ω over γ* is

$$\int_\gamma \omega := \int_{[a,b]} \gamma^*\omega.$$

We now want to show that this is independent of the parameterization of the curve γ, i.e., that it is a true line integral.

Proposition 4.11. *For a smooth 1-form ν on M, $\gamma : [a, b] \to M$ a smooth curve, and $\phi : [c, d] \to [a, b]$ an (orientation-preserving) diffeomorphism,*

$$\int_{\gamma\circ\phi} \nu = \int_\gamma \nu.$$

Proof. Set $\gamma^*\nu =: \omega = f(t)dt$. Then $\phi^*\omega = f(\phi(s))\frac{d\phi}{ds}ds$. Thus,

$$\begin{aligned} \int_{\gamma\circ\phi} \nu &= \int_c^d \omega(\phi(s))\frac{d\phi}{ds}ds \\ &= \int_a^b \omega(t)dt \\ &= \int_\gamma \nu \end{aligned}$$

as desired. □

4.2 Multilinear Algebra

We now want to generalize the 1-forms — things we can integrate along curves — to k-forms — things we can integrate along k-dimensional subspaces of M. However, to do so, we must digress into some linear algebra.

If we think of our 1-forms as measuring something like a "signed length of a vector in a given direction at $p \in M$", then we might expect that we need to define something like a "signed volume of a parallelpiped in T_pM" to define k-dimensional integrals. From our knowledge of determinants, we would expect such a volume measurement to be a multilinear, alternating map. We will treat each of these properties algebraicly in turn.

4.2.1 Multilinearity

Fix an n-dimensional vector space V, and let W be any other vector space.

Definition 4.12. A map

$$f : V^{\times k} \longrightarrow W$$

is called (k-)multilinear if

$$f(v_1, \dots, v_i + \lambda w_i, \dots, v_k) = f(v_1, \dots, v_k) + \lambda f(v_1, \dots, w_i, \dots, v_k)$$

for all $v_1, \dots, v_k, w_i \in V$, all $1 \leq i \leq k$ and all $\lambda \in \mathbb{R}$. The set of k-multilinear maps from $V^{\times k}$ to W forms a vector space, which we denote by $\mathrm{Mult}^k(V, W)$.

Definition 4.13. Let $\mathrm{Free}(V^{\times k})$ denote the free vector space on the set $V^{\times k}$. We define the k^{th} *tensor power* of V to be the quotient space

$$V^{\otimes k} := \mathrm{Free}(V^{\times k})/L$$

where

$$L := \mathrm{Span}_{\mathbb{R}}\left\{(v_1, \dots, v_i + \lambda w_i, \dots, v_k) - (v_1, \dots, v_k) - \lambda(v_1, \dots, w_i, \dots, v_k)\Big|_{\substack{1\leq i\leq k \\ v_1,\dots,v_k,w_i\in V \\ \lambda\in\mathbb{R}}}\right\}$$

is the subspace generated by the multilinearity relations. We denote the image of $(v_1, \dots, v_k)$ in $V^{\otimes k}$ by $v_1 \otimes \cdots \otimes v_k$.

Proposition 4.14 (Universal property of the tensor power)**.** *Denote the composite* $V^{\times k} \hookrightarrow \mathrm{Free}(V^{\times k}) \to V^{\otimes k}$ *by* q*. Then* q *is multilinear. For any multilinear map* $f : V^{\times k} \to W$*, there is a unique linear map* $\overline{f} : V^{\otimes k} \to W$ *such that* $\overline{f} \circ q = f$*. That is, such that the diagram*

$$\begin{tikzcd} V^{\times k} \arrow[r, "f"] \arrow[d, "q"'] & W \\ V^{\otimes k} \arrow[ur, dashed, "\exists ! \overline{f}"'] & \end{tikzcd}$$

commutes.

Proof. We leave it to the reader to verify that q is multilinear.

To prove the universal property, let $f : V^{\times k} \to W$ be a multilinear map. The universal property of the free vector space shows that f induces a unique linear map $\widetilde{f} : \mathrm{Free}(V^{\times k}) \to W$ which restricts to f on $V^{\times k}$.

Since f is multilinear, we have

$$\begin{aligned} &\widetilde{f}((v_1, \ldots, v_i + \lambda w_i, \ldots, v_k) - (v_1, \ldots, v_k) - \lambda(v_1, \ldots, w_i, \ldots, v_k)) \\ =&f(v_1, \ldots, v_i + \lambda w_i, \ldots, v_k) - f(v_1, \ldots, v_k) - \lambda f(v_1, \ldots, w_i, \ldots, v_k) \\ =&0 \end{aligned}$$

so $\widetilde{f}$ vanishes on the subspace L. Thus, by the universal property of the quotient, $\widetilde{f}$ induces a unique linear map $\overline{f} : V^{\otimes k} \to W$. This map makes the diagram above commute, and is the unique map with this property, completing the proof. □

Corollary 4.15. *Composition with* q *induces an isomorphism*

$$\mathrm{Mult}^k(V, W) \cong \mathrm{Lin}(V^{\otimes k}, W).$$

For ease of notation, we denote by $\underline{n} := \{1, 2, \ldots, n\}$.

Lemma 4.16. *Let* $\{v_1, \ldots, v_k\}$ *be a basis of* V*. Then the set*

$$\{v_{i_1} \otimes \cdots \otimes v_{i_k}\}_{\vec{i} \in \underline{n}^{\times k}}$$

is a basis of $V^{\otimes k}$*. In particular* $\dim(V^{\otimes k}) = n^k$

Proof. To see that the desired set spans $V^{\otimes k}$, note that the set of tensors $w_1 \otimes \cdots w_k$ which ranges over *all* $w_1, \ldots, w_k \in V$ spans $V^{\otimes k}$ by construction. If we write

$$w_i = \lambda_i^j v_j$$

(using the Einstein summation convention) then we see that

$$\begin{aligned} w_1 \otimes \cdots w_k &= (\lambda_1^j v_j) \otimes \cdots \otimes (\lambda_k^j v_j) \\ &= \lambda_1^{i_1} \cdots \lambda_k^{i_k} v_{i_1} \cdots v_{i_k} \end{aligned}$$

is a linear combination of our purported basis vectors, and thus the proposed basis spans $V^{\otimes k}$.

On the other hand, given $\vec{j} \in \underline{n}^{\times k}$ we can define a multilinear map

$$\begin{aligned} \psi_{\vec{j}} : \quad & V^{\times k} \longrightarrow \mathbb{R} \\ & (\lambda_1^i v_i, \ldots, \lambda_k^i v_i) \longmapsto \lambda_1^{j_1} \cdots \lambda_k^{j_k} \end{aligned}$$

We thus obtain a linear map $\psi_{\vec{j}} : V^{\otimes k} \to \mathbb{R}$ by universal property.

Thus, if

$$u := \sum_{\vec{i} \in \underline{n}^{\times k}} \mu^{\vec{i}} v_{i_1} \otimes \cdots \otimes v_{i_k} = 0$$

is a linear relation among our proposed basis, applying $\psi_{\vec{j}}$ shows that

$$\psi_{\vec{j}}(u) = \mu^{\vec{j}} = 0$$

and so the proposed basis is linearly independent. □

Exercise 4.17. Show that if $v_1 \otimes \cdots \otimes v_k = 0$, then there exists an $1 \leq i \leq k$ such that $v_i = 0$.

Proposition 4.18. *There is an isomorphism*

$$\Psi : (V^*)^{\otimes k} \xrightarrow{\cong} \mathrm{Mult}^k(V, \mathbb{R})$$

defined by

$$\Psi(f^1 \otimes \ldots \otimes f^k)(v_1, \ldots, v_k) = f^1(v_1) f^2(v_2) \cdots f^k(v_k).$$

Proof. We leave it to the reader to check that the map Ψ so defined is, indeed a linear map.

To see that Ψ is injective, suppose that $f^1 \otimes \ldots \otimes f^k$ is non-zero, so that each f^i is non-zero. Then choose a vector v_i such that $f^1(v_i) \neq 0$. We thus have

$$\Psi(f^1 \otimes \ldots \otimes f^k)(v_1, \ldots, v_k) = \prod_i f^i(v_i) \neq 0$$

and so $\Psi(f^1 \otimes \ldots \otimes f^k)$ is non-zero. Thus Ψ is injective, and since the two vector spaces have the same dimension, Ψ is an isomorphism. □

4.2.2 Alternation

We now turn to alternating multilinear maps. We fix the n-dimensional vector space V.

Definition 4.19. A k-multilinear map $f : V^{\times k} \to W$ is said to be *alternating* if, for $1 \leq i < j \leq k$,

$$f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Equivalently, we say that a linear map $\overline{f} : V^{\otimes k} \to W$ is alternating if the associated multilinear map is alternating, i.e. if

$$f(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_k) = -f(v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k)$$

We will write $\mathrm{Alt}^k(V, W)$ for the vector space of alternating multilinear maps $V^{\times k} \to W$.

Definition 4.20. We define the k^{th} exterior power $\bigwedge^k V$ of V to be the quotient of $V^{\otimes k}$ by the subspace L spanned by

$$\left\{ v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_k + v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k \middle|_{v_\ell \in V}^{1 \le i < j \le k} \right\}.$$

We write $v_1 \wedge \cdots \wedge v_k$ for the image of $v_1 \otimes \cdots \otimes v_k$ in $\bigwedge^k V$.

Exercise 4.21. Prove the *universal property of the exterior power.* Write $q : V^{\otimes k} \to \bigwedge^k V$ for the quotient map. Then q is alternating and linear. Moreover, given any alternating linear map $f : V^{\otimes k} \to W$, there is a unique linear map $\overline{f} : \bigwedge^k V \to W$ such that $\overline{f} \circ q = f$. Conclude there is an isomorphism

$$\mathrm{Alt}^k(V, W) \to \mathrm{Lin}(\bigwedge^k V, W)$$

induced by composition with q.

Exercise 4.22. Show that for any $w_1, \ldots, w_k \in V$ and $\sigma \in S_k$,

$$w_1 \wedge \cdots \wedge w_k = \mathrm{sgn}(\sigma) w_{\sigma(1)} \wedge \cdots \wedge w_{\sigma(k)}$$

We now consider what a basis of $\bigwedge^k V$ could possibly be. Since

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots v_j \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k,$$

if an entry is repeated then

$$v_1 \wedge \cdots \wedge v_k = 0.$$

moreover, we can always multiply by ± 1 to reorder the entries in $v_1 \wedge \cdots \wedge v_k$.

Given a basis $\{v_1, \ldots, v_n\}$ of V, we thus might expect that the elements

$$v_{i_1} \wedge \cdots \wedge v_{i_k}$$

where $i_1 < i_2 < \cdots < i_k$ will form a basis of $\bigwedge^k V$. This is precisely what we will show next.

Lemma 4.23. *Let $\{v_1, \ldots, v_n\}$ be a basis of V. The set*

$$\left\{ v_{i_1} \wedge \cdots \wedge v_{i_k} \middle|_{i_1 < i_2 < \cdots < i_k}^{\{i_1, \ldots, i_k\} \subset \underline{n}} \right\}$$

is a basis of $\bigwedge^k V$. In particular, $\dim(\bigwedge^k V) = \binom{n}{k}$

Proof. It is clear that the desired elements span $\bigwedge^k V$, since the image of any of the basis elements of $V^{\otimes k}$ can be written as 0, +1, or −1 times one of these elements.

To see linear independence, suppose $J \subset \underline{n}$ with $J = \{j_1 < \ldots < j_k\}$. Then define a map

$$\phi_J : V^{\otimes k} \longrightarrow \mathbb{R}$$

which sends the tuple

$$(\sum_{i_1=1}^{n} \lambda_1^{i_1} v_{i_1}) \otimes \cdots \otimes (\sum_{i_k=1}^{n} \lambda_k^{i_k} v_{i_k})$$

to

$$\sum_{i_1,\ldots,i_k=1}^{n} (\lambda_1^{i_1} \cdots \lambda_k^{i_k}) \operatorname{sgn}(\sigma_I)\delta_{j_1}^{i_1} \cdots \delta_{j_k}^{i_k}$$

where σ_I denotes the unique permutation which puts the ordered tuple $(i_1, \ldots, i_k)$ into the order induced by that of $\mathbb{N}$. This map is multilinear and alternating, and so defines a unique linear map

$$\overline{\phi}_J : \bigwedge^k V \longrightarrow \mathbb{R}.$$

Now, suppose that

$$u = \sum_{I \subset \underline{n} // |I| = k} \mu^I v_{i_1} \wedge \cdots \wedge v_{i_k} = 0$$

Then for any $J \subset \underline{n}$ with $|J| = k$

$$\overline{\phi}(u) = \mu^J = 0$$

and so all of the coefficients of the sum are zero. Thus, the chosen set is, indeed, linearly independent. $\square$

Exercise 4.24. Show that if $w_1 \wedge \cdots \wedge w_k = 0$, then the set $\{w_1, \ldots, w_k\}$ is linearly dependent.

Proposition 4.25. *There is an isomorphism*

$$\Psi : \bigwedge^k V^* \longrightarrow \operatorname{Alt}^k(V, \mathbb{R})$$

defined by

$$\Psi(f^1 \wedge \cdots \wedge f^k)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) f^1(v_{\sigma(1)}) \cdots f^k(v_{\sigma(k)}).$$

Proof. We leave to the reader the task of checking that the map Ψ is linear and has the correct source and target.

To see that Ψ is injective, choose a basis $\{v_1, \ldots, v_n\}$ of V and write $v_i^\star$ for the elements of the dual basis.

Let

$$u = \sum_I \mu^I v_{i_1}^* \wedge \cdots \wedge v_{i_k}^*$$

so

$$\Psi(u) = \sum_I \mu^I \Psi(v_{i_1}^* \wedge \cdots \wedge v_{i_k}^*)$$

Given $J = \{j_1 < \ldots < j_k\} \subset \underline{n}$, we then note that

$$\Psi(v_{i_1}^* \wedge \cdots \wedge v_{i_k}^*)(v_{j_1}, \ldots, v_{j_k}) = \delta_I^J.$$

Thus, if $\Psi(u) = 0$, the $\mu^I = 0$ for all $I \subset \underline{n}$ with $|I| = k$, and so Ψ is injective. Dimension-counting again shows that Ψ is an isomorphism. $\square$

Definition 4.26. The *exterior algebra* (or *Graßmann algebra*) of an n-dimensional vector space V is

$$\bigwedge V := \bigoplus_{k=0}^{n} \bigwedge^{k} V$$

(note that $\bigwedge^0 V := \mathbb{R}$). The *wedge product* is the unique bilinear map

$$(-) \wedge (-) : \bigwedge V \times \bigwedge V \longrightarrow \bigwedge V$$

which sends $w_1 \wedge \cdots \wedge w_k \in \bigwedge^k V$ and $v_1 \wedge \cdots \wedge v_\ell \in \bigwedge^\ell V$ to

$$w_1 \wedge \cdots \wedge w_k \wedge v_1 \wedge \cdots \wedge v_\ell \in \bigwedge^{k+\ell} V.$$

Exercise 4.27. Verify that the wedge product has the following properties:

1. Associativity: $\omega \wedge (\eta \wedge \nu) = (\omega \wedge \eta) \wedge \nu$
2. Graded-commutativity: for $\omega \in \bigwedge^k V$ and $\eta \in \bigwedge^\ell V$

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

4.3 Forms, orientations, and integration

We now have all the pieces in place to define k-forms and the attendant (oriented) integral.

Definition 4.28. The k^{th} *exterior power of the cotangent bundle* is

$$\bigwedge^{k} T^*M := \left\{ (p, \omega) \middle|_{\omega \bigwedge^k T_p^* M \cong \mathrm{Alt}^k(T_p M, \mathbb{R})}^{p \in M} \right\}.$$

Given a chart (U, x) on M, $\omega \in \bigwedge^k T_p^* M$ can be written with respect to the coordinate basis as

$$\omega = \omega_I^x dx^I$$

where, given $I = \{i_1 < \cdots < i_k\}$, we define

$$dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

We still use the Einstein summation convention, so that, writing out the sum, we have

$$\omega_I^x dx^I = \sum_{i_1, \ldots, i_k = 1}^{\infty} \omega_{i_1, \ldots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Thus, the chart (U, x) allows us to define a chart

$$\begin{aligned} \phi_{(U,x)} : \textstyle\bigwedge^k T^*M|_U &\longrightarrow x(U) \times \mathbb{R}^{\binom{m}{k}} \\ (p, \omega) &\longmapsto (x(p), (\omega_I^x)_{I \subset \underline{m}}) \end{aligned}$$

Exercise 4.29. Note that for $\omega \in T_p^*M$ and a chart (U, x), we have

$$\omega_I^x = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right).$$

Show that for another chart (V, y) around p,

$$\omega_I^y = \sum_{\substack{J \subset \underline{m} \\ |J|=k}} \det\left(\left(\frac{\partial x^{j_a}}{\partial y^{i_b}}\right)_{a,b}\right) \omega_J^x.$$

Conclude that $\bigwedge^k T^*M$ is a smooth manifold and that the projection

$$\begin{aligned} \pi \bigwedge^k T^*M &\longrightarrow M \\ (p, \omega) &\longmapsto p \end{aligned}$$

is smooth.

Definition 4.30. A *smooth k-form on M* is a smooth section of $\bigwedge^k T^*M$, i.e. a smooth map

$$\omega : M \longrightarrow \bigwedge^k T^*M$$

such that $\pi \circ \omega = \mathrm{Id}_M$. We write $\Omega^k(M)$ for the vector space of smooth k-forms on M. Note that the identification $\bigwedge^0 V \cong \mathbb{R}$ identifies 0-forms with smooth functions $M \to \mathbb{R}$.

In coordinates (U, x), we can write

$$\omega(p) = \omega_I(p) dx^I$$

where the ω_I are smooth functions from U to $\mathbb{R}$.

Definition 4.31. Given $f : M \to N$ a smooth map and $\omega \in \Omega^k(N)$, we define the *pullback* k-form $f^*\omega \in \Omega^k(M)$ by

$$(f^*\omega)(v_1, \dots, v_k) = \omega(df(v_1), \dots, df(v_k))$$

for vectors $v_1, \dots, v_k \in T_pM$. For a 0-form $f : M \to \mathbb{R}$, $f^*h = h \circ f$.

We state without proof the following computation rules for the pullback.

Proposition 4.32. *For $f : M \to N$ a smooth map, $\omega, \eta \in \Omega^k(N)$, and $a, b \in \mathbb{R}$*

1. $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$
2. $f^*(a\omega + b\eta) = af^*(\omega) + bf^*(\eta)$
3. *For (U, x) and (V, y) charts on M and N respectively,*

$$f^*(dy^i) = \frac{\partial f^i}{\partial x^j} dx^j$$

on $f(U) \cap V$.

4. For a smooth map $g : N \to L$,

$$(g \circ f)^* = f^* \circ g^*.$$

Together, these rules determine f^* uniquely, and moreover allow us to compute f^* quite efficiently in coordinates. For instance, in the special case of a diffeomorphism $f : M \to N$ between m-manifolds and a top-dimensional form

$$\omega = g dy^1 \wedge \cdots \wedge dy^k$$

we can compute

$$\begin{aligned} f^*\omega &= (g \circ f) \left(\frac{\partial f^1}{\partial x^{i_1}} dx^{i_1} \right) \wedge \cdots \wedge \left(\frac{\partial f^m}{\partial x^{i_m}} dx^{i_m} \right) \\ &= (g \circ f) \sum_{\sigma \in S_m} \frac{\partial f^1}{\partial x^{\sigma(1)}} \cdots \frac{\partial f^m}{\partial x^{\sigma(m)}} dx^{\sigma(1)} \wedge \cdots dx^{\sigma(m)} \\ &= (g \circ f) \det(J(y \circ f \circ x^{-1})) dx^1 \wedge \cdots dx^m. \end{aligned}$$

It is this relation to the Jacobian determinant which will become key as we define integration.

Definition 4.33. Let M be a smooth m-manifold, (U, x) a coordinate chart, and $A \subset U$ such that $x(A)$ is Lebesgue measurable. Let ω be a smooth m-form on M such that on U

$$\omega = f dx^1 \wedge \cdots \wedge dx^m.$$

The *integral* of ω over A is defined to be

$$\int_A \omega = \int_{x(A)} (f \circ x^{-1}) d\mu$$

where μ denotes the Lebesgue measure on $\mathbb{R}^m$.

We would like this definition to not depend on the choice of chart. However, if (U, y) is another chart with the same domain, we can use the change of coordinates for Lebesgue integration to find

$$\begin{aligned} \int_A \omega &= \int_{x(A)} (f \circ x^{-1}) d\mu \\ &= \int_{x(y^{-1}(y(A)))} (f \circ x^{-1}) d\mu \\ &= \int_{y(A)} (f \circ y^{-1}) \left| \det(J(x \circ y^{-1})) \right| d\mu \end{aligned}$$

This latter is plus or minus the value of the integral calculated using the chart y, depending on whether the determinant of $J(x \circ y^{-1})$ is positive or negative. To resolve this problem, we restrict to manifolds with some additional structure.

Definition 4.34. Two charts (U, x) and (V, y) on a manifold with $U \cap V \neq 0$ are said to be *compatibly oriented* if

$$\det(J(x \circ y^{-1})) > 0$$

on $y(U \cap V)$. An *oriented atlas* on M is an atlas consisting of charts which are pairwise compatibly oriented. An *orientation* on a smooth manifold M is a choice of a maximal oriented sub-atlas of the atlas of M. We call a chart the orientation of M an *oriented chart*

We say M is *orientable* if it admits an orientation.

Exercise 4.35. If M is orientable, there are precisely two orientations on M.

Exercise 4.36. Show that S^2 is orientable and that $\mathbb{R}P^2$ is not.

Our computation above then shows the following.

Lemma 4.37. *Let M be an oriented m-manifold, (U, x) and (V, y) charts on M, $A \subset U \cap V$, and $\omega \in \Omega^m(M)$. Then the value of*

$$\int_A \omega$$

does not depend on whether it is computed using (U, x) or (V, y).

With this independence result, we are able to define integration more generally.

Definition 4.38. Let M be an m-manifold, $\omega \in \Omega^m(M)$, and $A \subset M$. Let $\{(U_\alpha, x_\alpha)\}_{\alpha=1}^\infty$ be a locally finite cover of M by oriented coordinate balls such that $x_\alpha(A \cap U_\alpha)$ is Lebesgue measureable for any α. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. The *integral* of ω over A is

$$\int_A \omega = \sum_{\alpha=1}^{\infty} \int_{U_\alpha \cap A} \psi_\alpha \omega.$$

Proposition 4.39. *The value of*

$$\int_A \omega$$

is independent of the cover and partition of unity chosen.

Proof. Let $\{(V_\beta, y_\beta)\}$ be another such cover, and $\{\phi_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Then

$$\begin{aligned}
\int_A \omega &= \sum_\alpha \int_{U_\alpha \cap A} \psi_\alpha \\
&= \sum_\alpha \int_{U_\alpha \cap A} (\sum_\beta \phi_\beta) \psi_\alpha \omega \\
&= \sum_{\alpha,\beta} \int_{A \cap U_\alpha \cap V_\beta} \phi_\beta \psi_\alpha \omega \\
&= \sum_{\alpha,\beta} \int_{A \cap V_\beta} \phi_\beta \psi_\alpha \omega \\
&= \sum_\beta \int_{A \cap V_\beta} \sum_\alpha \phi_\beta \psi_\alpha \omega \\
&= \sum_\beta \int_{A \cap V_\beta} \phi_\beta \left(\sum_\alpha \psi_\alpha \right) \omega \\
&= \sum_\beta \int_{A \cap V_\beta} \phi_\beta \omega
\end{aligned}$$

so that the values agree. □

We can then bootstrap this to a more general case: integration of a k-form over a k-dimensional submanifold.

Definition 4.40. A k-dimensional *submanifold* of an m-manifold M is a subspace $N \subset M$ and a smooth atlas on N such that the inclusion

$$\iota : N \longrightarrow M$$

is smooth and $d\iota_p : T_pN \to T_pM$ is injective for all $p \in N$.

Remark 4.41. Though we will not prove this in this class, being a submanifold is a *property* of $N \subset M$, not an additional structure. The smooth structure is induced by the smooth structure on M.

Definition 4.42. Let M be a smooth m-manifold, $\omega \in \Omega^k(M)$, and $\iota : N \to M$ an oriented k-dimensional submanifold. The *integral of* ω *over* N is

$$\int_N \omega := \int_N \iota^*\omega.$$

One final sanity check awaits this definition:

Exercise 4.43. Let $\phi : M \to N$ be an orientation-preserving diffeomorphism. Prove

$$\int_N \omega = \int_M \phi^*\omega.$$

4.3.1 Understanding forms on $\mathbb{R}^3$

To put all of this abstraction into context, let's return to the world of multivariable calculus, and try to understand what the integrals of k-forms on $\mathbb{R}^3$ are.

EXAMPLE 0: A 0-form on $\mathbb{R}^3$ is a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, and an oriented 0-submanifold is a discrete set of points in $\mathbb{R}^3$ with $+$ or $-$ signs assigned to them. The integral of f over $A \subset \mathbb{R}^3$ is thus

$$\int_A f = \sum_{a \in A} \operatorname{sign}(a) f(a).$$

The usual notion of a discrete integral.

EXAMPLE 1: A 1-form on $\mathbb{R}^3$ can be written as

$$\omega = \omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3$$

for smooth functions ω_i on $\mathbb{R}^3$. A 1-dimensional submanifold is a curve $\gamma : [a, b] \to$

$\mathbb{R}^3$. Thus the integral is

$$\begin{aligned}\int_\gamma \omega &= \int_{[a,b]} \gamma^*\omega \\ &= \int_a^b \left(\omega_1(\gamma(t))\frac{d\gamma^1}{dt}dt + \omega_2(\gamma(t))\frac{d\gamma^2}{dt}dt + \omega_3(\gamma(t))\frac{d\gamma^3}{dt}dt\right) \\ &= \int_a^b \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \cdot \gamma'(t)dt\end{aligned}$$

which is precisely the integral of the vector field $(\omega_1, \omega_2, \omega_3)$ along the curve γ. In Calculus III notation:

$$\int_\gamma (\omega_1, \omega_2, \omega_3) \cdot d\vec{r}.$$

Example 2: A 2-form on $\mathbb{R}^3$ can be written as

$$\omega = \omega_z dx^1 \wedge dx^2 + \omega_y dx^3 \wedge dx^1 + \omega_x dx^2 \wedge dx^3$$

and a 2-submanifold is a surface

$$f : S \longrightarrow \mathbb{R}^3$$

If we consider the case where the surface is covered by a single patch: $S \subset \mathbb{R}^2$ with coordinates (u^1, u^2), then we can compute

$$\begin{aligned}f^*\omega = \omega_z &\left(\frac{\partial f^1}{\partial u^1}\frac{\partial f^2}{\partial u^2} - \frac{\partial f^1}{\partial u^2}\frac{\partial f^2}{\partial u^1}\right) du^1 \wedge du^2 \\ &+ \omega_y \left(\frac{\partial f^1}{\partial u^2}\frac{\partial f^3}{\partial u^1} - \frac{\partial f^1}{\partial u^1}\frac{\partial f^3}{\partial u^3}\right) du^1 \wedge du^2 \\ &+ \omega_x \left(\frac{\partial f^2}{\partial u^1}\frac{\partial f^3}{\partial u^2} - \frac{\partial f^2}{\partial u^2}\frac{\partial f^3}{\partial u^1}\right) du^1 \wedge du^2 \\ &= \begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} \cdot \left[\frac{\partial f}{\partial u^1} \times \frac{\partial f}{\partial u^2}\right] du^1 \wedge du^2\end{aligned}$$

so

$$\int_{f(S)} \omega = \int_S f^*\omega = \iint_S \begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} \cdot d\vec{S}$$

where the latter is the usual surface integral of a vector field from Calculus III.

Example 3: A 3-form on $\mathbb{R}^3$ can be written

$$\omega = f dx^1 \wedge dx^2 \wedge dx^3$$

and a 3-submanifold is a volume $B \subset \mathbb{R}^3$. Then

$$\int_B \omega = \iiint_B f dV$$

is the usual volume integral of f over B.

4.4 *The exterior derivative*

Sticking for a moment with our Calculus III theme, let's recall some key theorems about the integrals of forms we just explored.

Theorem 4.44 (Fundamental Theorem of Line integrals)**.** *For a curve $\gamma : [a, b] \to \mathbb{R}^3$ and a function $f : \mathbb{R}^3 \to \mathbb{R}$,*

$$\int_\gamma \underbrace{\mathrm{grad}(f) \cdot d\vec{s}}_{1\text{-}form} = \int_{\{\gamma(a)^-, \gamma(b)^+\}} \underbrace{f}_{0\text{-}form} = f(\gamma(b)) - f(\gamma(a))$$

Theorem 4.45 (Stokes' Theorem)**.** *For X a vector field on $\mathbb{R}^3$, $S \subset \mathbb{R}^3$ an oriented surface with oriented boundary ∂S*

$$\int_{\partial S} \underbrace{X \cdot d\vec{s}}_{1\text{-}form} = \int_S \underbrace{\mathrm{curl}(X) \cdot d\vec{S}}_{2\text{-}form}.$$

Theorem 4.46 (Divergence Theorem)**.** *For $M \subset \mathbb{R}^3$ a volume with oriented boundary ∂M and X a vector field*

$$\int_{\partial M} \underbrace{X \cdot d\vec{S}}_{2\text{-}form} = \int_M \underbrace{\mathrm{div}(X) dV}_{3\text{-}form}.$$

This means that we can reinterpret the gradient, curl, and divergence as operators which move us from k-forms to $k + 1$-forms.

$$\Omega^0(\mathbb{R}^3) \xrightarrow{\text{grad}} \Omega^1(\mathbb{R}^3) \xrightarrow{\text{curl}} \Omega^2(\mathbb{R}^3) \xrightarrow{\text{div}} \Omega^3(\mathbb{R}^3)$$

The aim of this section is to define a generalization of these maps which subsumes the gradient, divergence, and curl.

Definition 4.47. Let M be a smooth manifold, and let (U, x) be a chart on M. Given $\omega \in \Omega^k(U)$ with coordinate expression

$$\omega = \omega_I dx^I$$

we define the *exterior derivative of ω on U* to be the $k + 1$-form

$$d\omega := \frac{\partial \omega_I}{\partial x^j} dx^j \wedge dx^I \in \Omega^{k+1}(U).$$

Exercise 4.48. Show that the curl, divergence and gradient on $\mathbb{R}^3$ are all special cases of the exterior derivative.

We must now show that the exterior derivative does not depend on the choice of chart.

Lemma 4.49. *Let (U, x) be a coordinate chart on M. There is a unique collection of linear maps*

$$\{\ d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U)\ \}_{k \geq 0}$$

which satisfy the following conditions:

1. *If $f \in \Omega^0(U)$ and V is a tangent vector field on U, then*

$$df(V) = V(f).$$

2. *If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^\ell(U)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

3. *For any $k \geq 0$, the composite of*

$$\Omega^k(U) \xrightarrow{d} \Omega^{k+1}(U) \xrightarrow{d} \Omega^{k+2}(U)$$

is the zero homomorphism.

Proof. We first show that the exterior derivative d has these properties. To show (1), we note that, for $V = V^i \frac{\partial}{\partial x^i}$, we have

$$\begin{aligned} df(V) &= \frac{\partial f}{\partial x^j} dx^j \left(V^i \frac{\partial}{\partial x^i} \right) \\ &= V^i \frac{\partial f}{\partial x^j} \delta^i_j \\ &= V^i \frac{\partial f}{\partial x^i} \\ &= V(f). \end{aligned}$$

To show (2) we check on basis elements. Let $\omega = f dx^I$ and $\eta = g dx^J$. Then

$$\begin{aligned} d(\omega \wedge \eta) &= d(fg dx^I \wedge dx^J) \\ &= \frac{\partial (fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= g \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \wedge dx^J + f \frac{\partial g}{dx^i} dx^i \wedge dx^I \wedge dx^J \end{aligned}$$

The first of these terms is $d(\omega) \wedge \eta$. To turn the second in $\omega \wedge d\eta$, we must commute the dx^i past dx^I, adding a factor of $(-1)^k$.

Finally to show (3), we simply compute on basis elements. Set $\omega = f dx^I$. Then

$$\begin{aligned} d(d\omega) &= d\left(\frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i \wedge dx^I \end{aligned}$$

The terms where $i = j$ are zero, since $dx^i \wedge dx^i = 0$. By the equality of mixed partials for smooth functions, the terms with $i < j$ and $j < i$ cancel one another out. Thus (3) holds.

To see uniqueness, suppose that δ is another such function. First note that, on functions $f \in \Omega^0(U)$, the condition

$$\delta f(V) = V(f)$$

uniquely determines δ, so δ and d agree on 0-forms. We then note that, since $\delta \circ \delta = 0$, we must have
$$0 = \delta(\delta f) = \delta(\frac{\partial f}{\partial x^i} dx^i)$$
and since δ must respect the wedge product, this is
$$0 = \delta(\delta f) = \delta(\frac{\partial f}{\partial x^i}) \wedge dx^i + \frac{\partial f}{\partial x^i}\delta(dx^i)$$
Taking the specific smooth function $f(x) = x^j$, all double partial derivatives vanish, and so we obtain
$$\delta(dx^j) = 0.$$
Inductively, we find that
$$\delta(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \delta(dx^{i_1}) \wedge (dx^{i_2} \wedge \cdots \wedge dx^{i_k}) - dx^{i_1} \wedge d(dx^{i_2} \wedge \cdots \wedge dx^{i_k}) = 0$$
Thus, on a basis element $f dx^I$ we see that
$$\begin{aligned}
\delta(f dx^I) &= \delta(f) \wedge dx^I + (-1)^k f \delta(dx^I) \\
&= \delta(f) \wedge dx^I \\
&= d(f) \wedge dx^I \\
&= \frac{\partial f}{\partial x^j} dx^j \wedge dx^I
\end{aligned}$$
so that $\delta = d$ as desired. □

Corollary 4.50. *The exterior derivative is independent of the choice of coordinates, and thus for any manifold M defines linear maps*
$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$
for $k \geq 0$, which satisfy (1)-(3) of the previous proposition.

Proof. Given (U, x) and (V, y) coordinate charts on M, write d^x and d^y for the exterior derivatives associated to the two charts. On $U \cap V$, both d^x and d^y satisfy conditions (1)-(3), and so must agree. Thus $d^x|_{U\cap V} = d^y|_{U\cap V}$, as desired. □

There are many uses one can make of the exterior derivative. The first we will explore is defining a powerful invariant of smooth manifolds, called the *de Rham cohomology*.

Definition 4.51. We call $\omega \in \Omega^k(M)$ *closed* if $d\omega = 0$. We call $\omega \in \Omega^k(M)$ *exact* if there is an $\eta \in \Omega^{k-1}(M)$ with $d\eta = \omega$. Equivalently, we define the space of *closed k-forms* on M to be
$$Z^k(M) := \ker\left(\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \right)$$
and the space of *exact k-forms* on M to be
$$B^k(M) := \operatorname{Im}\left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \right).$$
The k^{th} de Rham cohomology of M is quotient
$$H^k_{dR}(M) := Z^k(M)/B^k(M).$$

The basic idea of the de Rham cohomology is that it should measure "how different M is from $\mathbb{R}^n$". This makes sense owing to the Poincaré lemma which we will prove later. This lemma says that, on $\mathbb{R}^n$, every closed k-form with $k \geq 1$ is exact, and thus, the de Rham cohomology spaces are trivial for $k \geq 1$.

On the other hand, we can identify a case in which the de Rham cohomology is not trivial:

Example 4.52. Consider the manifold $M = \mathbb{R}^2 \setminus \{0\}$. Define

$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx.$$

Then a quick computation shows $d\omega = 0$, so $\omega \in Z^1(M)$.

On the other hand, let

$$\begin{aligned} \gamma : [0, 2\pi] &\longrightarrow M \\ t &\longmapsto (\cos(t), \sin(t)) \end{aligned}$$

Then we can compute

$$\int_\gamma \omega = 2\pi.$$

But the fundamental theorem of line integrals tells us that if ω were exact, then

$$\int_\gamma \omega = 0.$$

Thus, $H^1_{dR}(M) \ncong 0$.

Example 4.53. We can compute the zeroth de Rham cohomology fairly easily. Let M be a connected manifold. Since there are no exact 0-forms, it will suffice to compute $Z^0(M)$. If $f \in Z^0(M)$, then for any coordinate chart (U, x),

$$df = \frac{\partial f}{\partial x^i} dx^i = 0.$$

Since the dx^i are linearly independent, this implies that

$$\frac{\partial f}{\partial x^i} \equiv 0$$

for all $1 \leq i \leq m$. Thus, f is constant on U. Since M is connected, this implies that f is constant on M, and thus $H^0_{dR}(M) \cong \mathbb{R}$.

Exercise 4.54. Show that the map

$$\begin{aligned} \phi\, \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto (\cos(t), \sin(t)) \end{aligned}$$

identifies 1-forms on S^1 with 1-forms $f(t)dt$ on $\mathbb{R}$ such that f is 2π-periodic.

Example 4.55. We can also compute $H^*_{dR}(S^1)$ directly. The fundamental theorem of line integrals implies that, if $\omega = f(t)dt \in \Omega^1(S^1)$ is exact, then

$$\int_0^{2\pi} f(t)dt = 0.$$

Thus, the map

$$\begin{aligned} \Omega^1(S^1) &\longrightarrow \mathbb{R} \\ f(t)dt &\longmapsto \textstyle\int_0^{2\pi} f(t)dt \end{aligned}$$

descends to a map

$$H^1_{dR}(S^1) \longrightarrow \mathbb{R}.$$

If $\omega = f(t)dt$ is in the kernel of this map, then

$$\int_0^2 \pi f(t)dt = 0.$$

Thus, we can define a smooth 2π-periodic function

$$g(s) = \int_0^s f(t)dt.$$

We compute

$$d(g) = f(t)dt$$

and so ω is exact. Thus $H^1_{dR}(S^1) \cong \mathbb{R}$.

To better understand the de Rham Cohomology, we will put it into a general algebraic framework.

Definition 4.56. A *cochain complex* $V_\bullet = (\{V_i\}, \delta_V)$ is a sequence $\{V_i\}_{i \geq 0}$ of vector spaces with *differentials*

$$\delta_V : V_k \longrightarrow V_{k+1}$$

for $k \geq 0$ such that $\delta_V \circ \delta_V = 0$. A *k-cocycle* in $V_\bullet$ is an element of

$$Z^k(V_\bullet) := \ker(\delta_V : V_k \to V_{k+1})$$

and a *k-coboundary* is an element of

$$B^k(V_\bullet) := \mathrm{Im}(\delta_V : V_{k-1} \to V_k).$$

Since $\delta_V \circ \delta_V =$, we have $B^k(V_\bullet) \subset Z^k(V_\bullet)$. Thus we can define the *cohomology* of $V_\bullet$ to be

$$H^k(V_\bullet) := \frac{Z^k(V_\bullet)}{B^k(V_\bullet)}$$

Example 4.57. The reason we are interested in cochain complexes is that, as we have already seen,

$$\Omega^\bullet(M) := \big(\{\Omega^k(M)\}, d\big)$$

is a cochain complex, and the de Rham cohomology is the cohomology of this complex:

$$H^k_{dR}(M) = H^k(\Omega^\bullet(M)).$$

As with any mathematical structure, the key to understanding cochain complexes is understanding how they relate to each other.

Definition 4.58. A *morphism of cochain complexes* (*cochain map* for short) from $V_\bullet$ to $W_\bullet$ is a collection

$$f = \{f_i : V_i \to W_i\}_{i\geq 0}$$

of linear maps such that, for every $i \geq 0$, the diagram

$$\begin{tikzcd} V_i \arrow[r, "f_i"] \arrow[d, "\delta_V"'] & W_i \arrow[d, "\delta_V"] \\ V_{i+1} \arrow[r, "f_{i+1}"'] & W_{i+1} \end{tikzcd}$$

commutes.

The reason we take this as our definition of cochain maps is the following lemma.

Lemma 4.59. *If $f : V_\bullet \to W_\bullet$ is a cochain map, then the map*

$$\begin{aligned} H^k(f) : H^k(V_\bullet) &\longrightarrow H^k(W_\bullet) \\ [v] &\longrightarrow [f(v)] \end{aligned}$$

is well-defined.

Proof. Suppose that $[v] = [u]$ in $H^k(V_\bullet)$. Then there is an $x \in V_{k-1}$ such that $v - u = \delta_V(x)$. Then

$$f_k(v) - f_k(u) = f_k(\delta_V(x)) = \delta_W(f_{k-1}(x)).$$

Since $\delta_W(f_{k-1}(x)) \in B^k(W_\bullet)$, this means that $[f_k(v)] = [f_k(u)]$. $\square$

Lemma 4.60. *For $f : M \to N$ a smooth map of smooth manifolds, the pullback of forms along f, $f^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ is a cochain map.*

Proof. It suffices to check in coordinates that $d(f^*\omega) = f^*(d\omega)$. Let (V, y) be coordinates on N and (U, x) coordinates on M.

Firstly, for $h \in \Omega^0(N)$, we have

$$\begin{aligned} d(h \circ f) &= \frac{\partial h}{\partial y^i}\frac{\partial f^i}{\partial x^j}dx^j \\ &= f^*\left(\frac{\partial h}{\partial y^i}dy^i\right) \\ &= f^*(d(h)). \end{aligned}$$

Second, we can compute

$$f^*(dy^j) = \frac{\partial f^j}{\partial x^i}dx^i = d(f^i).$$

Finally, for an arbitrary form $\omega_I dy^I \in \Omega^k(N)$, we have

$$\begin{aligned} d(f^*\omega) &= d(\omega_I \circ f) \wedge df^{i_1} \wedge \cdots \wedge df^{i_k} \\ &\quad + \underbrace{(-1)(\omega_i \circ f) d(df^{i_1} \wedge \cdots \wedge df^{i_k})}_{=0} \\ &= f^*(d\omega_I) \wedge f^*(dx^{i_1}) \wedge \cdots \wedge f^*(dx^{i_k}) \\ &= f^*(d\omega) \end{aligned}$$

as desired. □

Corollary 4.61. *For $f : M \to N$ smooth, we obtain linear maps*

$$\begin{aligned} f^* : H^k_{dR}(N) &\longrightarrow H^k_{dR}(M) \\ [\omega] &\longmapsto [f^*\omega]. \end{aligned}$$

Given composable smooth maps f and g, then $(f \circ g)^* = g^* \circ f^*$, and thus if f is a diffeomorphism, then f^* is an isomorphism on cohomology.

We now come to the question: When do two cochain maps induce the same morphism on cohomology? The answer we will provide is familiar from our contemplation of the fundamental group, but requires one more technical preliminary.

Definition 4.62. If $f, g : V_\bullet \to W_\bullet$ are cochain maps, a *cochain homotopy* from f to g is a collection of maps

$$\{h : V_k \to W_{k-1}\}_{k \geq 0}$$

such that

$$\delta_W \circ h = h \circ \delta_V = f - g.$$

Lemma 4.63. *If h is a cochain homotopy between $f, g : V_\bullet \to W_\bullet$, then f and g induce the same map of cohomology, i.e., $H^k(f) = H^k(g)$.*

Proof. It suffices to check on k-cycles. If $v \in Z^k(V)$, then

$$f(v) - g(v) = \delta_W(h(v)) + h(\delta_V(v))$$

but since $v \in Z^k(V)$, $\delta_V(v) = 0$, and thus,

$$f(v) - g(v) = \delta_W(h(v)) \in B^k(W).$$

So we see that $[f(v)] = [g(v)]$, and f and g induce the same map. □

The situation in which smooth maps will induce chain-homotopic pullbacks is a familiar one: when they are homotopic. The one fly in the ointment is that we now need a *smooth* homotopy.

Definition 4.64. A *smooth homotopy* between smooth maps $f, g : M \to N$ is a smooth map[4]

$$H : M \times [0,1] \longrightarrow N$$

such that $H|_{M\times\{0\}} = f$ and $H|_{M\times\{1\}} = g$.

[4] I'm glossing over some complexities here, since we have not defined what a smooth map on $M \times [0,1]$ is. However, $M \times \mathbb{R}$ is a smooth manifold as we have defined them, and without losing much, one can think of a smooth map on $M \times [0,1]$ as a map defined on $M \times [0,1]$ which can be extended to a smooth map on an open neigborhood of $M \times [0,1]$ in $M \times \mathbb{R}$.

We write the points of $M \times \mathbb{R}$ as pairs (p,t), and given a chart (U,x) on M, we will make use of the corresponding chart[5]

$$\begin{aligned} (U \times \mathbb{R}) &\longrightarrow x(U) \times \mathbb{R} \subset \mathbb{R}^{k+1} \\ (p,t) &\longmapsto (x(p),t). \end{aligned}$$

[5] Note, the following proof of homotopy invariance follows the proof in Werner Ballmann's "Introduction to Geometry and Topology", and the interested reader may wish to read further there.

The basis vector field $\frac{\partial}{\partial t}$ corresponding to the coordinate t is the same, regardless of what other coordinates we choose, and so we obtain a global tangent vector field $\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$. The corresponding coordinate 1-form is dt.

Lemma 4.65. *Let $k \geq 1$, and let $\omega \in \Omega^k(M \times [0,1])$. Then there are unique $\eta \in \Omega^{k-1}(M \times [0,1])$ and $\xi \in \Omega^k(M \times [0,1])$ such that (1)*

$$\omega = dt \wedge \eta + \xi$$

(2) $\eta(v_1,\dots,v_{k-1}) = 0$ whenever one of the v_i is a scalar multiple of $\frac{\partial}{\partial t}$, and (3) $\xi(v_1,\dots,v_k) = 0$ whenever one of the v_i is a scalar multiple of $\frac{\partial}{\partial t}$.

Proof. Given ω, we can define

$$\eta\left(v_1,\dots,v_{k-1}\right) := \omega\left(\frac{\partial}{\partial t}, v_1,\dots,v_{k-1}\right)$$

and

$$\xi = \omega - dt \wedge \eta.$$

Which yields the desired pair of forms.

On the other hand, if $\overline{\eta}$ and $\overline{\xi}$ are another two such forms, we have

$$\begin{aligned} \omega\left(\frac{\partial}{\partial t}, v_1,\dots,v_{k-1}\right) &= (dt \wedge \overline{\eta})\left(\frac{\partial}{\partial t}, v_1,\dots,v_{k-1}\right) \\ &\quad + \overline{\xi}\left(\frac{\partial}{\partial t}, v_1,\dots,v_{k-1}\right) \\ &= \overline{\eta}(v_1,\dots,v_{k-1}). \end{aligned}$$

Thus $\overline{\eta} = \eta$, proving uniqueness. □

In coordinates (U,x) we can break up ω into terms containing dt and terms not containing dt, so that

$$\omega = \eta_I dt \wedge dx^I + \xi_J dx^J$$

where η_I and ξ_J are the coefficients of η and ξ, respectively.

Construction 4.66. For $t \in [0,1]$, write $i_t : M \to M \times [0,1]$ for the corresponding inclusion.

Given $k \geq 1$ and $\omega \in \Omega^k(M \times [0,1])$, write

$$\omega = dt \wedge \eta + \xi$$

as in the lemma. Then define $I\omega \in \Omega^{k-1}(M)$ by

$$(I\omega)_p(v_1,\dots,v_k) := \int_0^1 \eta_{(p,t)}((i_t)_{*,p}v_1,\dots,(i_t)_{*,p}v_k)dt.$$

This defines an $\mathbb{R}$-linear map

$$I : \Omega^k(M \times [0,1]) \longrightarrow \Omega^{k-1}(M).$$

The coefficients $\overline{\omega}_J$ of $I\omega$ are then given by

$$\overline{\omega}_J(p) = \int_0^1 \eta_J(p,t)dt.$$

Proposition 4.67. *For $\omega \in \Omega^k(M \times [0,1])$, then*

$$d(I\omega) + I(d\omega) = i_1^*\omega - i_0^*\omega.$$

That is, I is a cochain homotopy from i_1^* to i_0^*.

Proof. Since f is linear, it suffices to consider basis elements locally in coordinates (U,x). We can consider two cases.

If the basis element does not have a factor of dt, then

$$\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Then $I\omega = 0$ (since $\eta = 0$), and

$$d\omega = \frac{\partial f}{\partial t} dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Thus

$$\begin{aligned} (Id\omega)_p &= \left(\int_0^1 \frac{\partial f}{\partial t}(p,t)dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \left(f(p,1) - f(p,0)\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ (i_1^*\omega)_p - (i_0^*\omega)_p \end{aligned}$$

On the other hand, if

$$\omega = f dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}.$$

We see that

$$I\omega = \left(\int_0^1 f(p,t)dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$$

and so

$$\begin{aligned} d(I\omega) &= \left[\frac{\partial}{\partial x^j}\left(\int_0^1 f(p,t)dt\right)\right] dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \\ &= \left(\int_0^1 \frac{\partial f}{\partial x^j}(p,t)dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \end{aligned}$$

On the other hand

$$d\omega = \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$$

so

$$I(d\omega) = \left(\int_0^1 \frac{\partial f}{\partial x^j}(p,t)dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$$

and the proof is complete. □

Corollary 4.68. *If $H : M \times [0,1] \to N$ is a smooth homotopy from f to g, then the induced maps*

$$f^*, g^* : H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$$

are equal.

Proof. Since i_0^* and i_1^* are cochain homotopic they induce the same map on homology. Since $f = H \circ i_0$ and $g = H \circ i_1$, we have equalities of maps on homology.

$$f^* = i_0^* \circ H^* = i_1^* \circ H^* = g^*$$

as desired. □

Definition 4.69. We say that smooth manifold M and N are *smoothly homotopy equivalent* if there are smooth maps $f : M \to N$ and $g : N \to M$ and smooth homotopies from $f \circ g$ to id_N and from $g \circ f$ to id_M.

Corollary 4.70. *If M and N are smoothly homotopy equivalent, then $H^k_{dR}(N) \cong H^k_{dR}(M)$ for all $k \geq 0$.*

Exercise 4.71. Show that $\mathbb{R}^n$ is smoothly homotopy equivalent to $\mathbb{R}^0 \cong *$.

From the exercise, we then can obtain a classical result

Lemma 4.72 (Poincaré's Lemma)**.** *For every $k \geq 1$ and $n \geq 0$, every closed k-form on $\mathbb{R}^n$ is exact, i.e.*

$$H^k_{dR}(\mathbb{R}^n) \cong 0.$$

4.5 The Mayer-Vietoris sequence

Long exact sequences are one of the key tools used in algebraic and differential topology to compute cohomology groups. In general, long exact sequences arise from the same algebraic set-up, so we will begin by exploring this set-up.

Definition 4.73. A sequence

$$\cdots \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} \cdots$$

of vector spaces and linear maps is said to be *exact* if

$$\mathrm{Im}(f_k) = \ker(f_{k+1})$$

for all k. An exact sequence

$$0 \longrightarrow V \longrightarrow W \longrightarrow U \longrightarrow 0$$

is called a *short exact sequence.*

Definition 4.74. Let $V_\bullet, W_\bullet$, and $U_\bullet$ be cochain complexes. A *short exact sequence of chain complexes* is a sequence

$$0 \longrightarrow V_\bullet \xrightarrow{g} W_\bullet \xrightarrow{f} U_\bullet \longrightarrow 0$$

of chain complexes such that for each $k \geq 0$

$$0 \longrightarrow V_k \xrightarrow{g} W_k \xrightarrow{f} U_k \longrightarrow 0$$

is a short exact sequence of vector spaces.

Construction 4.75. Given a short exact sequence

$$0 \longrightarrow V_\bullet \xrightarrow{g} W_\bullet \xrightarrow{f} U_\bullet \longrightarrow 0$$

of cochain complexes, we construct *connecting homomorphisms*

$$\kappa_k : H^k(U_\bullet) \longrightarrow H^k(V_\bullet)$$

as follows. Consider the following segment of the short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & V_k \arrow[r, "g"] \arrow[d, "\delta_V"] & W_k \arrow[r, "f"] \arrow[d, "\delta_W"] & U_k \arrow[r] \arrow[d, "\delta_U"] & 0 \\
0 \arrow[r] & V_{k+1} \arrow[r, "g"] \arrow[d, "\delta_V"] & W_{k+1} \arrow[r, "f"] \arrow[d, "\delta_W"] & U_{k+1} \arrow[r] \arrow[d, "\delta_U"] & 0 \\
0 \arrow[r] & V_{k+2} \arrow[r, "g"] & W_{k+2} \arrow[r, "f"] & U_{k+2} \arrow[r] & 0
\end{tikzcd}$$

Given $x \in Z^k(U_\bullet)$, we can choose $\overline{x} \in W_k$ such that $f_k(\overline{x}) = x$ by exactness. Since the top-right square commutes, we see that

$$f_{k+1}(\delta_W(\overline{x})) = \delta_U(f_k(\overline{x})) = \delta_U(x) = 0$$

where the last step follows because x is a cocycle. Then, by exactness, there is a unique element we will call $\kappa(x)$ in V_{k+1} such that $g_{k+1}(\kappa(x)) = \delta_W(\overline{x})$.

The commutativity of the bottom left square shows that

$$g_{k+2}(\delta_V(\kappa(x))) = \delta_W(g_{k+1}(\kappa(x))) = \delta_W(\delta_W(\overline{x})) = 0$$

and so, since the bottom row is short exact, $\delta_V(\kappa(x)) = 0$, i.e., $\kappa(x) \in Z^{k+1}(V_\bullet)$.

We define our putative map by

$$\begin{aligned} \kappa : H^k(U_\bullet) &\longrightarrow H^{k+1}(V_\bullet) \\ [x] &\longmapsto [\kappa(x)] \end{aligned}$$

We now must show that this is well-defined. Suppose that $x, y \in Z^k(U_\bullet)$, and $a \in U_{k-1}$ such that $\delta_U(a) = x - y$.

By similar reasoning to before, we can find $\overline{a} \in W_{k-1}$ such that $f_k(\delta_W(\overline{a})) = x - y$. Given choices of $\overline{x}$ and $\overline{y}$, we see that

$$f_k(\overline{x} - \overline{y} - \delta_W(\overline{a}) = x - y - (x - y) = 0.$$

Thus, by exactness, there is a $b \in V_k$ such that $g_k(b) = \overline{x} - \overline{y} - \delta_W(\overline{a})$. By the commutativity of the top left square above,

$$g_{k+1}(\delta_V(b)) = \delta_W(x - y - \delta_W(\overline{a})) = \delta_W(\overline{x}) - \delta_W(\overline{y}).$$

Since g_{k+1} is injective, this means that

$$\delta_V(b) = \kappa(x) - \kappa(y)$$

and so $[\kappa(x)] = [\kappa(y)]$.

Theorem 4.76. *Let*

$$0 \longrightarrow V_\bullet \xrightarrow{g} W_\bullet \xrightarrow{f} U_\bullet \longrightarrow 0$$

be a short exact sequence of cochain complexes. Then the sequence

$$\begin{tikzcd}
H^0(V_\bullet) \arrow[r, "g"] & H^0(W_\bullet) \arrow[r, "f"] & H^0(U_\bullet) \arrow[dll, rounded corners, "\kappa"] \\
H^1(V_\bullet) \arrow[r, "g"] & H^1(W_\bullet) \arrow[r, "f"] & H^1(U_\bullet) \arrow[dll, rounded corners, "\kappa"] \\
\cdots & & \cdots \arrow[dll, rounded corners, "\kappa"] \\
H^k(V_\bullet) \arrow[r, "g"] & H^k(W_\bullet) \arrow[r, "f"] & H^k(U_\bullet) \arrow[dll, rounded corners, "\kappa"] \\
H^{k+1}(V_\bullet) \arrow[r, "g"] & H^{k+1}(W_\bullet) \arrow[r, "f"] & \cdots
\end{tikzcd}$$

is exact.

Proof. First, we show exactness at $H^k(W_\bullet)$. Let $v \in Z^k(V_\bullet)$. Then by exactness $f_k(g_k(v)) = 0$, and so we have that $\mathrm{Im}(g_k) \subset \ker(f_k)$ in $H^k(W_\bullet)$. On the other hand, given $w \in Z^k(W_\bullet)$ such that $f_k(w) = \delta_U(y)$, we see that since f_{k-1} is injective, there is an $\overline{y}$ such that $f_{k-1}(\overline{y}) = y$. Then setting

$$u = \delta_W(\overline{y}) - w$$

we see that $f_k(u) = 0$. However, this in turn means that there exists a $\overline{u} \in V_k$ such that $g(\overline{u}) = u$. We now wish to show that $\overline{u}$ is a cocycle. However, since g is a cochain map,

$$g_{k+1}(\delta_V(\overline{u})) = \delta_W(u) = 0$$

and since g_{k+1} is injective, this implies that $\delta_V(\overline{u})$ is injective. Thus, the sequence is exact at $H^k(W_\bullet)$

For exactness at $H^k(U_\bullet)$, let consider a cocycle $w \in Z^k(W_\bullet)$, and let us compute $\kappa(f_k(w))$. In the construction of κ, starting from $x = f_k(w)$ we may thus take $\overline{x} = w$. The element $\kappa(x)$ is then uniquely characterized by $g_{k+1}(\kappa(x)) = \delta_W(w)$. However, because w is a cocycle, $\delta_W(w) = 0$, and thus, we must have $\kappa(x) = 0$ as well. Thus, the image of $H^k(f)$ is a subset of the kernel of κ in $H^k(U_\bullet)$.

On the other hand, suppose that $x \in Z^k(U)$, such that $\kappa(x) = \delta_V(y)$ is a coboundary.[6] Then, since g is a cochain map,

$$\delta_W(g_k(y) - \overline{x}) = \delta_W(\overline{x}) - \delta_W(\overline{x}) = 0$$

[6] This is saying precisely that $[x]$ is in the kernel of $\kappa : H^k(U_\bullet) \to H^{k+1}(V_\bullet)$.

so the element $u = g_k(y) - \overline{x}$ is a cocycle. But we also have

$$f_k(u) = f_k(g_k(y)) - f_k(\overline{x}) = x$$

so that $[x] = f_k([u])$. Thus, the sequence is exact at $H^k(U)$.

We then show exactness at $H^{k+1}(V_\bullet)$. First, let $x \in Z^k(U_\bullet)$. We wish to show that $g_{k+1}(\kappa(x))$ is a coboundary. However, this is immediate, since $\kappa(x)$ is *defined* by the requirement that $g_{k+1}(\kappa(x)) = \delta_W(\overline{x})$. Thus, the image of κ is a subset of the kernel of g_{k+1} in $H^k(V_\bullet)$.

Finally, suppose that $y \in Z^k(V_\bullet)$, and suppose that $g_{k+1}(y) = \delta_W(z)$. We wish to show that there is a cocycle x in $Z^k(U_\bullet)$ such that $\kappa(x)$ is cohomologous to y. To show this, note that $f_k(z)$ is a cocycle since f is a chain map, i.e.

$$\delta_U(f_k(z)) = f_{k+1}(\delta_W(z)) = f_{k+1}(g_{k+1}(y)) = 0.$$

And, by construction $\kappa(f_k(z)) = y$, which completes the proof. □

Definition 4.77. We call the exact sequence of Theorem 4.76 the *long exact sequence* associated to the short exact sequence

$$0 \longrightarrow V_\bullet \xrightarrow{g} W_\bullet \xrightarrow{f} U_\bullet \longrightarrow 0.$$

We now return to the de Rham cohomology. Let M be a smooth manifold, and let U and V be open subsets of M with $U \cup V = M$.

Write $i_U : U \to M$, $i_V : V \to M$, $j_U : U \cap V \to U$, and $j_V : U \cap V \to V$ be the inclusion maps.

Proposition 4.78. *The sequence*

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i_U^* \oplus i_V^*} \Omega^k(U) \oplus \Omega(V) \xrightarrow{j_U^* - j_V^*} \Omega^k(U \cap V) \longrightarrow 0$$

is short exact for any $k \geq 0$.

Proof. First, suppose that $\omega \in \Omega^k(M)$ such that

$$(i_U^*(\omega), i_V^*(\omega)) = (0, 0).$$

Then $i_U^*(\omega) = \omega|_U = 0$ and $i_V^*(\omega) = \omega|_V = 0$. Thus, since $U \cup V = M$, $\omega = 0$. This shows exactness at $\Omega^k(M)$.

We then note that $i_U \circ j_U = i_V \circ j_V$, and so

$$(j_U^* - j_V^*) \circ (i_U^* \oplus i_V^*) = j_U^* \circ i_U^* + j_V^* \circ i_V^* = 0.$$

Thus, $\operatorname{Im}(i_U^* \oplus i_V^*) \subset \ker(j_U^* - j_V^*)$. On the other hand, suppose that

$$(\omega, \eta) \in \Omega^k(U) \oplus \Omega^k(V)$$

with $j_U^* \omega = j_V^* \eta$. This means, precisely, that $\omega|_{U \cap V} = \eta|_{U \cap V}$. Thus, we can define a smooth form $\overline{\omega}$ on M by setting $\overline{\omega}|_U = \omega$ and $\overline{\eta}|_V = \eta$. This proves the exactness at $\Omega^k(U) \oplus \Omega^k(V)$.

Finally, suppose that $\eta \in \Omega^k(U \cap V)$, and let $\{\phi_U, \phi_V\}$ be a partition of unity subordinate to $\{U, V\}$. Then we can define forms

$$\omega_U = \phi_U \eta$$

and

$$\omega_V = \phi_V \eta$$

on U and V respectively, extending by 0 where necessary, such that

$$j_U^*(\omega_U) - j_V^*(-\omega_V) = \eta.$$

Thus $(j_U^* - j_V^*)$ is surjective. □

This is almost enough to derive the Mayer-Vietoris sequence. We need one more step to conclude.

Exercise 4.79. Given cochain complexes $V_\bullet$ and $W_\bullet$ define

$$(V \oplus W)_\bullet := (\{V_k \oplus W_k\}, (\delta_V, \delta_W)).$$

Show that $(V \oplus W)_\bullet$ is a cochain complex. Construct an isomorphism

$$H^k((V_\oplus W)_\bullet) \cong H^k(V_\bullet) \oplus H^k(W_\bullet).$$

Corollary 4.80. *Given open $U, V \subset M$ such that $U \cup V = M$, there is a long exact sequence, called the Mayer-Vietoris sequence, whose generic term is*

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_{dR}(M) \arrow[r, "i_U^* \oplus i_V^*"] & H^k_{dR}(U) \oplus H^k_{dR}(V) \arrow[r, "j_U^* - j_V^*"] & H^k(U \cap V) \arrow[dl, out=-30, in=150] \\
& & H^{k+1}_{dR}(M) \arrow[r] & \cdots
\end{tikzcd}$$

Notation 4.81. The connecting homomorphism in the Mayer-Vietoris sequence is often called the *connecting homomorphism* and denoted by δ.

Example 4.82. We can compute the de Rham cohomology of the n-sphere S^n whenever $n \geq 1$. We already know that

$$H^k_{dR}(S^1) \cong \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & \text{else} \end{cases}$$

We then claim that

$$H^k_{dR}(S^n) \cong \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{else} \end{cases}$$

We show this by induction. If it is true for S^{n-1}, consider the two open sets

$$U_+ := \{x \in S^n \mid x_{n+1} > -\frac{1}{5}\}$$

and

$$U_- := \{x \in S^n \mid x_{n+1} < \frac{1}{5}\}.$$

Their intersection is smoothly homotopy equivalent to S^{n-1}, and each is smoothly contractible. As such, we obtain the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^k_{dR}(U_+) \oplus H^k_{dR}(U_-) \longrightarrow H^k_{dR}(S^{n-1}) \longrightarrow H^{k+1}_{dR}(S^n) \longrightarrow H^{k+1}_{dR}(U_+) \oplus H^{k+1}_{dR}(U_-) \longrightarrow \cdots$$

The first and last term above are trivial vector spaces when $k \geq 1$, and so for $k \geq 1$, we have

$$H^k_{dR}(S^{n-1}) \cong H^k_{dR}(S^n).$$

We then analyze the case where $k = 0$ and $n \geq 2$ in detail. We obtain the sequence

$$\begin{tikzcd}
H^0_{dR}(S^n) \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] \arrow[d, "\cong"] & H^0_{dR}(U_+) \oplus H^0_{dR}(U_-) \arrow[r] \arrow[d, "\cong"] & H^0_{dR}(S^{n-1}) \arrow[r] \arrow[d, "\cong"] & H^1_{dR}(S^n) \arrow[r] \arrow[d, "\cong"] & H^1_{dR}(U_+) \oplus H^1_{dR}(U_-) \arrow[d, "\cong"] \\
\mathbb{R} \arrow[r, "a"'] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "b"'] & \mathbb{R} \arrow[r, "c"'] & H^1(S^n) \arrow[r, "d"'] & 0
\end{tikzcd}$$

Since the map a is rank 1, the map b must be as well (by exactness). Thus, $\ker(c) = \mathbb{R}$ and so $\mathrm{Im}(c) = 0$. But

$$\mathrm{Im}(c) = \ker(d) = H^1_{dR}(S^n)$$

so that $H^1_{dR}(S^n) \cong 0$ as desired.

Example 4.83. Consider the torus $T^2 = S^1 \times S^1$. We can divide this into two open sets for our Mayer-Vietoris sequence by setting $U = U_+ \times S^1$ and $V = U_- \times S^1$. Since U_+ and U_- are smoothly contractible, and $U_+ \cap U_-$ is smoothly homotopy equivalent to a pair of points, we obtain a long exact sequence

$$\begin{tikzcd}
H^0_{dR}(T^2) \arrow[r] & H^0_{dR}(S^1) \oplus H^0(S^1) \arrow[r] & H^0_{dR}(S^1 \amalg S^1) \arrow[dll] \\
H^1_{dR}(T^2) \arrow[r] & H^1_{dR}(S^1) \oplus H^1(S^1) \arrow[r] & H^1_{dR}(S^1 \amalg S^1) \arrow[dll] \\
H^2_{dR}(T^2) \arrow[r] & H^2_{dR}(S^1) \oplus H^2(S^1) \arrow[r] & H^2_{dR}(S^1 \amalg S^1)
\end{tikzcd}$$

Filling in the cohomology groups we already know, we have

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "a"] & \mathbb{R} \oplus \mathbb{R} \arrow[dll] \\
H^1_{dR}(T^2) \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "b"] & \mathbb{R} \oplus \mathbb{R} \arrow[dll] \\
H^2_{dR}(T^2) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

It now falls to us to understand the two maps labelled a and b above. It is not hard to see that both are identified with the map

$$\begin{aligned} \mathbb{R} \oplus \mathbb{R} &\longrightarrow \mathbb{R} \oplus \mathbb{R} \\ (a, b) &\longmapsto (a+b, a+b). \end{aligned}$$

This map has matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and thus is rank 1. This means that we can identify its kernel and cokernel with $\mathbb{R}$. As a result, we obtain two short exact sequences

$$0 \longrightarrow \mathbb{R} \longrightarrow H^1_{dR}(T^2) \longrightarrow \mathbb{R} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{R} \longrightarrow H^1_{dR}(T^2) \longrightarrow 0 \longrightarrow 0.$$

The first of these shows that

$$H^1_{dR}(T^2) \cong \mathbb{R} \oplus \mathbb{R}$$

and the second shows that

$$H^2_{dR}(T^2) \cong \mathbb{R}.$$