An Introduction to Quasi-Categories

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These notes were prepared for a lecture course held during fall semester of 2021 at the University of Virginia. Some exposition and figures have been taken with permission from lecture notes taken by Angus Rush during a lecture course on the same topic given by Tobias Dyckerhoff at Universität Hamburg. Those notes are available at https://github.com/angusrush/htt_precursor_notes. In addition to these notes, I have freely and heavily used the references cited.

Contents

Overture: Why higher categories?

Category theory is built on an observation: considering mathematical objects with a specified structure[1] is effectively the same thing as considering the set of all maps[2] which preserve that structure. This observation leads to the categorical perspective — considering structure-preserving maps to be the key to understanding mathematical structures.

[1] Groups, topological spaces, dg-algebras, or whatever you like.

[2] Morphisms, really.

While this perspective is of great utility, there are cases where simply considering morphisms is insufficient to fully understand the the mathematical structures under consideration. Ironically, one of the first places where this manifests is in the study of categories. If we consider the category Cat, whose objects are (small[3]) categories and whose morphisms are functors, our only notion of when two categories are "the same" is an *isomorphism of categories*: two categories C and D are *isomorphic* if there is a functor

[3] There's some set theoretic futzing about which is required to avoid Russell's Paradox-style issues with the "category of all categories". These notes will comprehensively neglect such issues outside this sidenote. We will work by assuming that there are *small* sets and *large* sets, such that the set of all small sets is a large set. This can be done far more rigorously with the language of *Grothendieck Universes*, which we will not discuss here.

$$F : \mathsf{C} \longrightarrow \mathsf{D}$$

which is bijective on objects, and induces a bijection on hom-sets.

As anyone who has studied categories can tell, this is a problem. The idea of isomorphism of categories is insanely rigid and restrictive. For instance, the category $\mathrm{Vect}_{\mathbb{R}}^{\mathrm{fd}}$ of finite-dimensional vector spaces over $\mathbb{R}$ should be effectively the same as the category $\mathrm{Vect}_{\mathbb{R}}^{\mathrm{euc}}$ whose objects are the spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$. However, these categories are *not* isomorphic — they are equivalent.

To rectify this issue, we need to consider *2-morphisms* — morphisms *between morphisms*. When our morphisms are functors, our 2-morphisms are *natural transformations*.[4] We can then define a *2-category* $\mathbb{C}\mathrm{at}$ whose objects are (small) categories, whose morphisms are functors, and whose 2-morphisms are natural transformations.

[4] Reminder: a *natural transformation* from $F : \mathsf{C} \to \mathsf{D}$ to $G : \mathsf{C} \to \mathsf{D}$ consists of a collection of morphisms $\mu_c : F(c) \to G(c)$ in D such that, for every $f : c \to d$ in C, the diagram

$$\begin{tikzcd} F(c) \arrow[r, "\mu_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(d) \arrow[r, "\mu_d"'] & G(d) \end{tikzcd}$$

commute.

In this 2-category, we get a *much* better notion of when "two categories are the same". We say that two categories C and D are *equivalent* when there are functors $F : \mathsf{C} \longrightarrow \mathsf{D}$ and $G : \mathsf{D} \longrightarrow \mathsf{C}$ and natural isomorphisms

$$\begin{aligned} \mu : G \circ F &\overset{\cong}{\Longrightarrow} \mathrm{Id}_{\mathsf{C}} \\ \nu : F \circ G &\overset{\cong}{\Longrightarrow} \mathrm{Id}_{\mathsf{D}} \end{aligned} \tag{1}$$

This is the usual notion of *equivalence of categories*, and much better suited to studying categories.

To get a sensible notion, we had to go one rung up a ladder — considering objects, morphisms[5] between objects, and 2-morphisms between morphisms, rather than just

[5] In a higher category, we often call morphisms between objects *1-morphisms*.

objects and morphisms between them. The idea of higher category theory is to extend this process. Instead of just considering objects and morphisms, we consider a hierarchy:

- Objects
- (1-)morphisms between objects
- 2-morphisms between 1-morphisms
- 3-morphisms between 2-morphisms
- $\vdots$

If we stop that process at n-morphisms, we get an n*-category*. If we continue it off to infinity, we get what is sometimes called an ω-category, or an (∞, ∞)-category.

When we have n-morphisms for any n, it no longer makes sense to require "equations" of morphisms to hold strictly. So instead of requiring $\mu \circ \nu = \eta$ for k-morphisms μ, ν, and η, we *choose* a $(k+1)$-isomorphism $\tau : \mu \circ \nu \xrightarrow{\cong} \eta$. We might then have to specify additional conditions on τ, now using $(k+2)$-isomorphisms.

This sort of data — specifying 2-isomorphisms instead of equalities, and then 3-isomorphisms between those, and so on off to infinity — is often called *coherence data*.[6] Notice that to define coherence data, we only need the n*-isomorphism*, rather than all n-isomorphism. It therefore suffices to consider an $(\infty, 1)$*-category*: an (∞, ∞)-category where all n-morphisms for $n > 1$ are invertible.[7] For the rest of these notes, we will be studying a model for $(\infty, 1)$-categories, and how coherence works in this model. Following [9], we will tend to refer to $(\infty, 1)$-categories as ∞*-categories* for brevity.

[6] We'll explore a more specific example of coherence data later in this chapter.

[7] "Invertible" in a higher-categorical setting means "Invertible up to coherence data". So, for instance, an equivalence of categories is invertible in this sense.

The fundamental metaphor and the homotopy hypothesis

Our understanding of ∞-categories is built on a metaphor Let's consider an ∞-category where all 1-morphisms are invertible up to coherent data. Such an ∞-category is called an $(\infty, 0)$-category or an ∞*-groupoid*. If we look at Equation 1 very closely, we can notice something surprising, this looks a lot like the definition of a homotopy equivalence of topological spaces.

This leads to the fundamental metaphor of higher category theory: ∞-groupoids look basically the same as topological spaces. An invertible 1-morphism between objects is kind of like a path between points, an invertible 2-morphism between 1-morphisms is kind of like a homotopy between paths, an invertible 3-morphism between 2-morphisms is kind of like a homotopy of homotopies, and so on.

This insight is sometimes phrased as a theorem in a given model of ∞-categories, but its really more of an axiom, requirement, or sanity check. If ∞-groupoids aren't basically the same thing as spaces, then our definition of ∞-categories is off the mark. Most higher category theorists refer to this sanity check as the

Homotopy Hypothesis. *An* ∞*-groupoid is the same thing as a topological space.*

There are, as always some technicalities involved[8], but in essence, this is the key observation which enables the study of higher categories.

[8] For instance, by a topological space, we don't mean *any* topological space, but rather a sufficiently good one — one that can be built as a cell-decomposition.

An example of coherent data

Now that we've got some sense of what we mean by an ∞-category, and how we think about ∞-categories, there's a natural question to ask: Why?

An ∞-category is a lot of data, as is even the simplest coherence condition, and it makes sense to wonder why any of this could be necessary. Rather than try to make a general argument that higher data is useful, lets look at a classical example.

Let X be a path-connected topological space, and $x \in X$ a point. We can consider the space[9] of loops in X based at x:

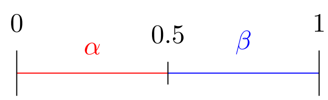
$$\Omega_x X := \left\{ \substack{\alpha:[0,1]\to X \text{ s.t.} \\ \alpha(0)=\alpha(1)=x} \right\}$$

[9] We won't define the topology on $\Omega_x X$ formally here — if you are familiar with the compact-open topology, view $\Omega_x X$ as a subspace of the space of continuous maps $[0,1] \to X$.

We can define something like a multiplication on $\Omega_x X$:

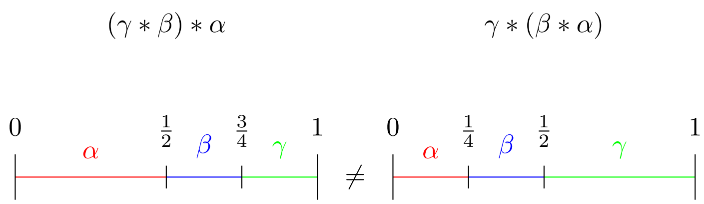
$$* : \Omega_x X \times \Omega_x X \longrightarrow \Omega_x X$$

which sends a pair of loops α and β to the loop $\beta * \alpha$ which we schematically depict as



This multiplication is probably familiar to you. If we take path components of the loop space, we get the underlying set of the fundamental group $\pi_0(\Omega_x X) = \pi_1(X, x)$. The map induced by $*$ on $\pi_0(\Omega_x X)$ is the associative multiplication in the fundamental group.

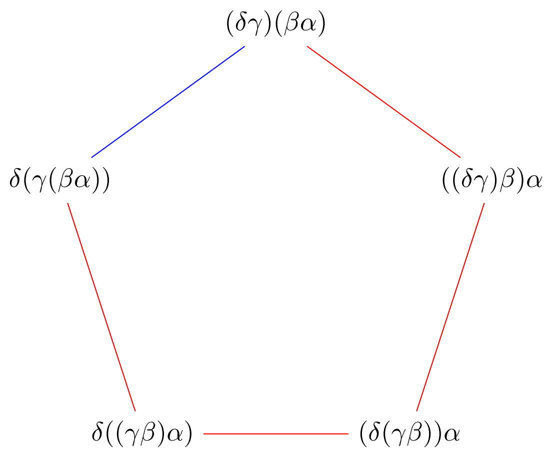
However, taking path components kills off a lot of the data contained in $\Omega_x X$, so lets try not to do that, and instead study $*$ as a multiplication on the loop space. We immediately run into a problem: this multiplication is not associative. We know that up to homotopy we get an associative multiplication, but if we draw $\gamma * (\beta * \alpha)$ and $(\gamma * \beta) * \alpha$, we get



However, since we know that the multiplication is associative up to homotopy, we can *choose* a homotopy for every triple γ, β, α of loops

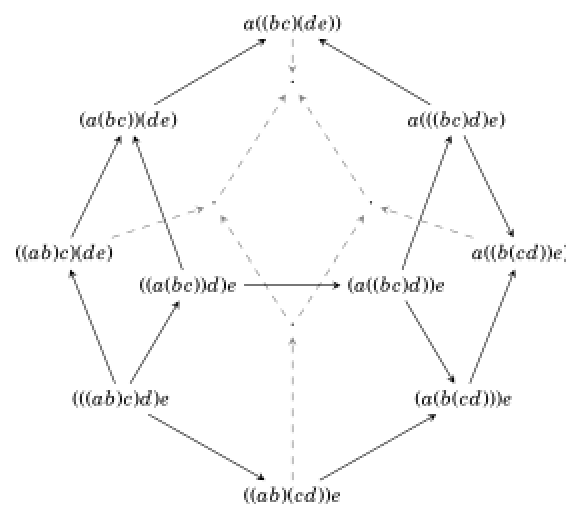
$$A_{\gamma,\beta,\alpha} : (\gamma * \beta) * \alpha \xrightarrow{\simeq} \gamma * (\beta * \alpha).$$

We then run into another problem: if we choose *four* loops δ, γ, β, and α, these homotopies might provide more than one way of getting from the product $(\delta * \gamma) * (\beta * \alpha)$ to the product $\delta * (\gamma * (\beta * \alpha))$. We draw these paths in blue and red below.



Fortunately, we can just play the same game one step up: we can choose a homotopy $H_{\delta,\gamma,\beta,\alpha}$ connecting the blue path to the red path. It turns out that if we choose *five* loops, we get a similar problem (now described by a 3-dimensional polyhedron[10]), and we can choose a homotopy between the homotopies. If we continue choosing these homotopies off to infinity, we get an algebro-topological structure on $\Omega_x X$, called an A_∞*-space structure.* Schematically, an A_∞-space structure on a space Y consists of

[10] This polyhedron is rather more complicated. It is sometimes called K_5 or the *3-dimensional associohedron*, and was discovered by James Stasheff in his study of A_∞-spaces [13]. If we draw this associohedron, we get something like:



(Image credit: Nilesj, retrieved from wikimedia commons.) While there are, of course, higher-dimensional associahedra, we cannot draw or visualize them as easily, and must content ourselves with combinatorial descriptions.

- A map $* : Y \times Y \to Y$
- For every $a, b, c \in Y$, chosen paths $A_{a,b,c}$ in Y displaying the homotopy associativity of $*$.
- For every triple $a, b, c, d \in Y$, chosen homotopies $H_{a,b,c,d}$ filling the pentagon as drawn above.
- $\quad \ldots .$

As with our previous descriptions of coherence, this is a *lot* of data to keep track of. More, perhaps, than we might want to. However, it turns out that this structure allows us to make a powerful statement:

Theorem 0.1 (Stasheff)**.** *Let Y be a topological space. The following are equivalent.*

1. *We can define the structure of an A_∞-space on Y endowing $\pi_0(Y)$ with the structure of a group.*
2. *There is a space X with basepoint x and a homotopy equivalence*
$$Y \simeq \Omega_x X.$$

We already knew one of the implications in this theorem, more or less: loop spaces carry "group-like" A_∞-structures. But the other direction allows us to *recognize* loop spaces in terms of algebraic structures.

It turns out that the language of ∞-categories is very well-suited to discuss A_∞-spaces. To whet your appetite for things to come, let me loosely state a theorem about A_∞-spaces:

Theorem 0.2. *An A_∞-space is a coherently associative monoid in the ∞-category of spaces.*

1

The theory of 1-categories

Before we can embark on the exploration of higher category theory, we will need a firm 1-categorical foundation. Much of the intuition behind quasi-categories rests on being able to visualize the underlying 1-categorical concept, and many proofs require subtle manipulations of 1-categories. We therefore begin with 1-category theory.

1 *Categories, Functors, Natural transformations*

In some sense, a category is a *setting in which to do mathematics*. It tells us what kind of mathematical objects we are considering, and how we relate them. More formally

Definition 1.1. A *category* C consists of

- A set $\mathrm{Ob}(\mathsf{C})$, called the *objects* of C.
- For each pair of objects $x, y \in \mathrm{Ob}(\mathsf{C})$, a set $\mathrm{Hom}_{\mathsf{C}}(x, y)$ called the *morphisms* from x to y. We will write $f : x \to y$ for a morphism $f \in \mathrm{Hom}_{\mathsf{C}}(x, y)$.
- For every triple of objects $x, y, z \in \mathrm{Ob}(\mathsf{C})$, a map of sets

$$\begin{aligned} \circ : \mathrm{Hom}_{\mathsf{C}}(y, z) \times \mathrm{Hom}_{\mathsf{C}}(x, y) &\longrightarrow \mathrm{Hom}_{\mathsf{C}}(x, z) \\ (f, g) &\longmapsto f \circ g \end{aligned}$$

  called *composition*.
- For every $x \in \mathrm{Ob}(\mathsf{C})$, a distinguished element $\mathrm{id}_x \in \mathrm{Hom}_{\mathsf{C}}(x, x)$ called *the identity on* x.

These data are then required to satisfy the following conditions.

- (Associativity) For every triple of morphisms $x \xrightarrow{h} y$, $y \xrightarrow{g} z$, and $z \xrightarrow{f} w$, the composites

$$f \circ (g \circ h) = (f \circ g) \circ h$$

  are equal.

- (Unitality) For $f : x \to y$ and $g : y \to x$, we have

$$f \circ \mathrm{id}_x = f \quad \text{and} \quad \mathrm{id}_x \circ g = g.$$

Example 1.2.

1. There is a category Set whose objects are (small) sets, and such that, for every two sets X and Y, we have

$$\mathrm{Hom}_{\mathsf{Set}}(X, Y) := \{\text{functions } f : X \to Y\}.$$

The composition is the usual composition of functions, and the identies are the identity maps.

2. There are categories Grp and Ab whose objects are (small) groups and (small)[1] abelian groups, respectively, and whose morphisms are group homomorphisms.

[1] We say that a group is small if its underlying set is small. The same holds for other types of objects — vector spaces, topological spaces, etc.

3. There is a category Top whose objects are (small) topological spaces, and whose morphisms are continuous maps of topological spaces.

4. Let k be a (small) field. There is a category Vect_k whose objects are (small) k-vector spaces, and whose morphisms are linear maps.

5. Let $(P, \geq)$ be a partially ordered set. We can define a category P with $\mathrm{Ob}(\mathsf{P}) := P$, and a unique morphism $p \to p'$ if and only if $p \leq p'$.

We will often have cause to sketch categories arising from posets. Let us consider, for instance, the category associated to the power set $\mathbb{P}(\{0,1\})$, ordered by inclusion. We will draw an arrow for every non-identity morphism:

$$\begin{tikzcd} \{0\} \arrow[r] & \{0,1\} \\ \emptyset \arrow[u] \arrow[r] \arrow[ur] & \{1\} \arrow[u] \end{tikzcd}$$

In the picture, we assume that the category is a poset, e.g. that the arrow from $\emptyset \to \{0,1\}$ is the composite of the arrows $\emptyset \to \{1\} \to \{0,1\}$ and the composite of the arrows $\emptyset \to \{0\} \to \{0,1\}$. For ease of drawing, we will sometimes omit composite arrows:

$$\begin{tikzcd} \{0\} \arrow[r] & \{0,1\} \\ \emptyset \arrow[u] \arrow[r] & \{1\} \arrow[u] \end{tikzcd}$$

6. Let G be a group. Then there is a category BG with a single object $*$, and $\mathrm{Hom}_{\mathsf{BG}}(*, *) = G$. The composition is given by $g \circ f = g \cdot f$, and the identity is the neutral element of G.

7. (Key Example) We define *the simplex category* Δ to be the category whose objects are the totally ordered sets $[n] := \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms $[n] \to [m]$ are maps of sets $f : [n] \to [m]$ such that, if $a \leq b$, $f(a) \leq f(b)$.

8. Let C be a category. We can define the *opposite category* C^{op} to have $\mathrm{Ob}(\mathsf{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathsf{C})$, and, for every $x, y \in \mathrm{Ob}(\mathsf{C}^{\mathrm{op}})$

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathsf{C}}(y, x).$$

The composition and identities are the same maps as in C. One thinks of the opposite category as 'reversing the arrows' of C, i.e. reversing the direction of morphisms.

9. Given two categories C and D, we can form a new category $\mathsf{C} \times \mathsf{D}$, called the *product category*. The objects are $\mathrm{Ob}(\mathsf{C} \times \mathsf{D}) = \mathrm{Ob}(\mathsf{C}) \times \mathrm{Ob}(\mathsf{D})$, and the morphisms are given by

$$\mathrm{Hom}_{\mathsf{C}\times\mathsf{D}}((x_1, x_2), (y_1, y_2)) := \mathrm{Hom}_{\mathsf{C}}(x_1, y_1) \times \mathrm{Hom}_{\mathsf{D}}(x_2, y_2).$$

The identities are $\mathrm{id}_{(x,y)} = (\mathrm{id}_x, \mathrm{id}_y)$.

Definition 1.3. Let C be a category, and let $f : x \to y$ be a morphism in C. We call $g : y \to x$ a *right inverse*[2] of f if $f \circ g = \mathrm{id}_y$. We say that g is a *left inverse* of f if $g \circ f = \mathrm{id}_x$. If g is both a left and right inverse to f, then we call g the *inverse* of f, and say that f is an isomorphism in C.

[2] This is also sometimes called a *section* of f.

Exercise 1. Justify calling g *the* inverse of f by showing that any two inverses of f are equal. Show that every identity is an isomorphism.

Exercise 2. Show that the isomorphisms in Set, Top and Grp are, respectively, the bijections, homeomorphisms, and group isomorphisms. Show that the only isomorphisms in the simplex category Δ are the identities.

Definition 1.4. If every morphism in a category C is an isomorphism, we call C a *groupoid.*

Exercise 3. Let G be a finite group. Show that BG is a groupoid. Show that every groupoid G which has only one object $*$ arises in this way.

So how do we relate different categories to one another? We need to define a 'map of categories' which tells us not only how objects relate, but also how morphisms relate.

Definition 1.5. Let C and D be categories. A *functor* $F : \mathsf{C} \to \mathsf{D}$ consists of a map of sets $F : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$, and, for each $x, y \in \mathrm{Ob}(\mathsf{C})$ a map of sets $F : \mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(F(x), F(y))$ such that:

- For every $x \in \mathrm{Ob}(\mathsf{C})$, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.
- For every pair of composable morphisms $x \xrightarrow{g} y \xrightarrow{f} z$ in C, we have that $F(f \circ g) = F(f) \circ F(g)$.

One can compose functors in the obvious way — simply compose the defining maps of sets. We will leave it as an exercise to the reader to check that this is actually a functor.

Example 1.6.

1. For every category C, there is an 'identity' functor $\mathrm{id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$ which sends $x \mapsto x$ and $f \mapsto f$.

2. There is a functor $F : \mathsf{Top} \to \mathsf{Set}$ which sends each topological space to its underlying set, and each continuous map to the underlying map of sets.

3. Let G be a group. A functor
$$F : \mathsf{BG} \longrightarrow \mathsf{Set}$$
Specifies a set $X = F(*)$ and for each $g \in G$, an isomorphism (bijection) $F(g) : X \to X$ such that $F(g \cdot h) = F(g) \circ F(h)$. We thus see that specifying the functor F is equivalent to choosing a set X and a G-action on X.

4. There is a functor
$$\mathsf{Vect}_k^{\mathrm{op}} \longrightarrow \mathsf{Vect}_k$$
which sends each vector space V to the dual vector space $V^\vee := \mathrm{Hom}_{\mathsf{Vect}_k}(V, k)$ and each morphism $f : V \to W$ to the induced map
$$\begin{aligned} f^*\, W^\vee &\longrightarrow V^\vee \\ \alpha &\longmapsto \alpha \circ f \end{aligned}$$

of vector spaces.

Construction 1.7. Let C be a category, and $S \subset \mathrm{Ob}(\mathsf{C})$ be a set of objects. We define the *full subcategory of* C *on* S to be the category D with $\mathrm{Ob}(\mathsf{D}) = S$ and, for each $x, y \in S$

$$\mathrm{Hom}_{\mathsf{D}}(x, y) = \mathrm{Hom}_{\mathsf{C}}(x, y).$$

Definition 1.8. There is a category Cat whose objects are (small) categories, and whose morphisms are functors.[3]

[3] This runs into the same set-theoretic difficulties alluded to earlier. While we won't go into these here, it is important to note that Cat, though it is a category, is not an object of Cat. That is, the category of categories does not contain itself.

Exercise 4. The totally ordered sets $[n]$ from the definition of the simplex category are, in particular, posets, and thus give rise to categories $\overline{n}$. Show that the simplex category Δ can be identified with the full subcategory of Cat on the objects $\overline{n}$, $n \geq 0$.

Example 1.9. Let's define a category $\mathrm{Set}^{\mathrm{sm}}$ to have precisely one object of each cardinality, and morphisms the maps of sets. It shouldn't really matter whether we work in Set or $\mathrm{Set}^{\mathrm{sm}}$, but the two categories aren't isomorphic.

Definition 1.10. Let C and D be categories, and $F, G : \mathsf{C} \to \mathsf{D}$ be functors. A *natural transformation from* F *to* G $\alpha : F \Rightarrow G$ consists of a morphism $\alpha_x : F(x) \to G(x)$ in D for every $x \in \mathrm{Ob}(\mathsf{C})$ such that, for every morphism $f : x \to y$ in C the following diagram commutes:[4]

$$\begin{tikzcd} F(x) \arrow[r, "\alpha_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\alpha_y"'] & G(y). \end{tikzcd}$$

[4] When we say a diagram commutes, we mean that it follows the convention described above for drawing posets: if two composites can be equal, they are equal. In this case, this would mean in equations that $G(f) \circ \alpha_x = \alpha_y \circ F(f)$.

We call a natural transformation a *natural isomorphism* if each of its components α_x is an isomorphism in D.

Example 1.11.

1. There is an 'identity' natural isomorphism $\mathrm{Id}_F : F \to F$ for every $F : \mathsf{C} \to \mathsf{D}$. The components of Id_F are $(\mathrm{Id}_F)_x := \mathrm{id}_{F(x)}$

2. Consider the two functors $\mathrm{id}, C : \mathsf{Grp} \to \mathsf{Grp}$, where $C(G) = G/[G, G]$ is the abelianization. There is a natural transformation $\alpha : \mathrm{id} \Rightarrow C$ with component α_G given by the quotient map $G \to G/[G, G]$.

3. Consider a group G and two functors

$$F, G : BG \longrightarrow \mathrm{Set}$$

Corresponding to G-sets $X = F(*)$ and $Y = G(*)$. A natural transformation $F \Rightarrow G$ is equivalently a G-equivariant map $X \to Y$.

Construction 1.12. Let $\beta : F \Rightarrow G$ and $\alpha : G \Rightarrow H$ be natural transformations among functors $F, G, H : \mathsf{C} \to \mathsf{D}$. We define the composite $\alpha \circ \beta$ via the formula, for all $x \in \mathsf{C}$, $(\alpha \circ \beta)_x = \alpha_x \circ \beta_x$. To check that this is natural, note that for each morphism $f : x \to y$

in C we have a pair of squares:

$$\begin{tikzcd} F(x) \arrow[r, "\beta_x"] \arrow[d, "F(f)"'] & G(x) \arrow[r, "\alpha_x"] \arrow[d, "G(f)"] & H(x) \arrow[d, "H(f)"] \\ F(y) \arrow[r, "\beta_y"'] & G(y) \arrow[r, "\alpha_y"'] & H(y). \end{tikzcd}$$

where the left and right hand squares both commute. It is easy to verify from the diagram that this means the external square commutes as well[5]

[5] This is a kind of *pasting law*, and we will often make use of such arguments without comment.

Exercise 5. Let C and D be categories. Show that there is a category $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ with objects given by functors from C to D, and morphisms given by natural transformations.

Definition 1.13. We say that a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence of categories* if there is a functor $G : \mathsf{D} \to \mathsf{C}$, and natural isomorphisms $\alpha : G \circ F \cong \mathrm{Id}_{\mathsf{C}}$ and $\beta : F \circ G \cong \mathrm{Id}_{\mathsf{D}}$. We then say that the categories C and D are *equivalent*, and call G a *weak inverse* to F.[6]

[6] Notice that this is almost exactly the same as a homotopy equivalence of topological spaces. If we replace the word "functor" by "continuous map", the word "category" by "space", and the word "natural isomorphism" by "homotopy"

Definition 1.14. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. We call F

1. *full* if, for every $x, y \in \mathsf{C}$, the map $F : \mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(F(x), F(y))$ is surjective.
2. *faithful* if , for every $x, y \in \mathsf{C}$, the map $F : \mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(F(x), F(y))$ is injective.
3. *fully faithful* if it is both full and faithful.
4. *essentially surjective* if, for every $y \in \mathsf{D}$, there exists an $x \in \mathsf{C}$ such that $F(x)$ is isomorphic to y.

Proposition 1.15. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. Then the following are equivalent:*

1. *F is fully faithful and essentially surjective.*
2. *F is an equivalence of categories.*

Proof. We first show 2. $\Rightarrow$ 1. Let G be a weak inverse to F, and α and β the natural isomorphisms displaying the weak invertibility of F. Given y in D, the y-component of β is an isomorphism $\beta_y : F(G(y)) \xrightarrow{\cong} y$, showing that F is essentially surjective.

Let $f : x \to y$ be a morphism in C. Then, by the naturality of α, the diagram

$$\begin{tikzcd} G(F(x)) \arrow[r, "\alpha_x"] \arrow[d, "G(F(f))"'] & x \arrow[d, "f"] \\ G(F(y)) \arrow[r, "\alpha_y"'] & y \end{tikzcd}$$

commutes. Since α_x is invertible, this means that

$$f = \alpha_y \circ G(F(f)) \circ \alpha_x^{-1},$$

i.e., f is uniquely determined by $G(F(f))$. Thus $G \circ F : \mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{C}}(G(F(x)), G(F(y)))$ is injective, and hence, $F : \mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(F(x), F(y))$ is injective, and F is faithful. Note that we can make the same argument with β to find that G is faithful.

Now let $h : F(x) \to F(y)$ in D be a morphism, and define $f := \alpha_y \circ G(h) \circ \alpha_x^{-1}$. Again we find a commutative square:

$$\begin{tikzcd} G(F(x)) \arrow[r, "\alpha_x"] \arrow[d, "G(h)"'] & x \arrow[d, "f"] \\ G(F(y)) \arrow[r, "\alpha_y"'] & y \end{tikzcd}$$

Which implies that $G(F(f)) = G(h)$. However, since G is faithful, this implies $F(f) = h$, and thus, $F : \mathrm{Hom}_{\mathsf{C}}(x, y) \to \mathrm{Hom}_{\mathsf{D}}(F(x), F(y))$ is surjective and F is full.

We now show 1. $\Rightarrow$ 2. Suppose F is essentially surjective and fully faithful. For every $y \in \mathrm{Ob}(\mathsf{D})$, we choose[7] an object which we call $G(y) \in \mathrm{Ob}(\mathsf{C})$ together with an isomorphism $\beta_y : F(G(y)) \xrightarrow{\cong} y$ (by essential surjectivity). We then define, for every $h : x \to y$ in D, the morphism $G(h) := F^{-1}(\beta_y^{-1} \circ h \circ \beta_x) \in \mathrm{Hom}_{\mathsf{C}}(G(x), G(y))$.[8]

[7] Note that this requires the axiom of choice. As a rule, we will assume the axiom of choice in our categorical arguments.

[8] We can do this precisely because F is fully faithful, so the map of sets

$$F : \mathrm{Hom}_{\mathsf{C}}(G(x), G(y)) \to \mathrm{Hom}_{\mathsf{D}}(F(G(x)), F(G(y)))$$

is a bijection.

Since

$$G(\mathrm{id}_x) = F^{-1}(\beta_x^{-1} \circ \mathrm{id}_x \circ \beta_x) = F^{-1}(\beta_x^{-1} \circ \beta_x) = F^{-1}(\mathrm{id}_x),$$

we see that $G(\mathrm{id}_x) = \mathrm{id}_{G(x)}$. Moreover, since

$$G(f \circ g) = F^{-1}(\beta_z^{-1} \circ f \circ g \circ \beta_x) = F^{-1}(\beta_z^{-1} \circ f \circ \beta_y \circ \beta_y^{-1} \circ g \circ \beta_x)$$

we have $G(f \circ g) = G(f) \circ G(g)$. Thus, G defines a functor.[9] Tracing the definitions, it is immediate that the squares

[9] We have used the fully faithfulness of F for both computations here, as well as the functoriality of F.

$$\begin{tikzcd} F(G(x)) \arrow[r, "\beta_x"] \arrow[d, "F(G(f))"'] & x \arrow[d, "f"] \\ F(G(y)) \arrow[r, "\beta_y"'] & y \end{tikzcd}$$

commute for any $f : x \to y$ in D, so that $\beta : F \circ G \Rightarrow \mathrm{Id}_{\mathsf{D}}$ is a natural isomorphism.

We now seek to obtain the natural isomorphism $\alpha : G \circ F \Rightarrow \mathrm{Id}_{\mathsf{C}}$. Note that $\beta_{(F(x))} : F(G(F(x))) \to F(x)$ is an isomorphism. By the fully faithfulness of F, there is a unique isomorphism[10] $\alpha_x := F^{-1}(\beta_{F(x)}) : G(F(x)) \to x$. Since β is a natural transformation, for any $f : x \to y$ in C, we get a commutative square

[10] We leave it as an exercise to the interested reader to check that this is actually an isomorphism using the fully-faithfulness of F.

$$\begin{tikzcd} F(G(F(x))) \arrow[r, "\beta_{F(x)}"] \arrow[d, "F(G(F(f)))"'] & F(x) \arrow[d, "F(f)"] \\ F(G(F(y))) \arrow[r, "\beta_{F(y)}"'] & F(y) \end{tikzcd}$$

i.e. $F(f \circ \alpha_x) = F(\alpha_y \circ G(F(f)))$. But, since F is fully faithful, this implies

$$f \circ \alpha_x = \alpha_y \circ G(F(f))$$

and thus the square

$$\begin{tikzcd} G(F(x)) \arrow[r, "\alpha_x"] \arrow[d, "G(F(f))"'] & x \arrow[d, "f"] \\ G(F(y)) \arrow[r, "\alpha_y"'] & y \end{tikzcd}$$

commutes. This means that α is natural, and thus provides the desired natural isomorphism $\alpha : G \circ F \Rightarrow \mathrm{Id}_{\mathsf{C}}$. □

Exercise 6. Show that the two categories from Example 1.9 are equivalent.

2 *The Yoneda Lemma*

We now have enough background to state and prove what might be called the "Fundamental Theorem of Category theory" — Yoneda's Lemma.[11]

[11] Nobuo Yoneda, a computer scientist from Japan, formulated and proved the Yoneda lemma, but never published it. He communicated it to Saunders MacLane, one of the originators of category theory. The name 'the Yoneda Lemma' was given to the lemma by MacLane.

Definition 1.16. Let C be a category[12], and $c \in \mathrm{Ob}(\mathsf{C})$. We define a functor

$$\begin{aligned} h^c : \mathsf{C} &\longrightarrow \mathsf{Set} \\ d &\longmapsto \mathrm{Hom}_{\mathsf{C}}(c,d) \end{aligned}$$

called the *representable functor* associated to c. For a morphism $f : x \to y$ in C, we define $h^c(f)$ to send $g : c \to x$ to $f \circ g : c \to y$.

[12] Once again, set-theoretic technicalities come into play. Technically, what we need for this definition is a *locally small category*, that is, a category such that all of the Hom-sets are 'small sets.'

Similarly, we obtain a representable functor

$$\begin{aligned} h_c : \mathsf{C}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ d &\longmapsto \mathrm{Hom}_{\mathsf{C}}(d,c) \end{aligned}$$

This is sometimes called the *contravariant*[13] *representable functor* associated to c.

[13] This is an old terminological convention. A functor $F : \mathsf{C} \to \mathsf{D}$ is sometimes called a *covariant* functor from C to D, and a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ is called a *contravariant* functor from C to D. Covariant functors preserve the direct of morphisms, whereas contravariant functors reverse the direction of morphisms.

Exercise 7. Show that

$$\begin{aligned} \mathcal{Y} : \mathsf{C} &\longrightarrow \mapsto \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set}) \\ c &\mapsto h_c \end{aligned}$$

defines a functor. This functor is called the *Yoneda embedding*.

Notation 1.17. Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. We denote the set of natural transformations from F to G by $\mathrm{Nat}(F,G)$. Note that $\mathrm{Nat}(-,G) : \mathrm{Fun}(\mathsf{C},\mathsf{D})^{\mathrm{op}} \to \mathsf{Set}$ is the representable functor h_G on the category $\mathrm{Fun}(\mathsf{C},\mathsf{D})$.

Proposition 1.18 (The Yoneda Lemma)**.** *Let $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a functor and $c \in \mathrm{Ob}(\mathsf{C})$. There is an isomorphism*

$$\mathrm{Nat}(h_c, F) \overset{\cong}{\Rightarrow} F(c)$$

Moreover, this isomorphism is natural in both c and F.

Before beginning the proof, we first comment on what we mean by "natural in c and F". We view $\mathrm{Nat}(h_{(-)}, -)$ as a functor $\mathsf{C}^{\mathrm{op}} \times \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathsf{Set}$, which sends (x,F) to $\mathrm{Nat}(h_x, F)$, and sends a pair of morphisms $f : x \to y$ in C and $\alpha : F \Rightarrow G$ to the morphism

$$\begin{aligned} \mathrm{Nat}(h_y, F) &\longrightarrow \mathrm{Nat}(h_x, G) \\ \beta &\longmapsto (\alpha \circ \beta \circ \mathcal{Y}(f)) \end{aligned}$$

Similarly, we define a functor $(c,F) \mapsto F(c)$ with the obvious functoriality.

Proof. We first prove that there is an isomorphism for any c and F. We define a map

$$\begin{aligned} \chi_{(c,F)} : \mathrm{Nat}(h_c, F) &\longrightarrow F(c) \\ \alpha &\longmapsto \alpha_c(\mathrm{id}_c). \end{aligned}$$

We will show that this is a bijection (i.e. an isomorphism in Set.)

First, let $\alpha : h_c \Rightarrow F$ be an arbitrary natural transformation, and let $f : d \to c$ be a morphism in C (i.e., a morphism $c \to d$ in C^{op}). Then naturality yields a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(c,c) \arrow[r, "h_c(f)"] \arrow[d, "\alpha_c"'] & \mathrm{Hom}_{\mathsf{C}}(d,c) \arrow[d, "\alpha_d"] \\
F(c) \arrow[r, "F(f)"'] & F(d)
\end{tikzcd}$$

Since $h_c(f)(\mathrm{id}_c) = f$, this, in particular implies that $\alpha_d(f) = F(f)(\alpha_c(\mathrm{id}_c))$. Therefore, α is uniquely determined by $\alpha_c(\mathrm{id}_c)$, and thus, $\chi_{(c,F)}$ is injective.

Now, given $x \in F(c)$ we define a natural transformation β with by setting $\beta_d(f) := F(f)(c)$. We now check that β is natural. Let $g : e \to d$ be a morphism in C, and let $f \in \mathrm{Hom}_{\mathsf{C}}(d,c)$. We then have that

$$\alpha_e(h_c(g)(f)) = F(h_c(g)(f))(\mathrm{id}_c) = F(g \circ f)(\mathrm{id}_c) = F(g)(F(f)(\mathrm{id}_c))$$

and by definition

$$F(g)(\alpha_d(f)) = F(g)(F(f)(\mathrm{id}_c)).$$

So the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(d,c) \arrow[r, "h_c(g)"] \arrow[d, "\alpha_c"'] & \mathrm{Hom}_{\mathsf{C}}(e,c) \arrow[d, "\alpha_d"] \\
F(d) \arrow[r, "F(g)"'] & F(e)
\end{tikzcd}$$

commutes. Hence, β is a natural transformation with $\chi_{(c,F)}(\beta) = x$, and $\chi_{(c,F)}$ is surjective.

We now show naturality in c. Let $f : c \to d$ be a morphism in C. We wish to show that, for any $F : \mathsf{C}^{\mathrm{op}} \to \mathrm{Set}$ the diagram

$$\begin{tikzcd}
\mathrm{Nat}(h_d, F) \arrow[r, "- \circ \text{よ}(f)"] \arrow[d, "\chi_{(d,F)}"'] & \mathrm{Nat}(h_c, F) \arrow[d, "\chi_{(c,F)}"] \\
F(d) \arrow[r, "F(f)"] & F(c)
\end{tikzcd}$$

commutes. We therefore compute, for an arbitrary natural transformation $\alpha : h_d \Rightarrow F$,

$$\chi_{(c,F)}(\alpha \circ \text{よ}(f)) = (\alpha \circ \text{よ}(f))_c(\mathrm{id}_c) = \alpha_c(\text{よ}(f)_c(\mathrm{id}_c)) = \alpha_c(f \circ \mathrm{id}_c) = \alpha_c(f)$$

Similarly, we compute

$$F(f)(\chi_{(d,F)}(\alpha)) = F(f) \circ \alpha_d(\mathrm{id}_d).$$

However, since α is, itself a natural transformation, our work above shows that

$$F(f)(\alpha_d(\mathrm{id}_d)) = \alpha_c(f).$$

so the diagram commutes, and χ is natural in c.

Finally, we show naturality in F. Let $\beta : F \to G$ be a natural transformation, and c an arbitrary object of C. We want to show that the diagram

$$\begin{tikzcd} \mathrm{Nat}(h_c, F) \arrow[r, "\beta\circ -"] \arrow[d, "\chi_{(c,F)}"'] & \mathrm{Nat}(h_c, G) \arrow[d, "\chi_{(c,G)}"] \\ F(c) \arrow[r, "\beta_c"] & G(c) \end{tikzcd}$$

commutes. We again compute for an arbitrary natural transformation $\alpha : h_c \Rightarrow F$,

$$\chi_{(c,G)}(\beta \circ \alpha) = (\beta \circ \alpha)_c(\mathrm{id}_c)$$

and

$$\beta_c(\chi_{(c,F)}(\alpha)) = \beta_c(\alpha_c(\mathrm{id}_c))$$

This shows that the diagram commutes, and so χ is natural in F. □

Exercise 8. Show that the Yoneda embedding

$$\mathcal{Y} : \mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$$

is fully faithful.[14] In particular, if there is a natural isomorphism $\alpha : h_c \cong h_d$, then there is a unique isomorphism $c \cong d$ in C corresponding to α under the Yoneda embedding.

[14] This is, in fact, what we mean by an *embedding* of categories: a fully faithful functor.

Definition 1.19. A *universal property* on a category $\mathcal{C}$ is a functor

$$U : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}.$$

A pair (c, α), where $c \in \mathrm{Ob}(\mathcal{C})$ and $\alpha : h_c \Rightarrow U$ is a natural isomorphism is called a *representation of* U. We also say that (c, α) *satisfies the universal property* U. We often abuse notation by saying that the object c satisfies U, and leaving the natural isomorphism implicit.

Remark 1.20. We will also consider the dual case, that of a functor $U : \mathcal{C} \to \mathsf{Set}$, as a universal property on $\mathcal{C}$. Here an pair satisfying U is an object c and a natural isomorphism $h^c \cong U$.

Example 1.21. Define a functor

$$U_{V,W} : \mathsf{Vect}_k \to \mathsf{Set}$$

by mapping $Z \in \mathsf{Vect}_k$ to the set $\mathrm{Bilin}(V, W; Z)$ of bilinear maps $V \times W \to Z$ and mapping a linear map $f : X \to Z$ to the map

$$\begin{aligned} \mathrm{Bilin}(V, W; X) &\longrightarrow \mathrm{Bilin}(V, W; Z) \\ g &\longmapsto f \circ g \end{aligned}$$

A tensor product of V and W over k is then a representation of $U_{V,W}$.

Definition 1.22. Let $\mathcal{C}$ be a category, and define a functor

$$* : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$$

which sends every object of $\mathcal{C}$ to the singleton set $\{*\}$. A *terminal object* of $\mathcal{C}$ is a representation (c, α) for $*$.

An *initial object* is a terminal object in $\mathcal{C}^{\mathrm{op}}$, that is, a representation of the functor

$$* : \mathcal{C} \to \mathsf{Set}$$

which sends every object to the singleton set.[15]

Let's unpack what the definition means for a terminal object. Suppose $c \in \mathcal{C}$ is terminal. This means that α provides a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(-, c) = h_c(-) \cong \{*\}$$

That is, $\mathrm{Hom}_{\mathcal{C}}(d, c)$ is a singleton set for every $d \in c$. Since there is a unique bijection between any two singleton sets, naturality is automatic. The definition therefore amounts to saying that an object c is terminal if and only if there is a unique morphism $d \to c$ for every $d \in \mathcal{C}$.

An object c is then initial if and only if there is a unique morphism $f : c \to d$ for every object d in $\mathcal{C}$.

[15] An initial object is the *dual* concept to a terminal object — it is obtained by simply reversing the definitions of the arrows.

3 *Adjunctions*

Adjunctions are a key generalization of equivalences of categories. They consist, in effect, of a weakening of Definition 1.13.

Definition 1.23. An *adjunction* between two categories C and D consists of a functor $F : \mathsf{C} \to \mathsf{D}$ (the *left adjoint*), a functor $G : \mathsf{D} \to \mathsf{C}$ (the *right adjoint*), a natural transformation $\epsilon : \mathrm{id}_{\mathsf{C}} \Rightarrow G \circ F$ (the *unit*), and a natural transformation $\eta : F \circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$ (the *counit*). These data are required to satisfy two conditions.

1. The composite
$$F \xRightarrow{F \circ \epsilon} F \circ G \circ F \xRightarrow{\eta \circ F} F$$
is the identity transformation on F.

2. The composite
$$G \xRightarrow{\epsilon \circ G} G \circ F \circ G \xRightarrow{G \circ \eta} G$$
is the identity transformation on G.

There is an equivalent characterization of adjunctions, which is somewhat easier to keep track of:

Suppose we are given an adjunction (F, G, ϵ, η) between categories C and D, then for every $c \in C$ and $d \in \mathsf{D}$ we can obtain a map of sets

$$\phi_{c,d} : \mathrm{Hom}_{\mathsf{D}}(F(c), d) \xrightarrow{G} \mathrm{Hom}_{\mathsf{C}}(G(F(c)), G(d)) \xrightarrow{- \circ \epsilon_c} \mathrm{Hom}_{\mathsf{C}}(c, G(d)).$$

It turns out that this datum uniquely characterizes the adjunction

Lemma 1.24. *Given an adjunction (F, G, ϵ, η), the maps $\phi_{c,d}$ are isomorphisms, natural in c and d.*

Proof. We first prove naturality. Given $f : b \to c$ in C and $g : d \to e$ in D, we can form the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(F(c), d) \arrow[r, "G"] \arrow[d, "g \circ (-) \circ F(f)"'] & \mathrm{Hom}_{\mathsf{C}}(G(F(c)), G(d)) \arrow[r, "- \circ \epsilon_c"] \arrow[d, "G(g) \circ (-) \circ G(F(f))"] & \mathrm{Hom}_{\mathsf{C}}(c, G(d)) \arrow[d, "G(g) \circ (-) \circ f"] \\
\mathrm{Hom}_{\mathsf{D}}(F(b), e) \arrow[r, "G"] & \mathrm{Hom}_{\mathsf{C}}(G(F(b)), G(e)) \arrow[r, "- \circ \epsilon_b"] & \mathrm{Hom}_{\mathsf{C}}(b, G(e)).
\end{tikzcd}$$

The left-hand square commutes by the functoriality of G, and the right-hand square commutes by the naturality of ϵ.[16]

Now we show that $\phi_{c,d}$ is an isomorphism. We define an putative inverse by

$$\chi_{c,d} : \mathrm{Hom}_{\mathsf{C}}(c, G(d)) \xrightarrow{F} \mathrm{Hom}_{\mathsf{D}}(F(c), F(G(d))) \xrightarrow{\eta_d \circ -} \mathrm{Hom}_{\mathsf{D}}(F(c), d)$$

Our claim is then that $\chi_{c,d}$ is inverse to $\phi_{c,d}$.

To see that this is the case, consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(c, G(d)) \arrow[r, "F"] & \mathrm{Hom}_{\mathsf{D}}(F(c), F(G(d))) \arrow[r, "\eta_d \circ -"] \arrow[d, "G"] & \mathrm{Hom}_{\mathsf{D}}(F(c), d) \arrow[d, "G"] \\
& \mathrm{Hom}_{\mathsf{C}}(G(F(c)), G(F(G(d)))) \arrow[r, "G(\eta_d) \circ -"] & \mathrm{Hom}_{\mathsf{C}}(G(F(c)), G(d)) \arrow[d, "- \circ \epsilon_c"] \\
& & \mathrm{Hom}_{\mathsf{C}}(c, G(d))
\end{tikzcd}$$

Where the middle square commutes by functoriality. If we start with a morphism $f : c \to G(d)$ in C, we need only check that $G(\eta_d) \circ G(F(f)) \circ \epsilon_c = f$.

However, by the naturality of ϵ, we see that

$$G(\eta_d) \circ G(F(f)) \circ \epsilon_c = G(\eta_d) \circ \epsilon_{G(d)} \circ f$$

However, the second compatibility condition dictates that $G(\eta_d) \circ \epsilon_{G(d)} = \mathrm{id}_{G(d)}$. Thus $\phi_{c,d} \circ \chi_{c,d} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{C}}(c, G(d))}$. The statement that $\chi_{c,d} \circ \phi_{c,d} = \mathrm{id}$ is formally dual, completing the proof. □

Proposition 1.25. *Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors. The following conditions are equivalent:*

1. *There is the datum of an adjunction (F, G, ϵ, η) displaying F as left adjoint to G.*
2. *There is an isomorphism*

$$\phi_{c,d} : \mathrm{Hom}_{\mathsf{D}}(F(c), d) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{C}}(c, G(d))$$

natural in $c \in \mathsf{C}$ and $d \in \mathsf{D}$.

Proof. We have already see that (1) implies (2) in the previous lemma.

Now suppose that (2) holds. In particular, there are isomorphisms

$$\phi_{c,F(c)} : \mathrm{Hom}_{\mathsf{D}}(F(c), F(c)) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(c, F(G(c))) \quad \text{and}$$

$$\phi^{-1}_{G(d),d} : \mathrm{Hom}_{\mathsf{C}}(G(d), G(d)) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(F(G(d)), d)$$

We can define a unit $\epsilon_c := \phi_{c,F(c)}(\mathrm{id}_{F(c)})$ and a counit $\eta_d := \phi^{-1}_{G(d),d}(\mathrm{id}_{G(d)})$. Checking that these satisfy the identities for unit and counit is routine, but tedious — we leave this as an exercise. □

Corollary 1.26. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. If a right adjoint of F exists, it is unique up to unique natural isomorphism. Dually, if a left adjoint of F exists, it is unique up to unique natural isomorphism.*

[16] In particular, the naturality of epsilon means that the diagram

$$\begin{tikzcd}
c \arrow[r, "\epsilon_c"] \arrow[d, "f"'] & G(F(c)) \arrow[d, "G(F(f))"] \\
b \arrow[r, "\epsilon_b"'] & G(F(b))
\end{tikzcd}$$

commutes. Passing through the Yoneda embedding yields the commutativity of the last square.

Proof. This follows from the Yoneda lemma and the natural isomorphism of Hom-sets from Proposition 1.25. □

Example 1.27.

1. Consider the functor
$$F : \mathsf{Ab} \longrightarrow \mathsf{Set}$$
which forgets the underlying group structure.[17] This functor admits a left adjoint, called the *free Abelian group functor*
$$\begin{aligned} \mathbb{Z}[-] : \mathsf{Set} &\longrightarrow \mathsf{Ab} \\ S &\longmapsto \mathbb{Z}[S] \end{aligned}$$
which sends each set S to the free Abelian group $\mathbb{Z}[S]$ on S. We can define a map
$$\begin{aligned} \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}[S], G) &\longrightarrow \mathrm{Hom}_{\mathsf{Set}}(S, F(G)) \\ f &\longmapsto (s \mapsto f(1s)) \,. \end{aligned}$$
We leave to the reader the verification that this map is a natural isomorphism.

   This style of construction is a very general construction. In general one can say that a 'free' construction is a left adjoint to a forgetful functor.

2. Consider the functor
$$F : \mathsf{Top} \longrightarrow \mathsf{Set}$$
which sends each topological space to its underlying set, and each continuous map to the map of underlying sets. This admits a left adjoint Disc $: \mathsf{Set} \to \mathsf{Top}$ which sends each set X to the topological space $(X, \mathbb{P}(X))$, i.e. the discrete topology on X.

   The functor F also admits a right adjoint, Ind $: \mathsf{Set} \to \mathsf{Top}$, which sends each set X to the topological space $(X, \{\emptyset, X\})$ – the *indiscrete* or *trivial* topology on X.

3. ([12, Ex. 4.1.12])The forgetful functor
$$F : \mathsf{Field} \longrightarrow \mathsf{Set}$$
does not admit left or right adjoints. This is because there are no field homomorphisms between fields of different characteristic.

4. For any $X \in \mathsf{Set}$, the Hom-functor
$$\mathrm{Hom}_{\mathsf{Set}}(X, -) : \mathsf{Set} \longrightarrow \mathsf{Set}$$
has a left adjoint
$$- \times X : \mathsf{Set} \longrightarrow \mathsf{Set}$$
The associated natural isomorphism is given by
$$\begin{aligned} \mathrm{Hom}_{\mathsf{Set}}(Y, \mathrm{Hom}_{\mathsf{Set}}(X, Z)) &\xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Set}}(Y \times X, Z) \\ f &\longmapsto ((y, x) \mapsto f_y(x)) \end{aligned}$$
where we write $f_y : X \to Z$ for the image of y under f.

[17] In general, functors are called *forgetful* if they forget some underlying mathematical structure. In this case, we send each Abelian group to its underlying map of sets, and send each homomorphism to the map on underlying sets.

5. Let CW be the full subcategory of Top whose objects are CW-complexes. There is a functor
$$C(X,-) : \mathsf{CW} \longrightarrow \mathsf{Top}$$
which sends a space Z to the topological space $C(X,Z)$ of continuous maps from X to Z equipped with the compact-open topology. Then there is a natural isomorphism
$$\mathrm{Hom}_{\mathsf{Top}}(Y, C(X,Z)) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Top}}(Y \times X, Z)$$
obtained by restricting the natural isomorphism from the previous example to the continuous maps.

4 *Limits*

We will now consider a very special kind of adjunction. Let's consider a (small) category I — which we will call our *indexing category*, and a category C. Further denote by $[0]$ the category with a single object 0 and no non-identity morphisms. There is a unique (!) functor
$$T : \mathsf{I} \longrightarrow [0]$$
which then induces a unique functor
$$\mathrm{const} : \mathsf{C} \cong \mathsf{C}^{[0]} \longrightarrow \mathsf{C}^{I}$$
which we call the *constant diagram functor*. This functor takes an object $c \in \mathsf{C}$ and maps it to the functor $\mathrm{const}_c : \mathsf{I} \to \mathsf{C}$ which maps every object to c and every morphism to id_c.

Our aim is to investigate the left and right adjoints to this functor (if they exist). Such adjoints describe widely used constructions in category theory — the so-called *limits* and *colimits*.

Definition 1.28. If the constant diagram functor admits a left adjoint, we denote this adjoint by
$$\mathrm{colim}_I : \mathsf{C}^{\mathsf{I}} \longrightarrow \mathsf{C}$$
and call it the *(I-indexed) colimit functor*. For a diagram $F : \mathsf{I} \to \mathsf{C}$, we will call $\mathrm{colim}_I F$ a *colimit* of F. When the colimit functor exists, we say that C *admits* I*-indexed colimits*. If C admits I-indexed colimits for any (small) category I, we say that C is *cocomplete*.

Dual Definition 1.28. If the constant diagram functor admits a right adjoint, we denote this adjoint by
$$\lim : \mathsf{C}^{\mathsf{I}} \longrightarrow \mathsf{C}$$
and call it the *(I-indexed) limit functor*. For a diagram $F : \mathsf{I} \to \mathsf{C}$, we will call $\lim_I F$ a *limit* of F. When the limit functor exists, we say that C *admits* I*-indexed limits*. If C admits I-indexed limits for any (small) category I, we say that C is *complete*.

This definition is all well and good, but so far it's just so much abstract nonsense. Lets unwind the definition a little bit, to see what it is saying.

By the hom-set definition of an adjunction, const having a left adjoint means that there is an isomorphism
$$\psi : \mathrm{Hom}_{\mathsf{C}}(\operatorname*{colim}_I F, c) \cong \mathrm{Nat}(F, \mathrm{const}_c),$$
natural in $F \in \mathsf{C}^{\mathsf{I}}$ and $c \in \mathsf{C}$. There's something important to notice here: If we fix F, this isomorphism can be viewed as a universal property! $\mathrm{colim}_I F$ is just an object corepresenting the functor
$$\begin{aligned} \mathsf{C} &\longrightarrow \mathsf{Set} \\ c &\longmapsto \mathrm{Nat}(F, \mathrm{const}_c). \end{aligned}$$

One consequence of this is that it makes sense to talk about the colimit of a *specific* diagram regardless of whether or not the colimit functor exists more globally or not!

Now lets unwind this universal property.

Definition 1.29. A *cocone* over a diagram $F : \mathsf{I} \to \mathsf{C}$ is a natural transformation $\chi : F \Rightarrow \mathrm{const}_c$. More precisely, a cocone over F consists of:

- an object $c \in \mathsf{C}$ called the *tip* of the cocone; and
- for every object $i \in \mathsf{I}$, a morphism $\chi_i : F(i) \to c$.

The morphisms χ_i must be such that, for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} F(i) \arrow[rr, "F(f)"] \arrow[dr, "\chi_i"'] & & F(j) \arrow[dl, "\chi_j"] \\ & c & \end{tikzcd}$$

commutes.

Dual Definition 1.29. A *cone* over a diagram $F : \mathsf{I} \to \mathsf{C}$ is a natural transformation $\chi : \mathrm{const}_c \Rightarrow F$. More precisely, a cocone over F consists of:

- an object $c \in \mathsf{C}$ called the *tip* of the cone; and
- for every object $i \in \mathsf{I}$, a morphism $\chi_i : F(i) \Rightarrow c$.

The morphisms χ_i must be such that, for every morphism $f : i \to j$ in I, the diagram

$$\begin{tikzcd} & c \arrow[dl, "\chi_i"'] \arrow[dr, "\chi_j"] & \\ F(i) \arrow[rr, "F(f)"'] & & F(j) \end{tikzcd}$$

commutes.

What our universal property tells us is that there is a bijection between morphisms $\mathrm{colim}_{\mathsf{I}} F \to c$ and cocones over F with tip c. In particular, the identity $\mathrm{colim}_{\mathsf{I}} F \to \mathrm{colim}_{\mathsf{I}} F$ corresponds to a cocone over F with tip $\mathrm{colim}_I F$. Denote this cone by η.

Given any other cocone $\chi : F \Rightarrow \mathrm{const}_c$, there is a unique morphism $g : \mathrm{colim}_{\mathsf{I}} F \to c$ corresponding to g under ψ_c^{-1}. Writing down a naturality square

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(\mathrm{colim}_{\mathsf{I}} F, \mathrm{colim}_{\mathsf{I}} F) \arrow[r, "\psi"] \arrow[d, "g\circ -"] & \mathrm{Nat}(F, \mathrm{const}_{\mathrm{colim}_{\mathsf{I}} F}) \arrow[d, "\mathrm{const}(g)\circ -"] \\ \mathrm{Hom}_{\mathsf{C}}(\mathrm{colim}_{\mathsf{I}} F, c) \arrow[r, "\psi_c"'] & \mathrm{Nat}(F, \mathrm{const}_c) \end{tikzcd}$$

for ψ, and applying both composites to $\mathrm{id}_{\mathrm{colim}\, F}$, we find that $\mathrm{const}(g) \circ \eta = \chi$. We can thus reformulate the universal property of the colimit in terms of initial objects:

Definition 1.30. Let $F : \mathsf{I} \to \mathsf{C}$ be a diagram. The *category* $\mathsf{CoCone}(F)$ *of cocones over* F is defined by:

- The objects of $\mathsf{CoCone}(F)$ are cocones (c, χ) over F.
- A morphisms $(c, \chi) \to (d, \rho)$ are morphisms $f : c \to d$ in C such that $\mathrm{const}(f) \circ \chi = \rho$.

A *colimit cone* for F is an initial object in the category of cones.

Notice that, as with all universal properties, this defines a colimit cone up to unique isomorphism.

Proposition 1.31. *Suppose* C *admits* I*-indexed colimits. Denote by* $\mathrm{colim}_I : \mathsf{C}^{\mathsf{I}} \to \mathsf{C}$ *the colimit functor. Then* $(\mathrm{colim}_I F, \psi_{\mathrm{colim}_I F}(\mathrm{id}_{\mathrm{colim}\, F})_{)}$ *is a colimit cone for* F.

Proof. Follows from the unwinding of the universal property above. $\square$

Definition 1.32. Let $F : \mathsf{I} \to \mathsf{C}$ be a diagram. The *category* $\mathsf{Cone}(F)$ *of cones over* F is defined by:

- The objects of $\mathsf{Cone}(F)$ are cones (c, χ) over F.
- A morphisms $(c, \chi) \to (d, \rho)$ are morphisms $f : c \to d$ in C such that $\chi = \rho \circ \mathrm{const}(f)$.

A *limit cone* for F is an initial object in the category of cones.

Proposition 1.33. *Suppose* C *admits* I*-indexed limits. Denote by* $\lim_I : \mathsf{C}^{\mathsf{I}} \to \mathsf{C}$ *the limit functor. Then* $(\mathrm{colim}_I F, \psi_{\mathrm{colim}_I F}(\mathrm{id}_{\mathrm{colim}\, F}))$ *is a limit cone for* F.

Proposition 1.34. *Suppose that there exists a limit cone* (C_F, η^F) *for every diagram* $F : \mathsf{I} \to \mathsf{C}$*. Then the constant diagram functor* $\mathrm{const} : \mathsf{C} \to \mathsf{C}^{\mathsf{I}}$ *admits a right adjoint.*

Proposition 1.35. *Suppose that there exists a colimit cone (C_F, η^F) for every diagram $F : \mathsf{I} \to \mathsf{C}$. Then the constant diagram functor* const : $\mathsf{C} \to \mathsf{C}^{\mathsf{I}}$ *admits a left adjoint.*

Proof. We first define a functor $\operatorname{colim}_{\mathsf{I}} : \mathsf{C}^{\mathsf{I}} \to \mathsf{C}$ as follows. For every diagram $F : \mathsf{I} \to \mathsf{C}$, choose a colimit cone (C_F, η^F). On objects we define

$$\operatorname*{colim}_{\mathsf{I}} F := C_F$$

Given a natural transformation $\mu : F \Rightarrow G$, we can define a new cocone over F by $\eta^G \circ \mu$. Since η^F is initial in the category of cocones over F, there is thus a unique map $\hat{\mu} : C_F \to C_G$ such that $\mathrm{const}_{\hat{\mu}} \circ \eta^F = \eta^G \circ \mu$. We then define the induced morphism $\operatorname{colim}_{\mathsf{I}}(\mu)$ to be $\hat{\mu}$. The functoriality of this assignment follows immediately from the uniqueness of $\hat{\mu}$.

We now show that this is, indeed, a left adjoint. We define an isomorphism

$$\phi_{F,c} : \mathrm{Nat}(F, \mathrm{const}_c) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(\operatorname{colim}_{\mathsf{I}} F, c)$$

by sending each cone with tip c to the unique morphism of cones $f : \operatorname{colim}_{\mathsf{I}} F \to c$ (which exists and is unique because the colimit cone is initial). This is indeed an isomorphism, since for any map $f : \operatorname{colim}_{\mathsf{I}} F \to c$, the transformation $\mathrm{const}(f) \circ \eta^F$ is a cocone over F which is mapped to f by ϕ.

The final necessary check — naturality of ϕ in F and c — is left as an exercises to the interested reader. □

Example 1.36.

1. Let $\underline{n}$ denote the category with set of objects $\{1, 2, \dots, n\}$ and no non-identity morphisms. A functor $F : \underline{n} \to \mathsf{C}$ consists of a list of n objects c_i in C. The colimit of such a diagram is called a *coproduct* and is denoted by $c_1 \amalg c_2 \amalg \cdots \amalg c_n$ or $\coprod_{i \in \underline{n}} c_i$.
   - A coproduct of $X_1, \dots, X_n \in \mathsf{Set}$ is simply the disjoint union of the X_i. The universal cocone consists of the inclusions $X_i \to \coprod_{j \in \underline{n}} X_j$.
   - A coproduct in Top is similarly given by a disjoint union.
   - A coproduct of $G_1, \dots, G_n \in \mathsf{Ab}$ is the direct sum
   $$G_1 \oplus G_2 \oplus \cdots \oplus G_n.$$
   The universal cocone consists of the canonical inclusions $G_i \to \bigoplus_{j \in \underline{n}} G_j$
   - A coproduct of $G_1, \dots, G_n \in \mathsf{Grp}$ is the free product
   $$G_1 * G_2 * \cdots * G_n.$$
   The universal cocone consists of the canonical inclusions $G_i \to \bigstar_{j \in \underline{n}} G_j$.
2. Denote by Po the category which has three objects 0, 1, and 2, and only two non-identity morphisms $f : 0 \to 1$ and $g : 0 \to 2$. A functor $F : \mathsf{Pb} \to \mathsf{C}$ consists of a diagram
$$\begin{tikzcd} c_0 \arrow[r] \arrow[d] & c_1 \\ c_2 & \end{tikzcd}$$

Dual Example 1.36.

1. The *limit* of a functor $F : \underline{n} \to \mathsf{C}$ is called a product, and denoted by $F(1) \times F(2) \times \cdots \times F(n)$ or $\prod_{i \in \underline{n}} F(i)$.
   - Products in Set are given simply the Cartesian product of sets.
   - Products in Top are given by the Cartesian product equipped with the product topology.
   - A product in Grp is the direct product
   - A product in Ab is the direct product.
2. Denote by Pb the full subcategory of $\mathbb{P}(\{0,1\})$ on all the objects except $\{0,1\}$. A functor $F : \mathsf{Pb} \to \mathsf{C}$ consists of a diagram
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$
in C. The limit of such a functor is called a *pullback*, and denoted by $X \times_Z Y$.
   - In Set, the pullback of a diagram
   $$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$
   is given by the set
   $$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$
   - In Top, the pullback has the same underlying set as in Set, and is equipped with the subspace topology inherited from $X \times Y$.

in C. A colimit of such a functor is called a *pushout*, and denoted (somewhat abusively) by $c_1 \coprod_{c_0} c_2$.

- In Set, the pushout of a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Z \\ Y & \end{tikzcd}$$

  is the quotient set of $Z \amalg Y$ by the relation $f(x) \sim g(x)$ for all $x \in X$.
- In Top, the pushout of a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Z \\ Y & \end{tikzcd}$$

  is the quotient space of $Z \amalg Y$ by the same relation described above.
- In Grp, the pushout of a diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "g"'] & K \\ H & \end{tikzcd}$$

  is the amalgamated free product $K *_G H$.

3. Let $\varnothing$ denote the empty category (the category with no objects). By convention, there is a unique functor $\varnothing \to$ C. A colimit over this diagram is called an *initial object* in C. More precisely, it is an object $c \in$ C such that, for every object $d \in$ C, there is a unique morphism $c \to d$.

4. Consider the category CEq with two objects: 0 and 1, and two non-identity morphisms $f, g : 0 \to 1$. A functor $F :$ CEq is a diagram

$$\begin{tikzcd} X \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] & Y \end{tikzcd}$$

in C. A colimit over such a diagram is called a *coequalizer*.

- In Set, the coequalizer of a diagram

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

  Is the quotient of Y by the relation $f(x) \sim g(x)$ for any $x \in X$.
  Given an equivalence relation $\sim$ on a set Z, we can view this relation as a set of pairs $R \subseteq Z \times Z$. The two projections to Z give us maps

$$\begin{tikzcd} R \arrow[r, bend left, "p_1"] \arrow[r, bend right, "p_2"'] & Z \end{tikzcd}$$

  and the coequalizer of this diagram is $Z_{/\sim}$

3. Let $\varnothing$ denote the empty category. There is a unique functor $\varnothing \to$ C. A limit over this diagram is called a *final object* or *terminal object* of C.

- In Top, a coequalizer can be computed as the coequalizer in Set equipped with the quotient topology.

5 *Kan extensions*

In the previous section, we defined limits and colimits in terms of the canonical functor $T : \mathsf{I} \to [0]$. The theory developed from that starting point is surprisingly rich, but we can generalize it one step further.

Suppose that we start out with a functor

$$F : \mathsf{I} \longrightarrow \mathsf{J}.$$

As we did when we constructed (co)limits, we can use this to induce a functor

$$\begin{aligned} F^* : \mathsf{C}^{\mathsf{J}} &\longrightarrow \mathsf{C}^{\mathsf{I}} \\ \phi &\longmapsto \phi \circ F \end{aligned}$$

We can mimic our construction of (co)limits even further, and make the following

Definition 1.37. If the functor F^* admits a left adjoint, we denote said adjoint by

$$F_! : \mathsf{C}^{\mathsf{I}} \longrightarrow \mathsf{C}^{\mathsf{J}}$$

and call it the *(global) left Kan extension functor.*

Dual Definition 1.37. If the functor F^* admits a right adjoint, we denote said adjoint by

$$F_* : \mathsf{C}^{\mathsf{I}} \longrightarrow \mathsf{C}^{\mathsf{J}}$$

and call it the *(global) right Kan extension functor.*

As before, we can unwind the definition to get a sensible notion of left Kan extension

Definition 1.38. Let $F : \mathsf{I} \to \mathsf{J}$ and $G : \mathsf{I} \to \mathsf{C}$ be functors. A *left extension* of G along F consists of a functor

$$H : \mathsf{J} \longrightarrow \mathsf{C}$$

and a natural transformation $\mu : G \Rightarrow H \circ F$ as in the diagram

$$\begin{tikzcd} \mathsf{I} \arrow[r, "G"] \arrow[d, "F"'] & \mathsf{C} \\ \mathsf{J}. \arrow[ur, "H"'] \arrow[u, phantom, shift left=3, "\Swarrow"] & \end{tikzcd}$$

We will define a morphism of left extensions $(H_1, \mu) \to (H_2, \nu)$ to be a natural transformation $\gamma : H_1 \Rightarrow H_2$ such that the composite transformation

$$G \xRightarrow{\mu} H_1 \circ F \xRightarrow{\gamma \circ \mathrm{id}_F} H_2 \circ F$$

is ν.

We will call a left extension (H, μ) of G along F a *left Kan extension* if, for every other left extension (K, ν), there is a *unique* morphism of left extensions $(H, \mu) \to (K, \nu)$.[18]

Dual Definition 1.38. Let $F : \mathsf{I} \to \mathsf{J}$ and $G : \mathsf{I} \to \mathsf{C}$ be functors. A *right extension* of G along F consists of a functor

$$H : \mathsf{J} \longrightarrow \mathsf{C}$$

and a natural transformation $\mu : H \circ F \Rightarrow G$ as in the diagram

$$\begin{tikzcd} \mathsf{I} \arrow[r, "G"] \arrow[d, "F"'] & \mathsf{C} \\ \mathsf{J}. \arrow[ur, "H"'] \arrow[u, phantom, shift left=3, "\Nearrow"] & \end{tikzcd}$$

We will define a morphism of right extensions $(H_1, \mu) \to (H_2, \nu)$ to be a natural transformation $\gamma : H_1 \Rightarrow H_2$ such that the composite transformation

$$H_1 \circ F \xRightarrow{\gamma \circ \mathrm{id}_F} H_2 \circ F \xRightarrow{\nu} G$$

is μ.

We will call a right extension (H, μ) of G along F a *right Kan extension* if, for every other left extension (K, ν), there is a *unique* morphism of left extensions $(K, \nu) \to (H, \mu)$.

Note that the above universal property specifies left Kan extensions up to unique natural isomorphism. As with colimits, however, we need to check that our two definitions of left Kan extension agree.

[18] This can equivalently be formulated as saying that (H, μ) is an initial object in the category of left extensions. It can be a helpful exercise to see that in the case where $J = [0]$, this reduces to the universal property of the colimit from above.

Proposition 1.39. *Let* C *be a category, and suppose that* $F : \mathsf{I} \to \mathsf{J}$ *is a functor between small categories.*

1. *If for every functor* $G : \mathsf{I} \to \mathsf{C}$*, there is a left Kan extension of* G *along* F *in the sense of Definition 1.38, then the functor* $F^* : \mathsf{C}^\mathsf{J} \to \mathsf{C}^\mathsf{I}$ *admits a left adjoint.*
2. *Suppose the functor* $F^* : \mathsf{C}^\mathsf{J} \to \mathsf{C}^\mathsf{I}$ *admits a left adjoint* $F_!$ *with unit* $\epsilon_G : G \Rightarrow F^*(F_!G)$*. Then for every* $G : \mathsf{I} \to \mathsf{C}$*, the pair* (G, ϵ_G) *is a left Kan extension in the sense of Definition 1.38.*

Dual Proposition 1.39. Let C be a category, and suppose that $F : \mathsf{I} \to \mathsf{J}$ is a functor between small categories.

1. If for every functor $G : \mathsf{I} \to \mathsf{C}$, there is a right Kan extension of G along F in the sense of Definition 1.38, then the functor $F^* : \mathsf{C}^\mathsf{J} \to \mathsf{C}^\mathsf{I}$ admits a right adjoint.
2. Suppose the functor $F^* : \mathsf{C}^\mathsf{J} \to \mathsf{C}^\mathsf{I}$ admits a right adjoint F_* with counit $\eta_G : F^*(F_!G) \Rightarrow G$. Then for every $G : \mathsf{I} \to \mathsf{C}$, the pair (G, η_G) is a right Kan extension in the sense of Definition 1.38.

Proof. The proof is nearly identical to the proofs of Propositions 1.35 and 1.31. □

In practice, the majority of left Kan extensions one encounters will admit a simpler description in terms of colimits. The indexing categories for these colimits will, unsurprisingly, contain information about how the functor F relates the categories I and J.

Definition 1.40. Let $F : \mathsf{I} \to \mathsf{J}$ be a functor, and let $j \in \mathsf{J}$ be an object. We define the *slice category* or *overcategory* $\mathsf{I}_{/j}$ to have[19]

- Objects consisting of an object $i \in \mathsf{I}$ and a morphism $f : F(i) \to j$ in J.
- A morphism from $f : F(i) \to j$ to $g : F(k) \to j$ consisting of a morphism $h : i \to k$ in I such that the diagram

$$\begin{tikzcd} F(i) \arrow[dd, "F(h)"'] \arrow[dr, "f"] & \\ & j \\ F(k) \arrow[ur, "g"'] & \end{tikzcd}$$

  commutes.

The use of the notation $\mathsf{I}_{/j}$ is somewhat abusive, as it makes no reference to F. However, in practice, this rarely creates ambiguities.

Each category $\mathsf{I}_{/j}$ admits a canonical forgetful functor, $\mathsf{I}_{/j} \to \mathsf{I}$, which only remembers the objects/morphisms in I.

[19] There is an important special case of this construction, where F is simply the identity on I. In this case, each category $\mathsf{I}_{/i}$ has a terminal object id_i. This means, in particular, that the colimit of a diagram $H : \mathsf{I}_{/i} \to \mathsf{C}$ is simply the value $H(\mathrm{id}_i)$.

Dual Definition 1.40. Let $F : \mathsf{I} \to \mathsf{J}$ be a functor, and let $j \in \mathsf{J}$ be an object. We define the *slice category* or *undercategory* $\mathsf{I}_{j/}$ to have

- Objects consisting of an object $i \in \mathsf{I}$ and a morphism $f : j \to F(i)$ in J.
- A morphism from $f : j \to F(i)$ to $g : j \to F(k)$ consisting of a morphism $h : i \to k$ in I such that the diagram

$$\begin{tikzcd} F(i) \arrow[dd, "F(h)"'] & \\ & j \arrow[ul, "f"'] \arrow[dl, "g"] \\ F(k) & \end{tikzcd}$$

  commutes.

Each category $\mathsf{I}_{j/}$ admits a canonical forgetful functor, $\mathsf{I}_{j/} \to \mathsf{I}$, which only remembers the objects/morphisms in I.

Construction 1.41. Suppose that $F : \mathsf{I} \to \mathsf{J}$ and $G : \mathsf{I} \to \mathsf{C}$ are functors, and let (H, μ) be a left extension of G along F. We will denote by $G|_{\mathsf{I}_{/j}}$ the composite

$$\mathsf{I}_{/j} \longrightarrow \mathsf{I} \xrightarrow{G} \mathsf{C}$$

with the forgetful functor.

Our aim is to use (H, μ) to construct a cocone over $G|_{\mathsf{I}_{/j}}$, with tip $H(j)$. We do this as follows.

Given an object $f : F(i) \to j$ in $\mathsf{I}_{/j}$, the functoriality of H provides a morphism

$$H(f) : H(F(i)) \to H(j)$$

in C. Additionally, the natural transformation $\mu : G \Rightarrow H \circ F$ provides a component morphism

$$\mu_i : G(i) \to H(F(i))$$

in C. We define $\xi^j_f := H(f) \circ \mu_i : G(i) \to H(j)$.

We now need to check that ξ^j does, indeed, define a cocone over $G|_{\mathsf{I}_{/j}}$. Suppose we are given a morphism

$$\begin{tikzcd} F(i) \arrow[d, "F(h)"'] \arrow[dr, "f"] & \\ F(k) \arrow[r, "g"'] & j \end{tikzcd}$$

in $\mathsf{I}_{/j}$. Then we can write the following commutative diagram

$$\begin{tikzcd} G(i) \arrow[r, "\mu_i"] \arrow[d, "G(h)"'] & H(F(i)) \arrow[d, "H(F(h))"'] \arrow[dr, "H(f)"] & \\ G(k) \arrow[r, "\mu_k"'] & H(F(k)) \arrow[r, "H(g)"'] & H(j) \end{tikzcd}$$

Where the left-hand square commutes by the naturality of μ, and the right-hand triangle by the functoriality of H. Since the upper and lower composites are ξ^j_f and ξ^j_g, respectively, this diagram shows that ξ^j defines a cocone over $G|_{\mathsf{I}_{/j}}$.

Proposition 1.42. *Let $F : \mathsf{I} \to \mathsf{J}$ and $G : \mathsf{I} \to \mathsf{C}$ be functors.*

1. *Suppose that (H, μ) is a left extension of G along F, such that, for each $j \in \mathsf{J}$, the cocone ξ^j is a colimit cocone for $G|_{\mathsf{I}_{/j}}$. Then (H, μ) is a left Kan extension.*

2. *Suppose that, for every $j \in \mathsf{J}$ the diagram $G|_{\mathsf{I}_{/j}}$ admits a colimit. Then there is a left Kan extension (H, μ) of G along F such that, for every $j \in \mathsf{J}$, $H(j) \cong \operatorname{colim}_{\mathsf{I}_{/j}} G|_{\mathsf{I}_{/j}}$.*

Dual Proposition 1.42. Let $F : \mathsf{I} \to \mathsf{J}$ and $G : \mathsf{I} \to \mathsf{C}$ be functors.

1. Suppose that (H, μ) is a right extension of G along F, such that, for each $j \in \mathsf{J}$, the cocone ξ^j is a colimit cocone for $G|_{\mathsf{I}_{j/}}$. Then (H, μ) is a right Kan extension.

2. Suppose that, for every $j \in \mathsf{J}$ the diagram $G|_{\mathsf{I}_{j/}}$ admits a colimit. Then there is a right Kan extension (H, μ) of G along F such that, for every $j \in \mathsf{J}$, $H(j) \cong \operatorname{colim}_{\mathsf{I}_{j/}} G|_{\mathsf{I}_{j/}}$.

Proof. We begin with the proof of (1). Suppose we are given another left extension (K, ν) of G along F. For each $j \in \mathsf{J}$ (K, ν) will give rise to a cocone ψ^j with tip $K(j)$ over $G|_{\mathsf{I}_{/j}}$. Since ξ^j is a colimit cocone, this means that there is a unique morphism $\tau_j : H(j) \to K(j)$ which defines a morphism of cocones from ξ^j to ψ^j.

We now wish to show that τ defines a natural transformation $\tau : H \Rightarrow K$. Suppose that $g : j \to k$ is a morphism in J. The morphism g induces a functor

$$L_g : \mathsf{I}_{/j} \xrightarrow{g \circ -} \mathsf{I}_{/k}$$

which composes each object with g. We can then note that $G|_{\mathsf{I}_{/k}} \circ L_g = G|_{\mathsf{I}_{/j}}$.[20] Moreover, by construction, the diagrams

$$\begin{tikzcd} \mathrm{const}_{H(j)} \arrow[r, Rightarrow, "\mathrm{const}(H(g))"] & \mathrm{const}_{H(k)} \arrow[r, Rightarrow, "\mathrm{const}(\tau_k)"] & \mathrm{const}_{K(k)} \\ G|_{\mathsf{I}_{/j}} \arrow[u, Rightarrow, "\xi^j"] \arrow[ur, Rightarrow, "\xi^k \circ L_g"] \arrow[urr, Rightarrow, "\psi^k \circ L_g"'] & & \end{tikzcd}$$

[20] Precisely because the diagram

$$\begin{tikzcd} \mathsf{I}_{/j} \arrow[d, "L_g"'] \arrow[dr] & \\ \mathsf{I}_{/k} \arrow[r] & \mathsf{I} \end{tikzcd}$$

commutes.

and

$$\begin{tikzcd}
\mathrm{const}_{H(j)} \arrow[r, Rightarrow, "\mathrm{const}(\tau_j)"] & \mathrm{const}_{K(j)} \arrow[r, Rightarrow, "\mathrm{const}(K(g))"] & \mathrm{const}_{K(k)} \\
G|_{\mathsf{I}_{/j}} \arrow[u, Rightarrow, "\xi^j"] \arrow[ur, Rightarrow, "\psi^j"] \arrow[urr, Rightarrow, "\psi^k \circ L_g"'] & &
\end{tikzcd}$$

both commute. Consequently, by the universal property of colimit cocone ξ^j, we see that $K(g) \circ \tau_j = \tau_k \circ H(g)$, which proves the naturality of τ.

The penultimate step is to see that τ defines a morphism of extensions. To this end, given an object $i \in \mathsf{I}$ we must see that the diagram

$$\begin{tikzcd}
H \circ F \arrow[rr, Rightarrow, "\tau \circ \mathrm{id}_F"] & & K \circ F \\
& G \arrow[ul, Rightarrow, "\mu"] \arrow[ur, Rightarrow, "\nu"'] &
\end{tikzcd}$$

commutes. This amounts to seeing that, for every $i \in \mathsf{I}$, we have $\tau_{F(i)} \circ \mu_i = \nu_i$. Applying the definition of τ to the object $\mathrm{id}_{F(i)} : F(i) \to F(i)$ in $\mathsf{I}_{/F(i)}$, we obtain the desired equality.

Finally, we are left to check that there is no other morphism of extensions $\chi : H \Rightarrow K$. Suppose we have such a χ. Then, in particular, for each $j \in \mathsf{J}$, the morphism $\chi_j : H(j) \to K(j)$ will define a morphism of cocones from ξ^j to ψ^j. Since ξ^j is a colimit cocone, this morphism is uniquely, so $\chi = \tau$.

We now sketch the proof of part (2). By part (1), we need only construct a left extension (H, μ) such that each associated cone over $G|_{\mathsf{I}_{/j}}$ is a colimit cocone. We first define the functor H. Fix a colimit cocone (c_j, η^j) for each $G|_{\mathsf{I}_{/j}}$.

- Given an object $j \in \mathsf{J}$, we define $H(j) = c_j$.
- Given a morphism $g : j \to k$ in J, we can pull back the cocone $\eta^k : G|_{\mathsf{I}_{/k}} \Rightarrow \mathrm{const}\, c_k$ via $L_g : \mathsf{I}_{/j} \to \mathsf{I}_{/k}$ to get a cocone $\eta^k \circ L_g : G|_{\mathsf{I}_{/j}} \Rightarrow \mathrm{const}\, c_k$. The universal property of the colimi then yields a *unique* morphism $H(g) : c_k \to c_j$ which defines a morphism of cocones $\eta^j \Rightarrow \eta^k \circ L_g$.
- Functoriality follows from the uniqueness of the transformation defined above.

We leave it as an exercise to construct μ such that (H, μ) is a left extension. $\square$

Corollary 1.43. *Let* C *be a cocomplete category, and* $F : \mathsf{I} \to \mathsf{J}$ *a functor of small categories. Then the functor*

$$F^* : \mathsf{C}^{\mathsf{J}} \longrightarrow \mathsf{C}^{\mathsf{I}}$$

admits a left adjoint.

Dual Corollary 1.43. Let C be a complete category, and $F : \mathsf{I} \to \mathsf{J}$ a functor of small categories. Then the functor

$$F^* : \mathsf{C}^{\mathsf{J}} \longrightarrow \mathsf{C}^{\mathsf{I}}$$

admits a right adjoint.

Remark 1.44. Since adjunctions compose to give adjunctions, we can often compute Kan extensions in two stages. If we factor $F : \mathsf{I} \to \mathsf{J}$ as

$$\mathsf{I} \xrightarrow{\ \phi\ } \mathsf{K} \xrightarrow{\ \psi\ } \mathsf{J}$$

we can compute, e.g., $F_! G$ by first taking $\phi_! G$, and then computing $\psi_!(\phi_! G)$.

Example 1.45.

1. Consider the category $\mathsf{J} := \mathbb{P}(\{0,1\})$, and let I be the full subcategory of J on all the objects other that $\{0,1\}$. Let $F : \mathsf{I} \to \mathsf{J}$ be the inclusion. A functor $G : \mathsf{I} \to \mathsf{C}$ is a diagram

$$\begin{tikzcd} G(\varnothing) \arrow[r] \arrow[d] & G(\{1\}) \\ G(\{0\}) & \end{tikzcd}$$

   A left Kan extension $F_!G : \mathsf{J} \to \mathsf{C}$ is a pushout square

$$\begin{tikzcd} G(\varnothing) \arrow[r] \arrow[d] & G(\{1\}) \arrow[d] \\ G(\{0\}) \arrow[r] & G(\{0\}) \coprod_{G(\varnothing)} G(\{1\}). \end{tikzcd}$$

2. Consider the category $\mathsf{B}\mathbb{N}$ with a single object $*_{\mathbb{N}}$ and, for every $n \in \mathbb{N}$, a morphism $f_n : *_{\mathbb{N}} \to *_{\mathbb{N}}$, such that $f_n \circ f_k = f_{n+k}$, and $f_0 = \mathrm{id}_{*_{\mathbb{N}}}$. There is a canonical functor $\mathsf{B}\mathbb{N} \to \mathsf{B}\mathbb{Z}$. A functor $X : \mathsf{B}\mathbb{N} \to \mathsf{Set}$ consists of an object $X \in \mathsf{Set}$, together with a map $f : X \to X$, and the composites f^n for any n.

   TODO: finish writing up.

3. Let $G \subset H$ be groups, $\phi : G \to H$ a the inclusion, and let $X : \mathsf{B}G \to \mathsf{Set}$ be a G-set. The homomorphism ϕ gives rise to a functor

$$\Phi : \mathsf{B}G \to \mathsf{B}H.$$

   We aim to compute the left Kan extension $\Phi_!X$. Since there is only one object — $*_H$ in $\mathsf{B}H$, we can compute the value of $\mathsf{B}H$ on objects by computing the colimit of $X|_{\mathsf{B}G_{/*_H}}$.

   We first unpack the diagram category $\mathsf{B}G_{/*_H}$. An object will be an element of H, i.e. a morphism $h : F(*_G) = *_H \to *_H$. A morphism from h_1 to h_2 will be an element $g \in G$ such that the diagram

$$\begin{tikzcd} *_H \arrow[rr, "\phi(g)"] \arrow[dr, "h_1"'] & & *_H \arrow[dl, "h_2"] \\ & *_H & \end{tikzcd}$$

   i.e., it will be an element $g \in G$ such that $h_2\phi(g) = h_1$.

   Each isomorphism class in $\mathsf{B}G_{/*_H}$ is thus a left coset hG of G in H. Given any two elements $h_1, h_2 \in \mathsf{B}G_{/*_H}$ in the same equivalence class, we can notice that there is precisely one morphism $h_1 \to h_2$ in $\mathsf{B}G_{/*_H}$. Thus, each connected component of $\mathsf{B}G_{/*_H}$ consists only of terminal elements. We therefore see that

$$\Phi_!X(*_H) := \operatorname*{colim}_{\mathsf{B}G_{/*_H}} X = \coprod_{G/H} X$$

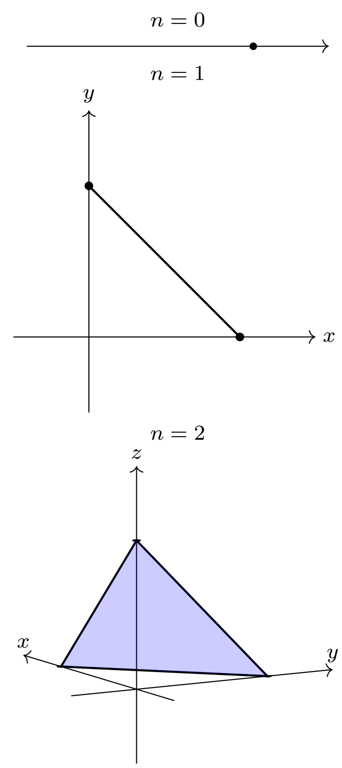
If we fix a set of coset representatives $[h_i]$, for $i = 1, \ldots, [H : G]$, we can describe the induced H-action on $\Phi_! X(*_H)$ explicitly: for each i, we can write $hh_i = h_{\sigma_h(i)} g_h^i$. The element h then acts by permuting the components of the disjoint union by σ_i, and acting by the map $g_h^i \cdot (-) : X_i \to X_j$.

2

Simplicial sets

If we take the Homotopy Hypothesis from the overture seriously, we will need to come up with a model that is flexible enough to include topological spaces and 1-categories. The source of such a model is a familiar way of building topological spaces: simplices.

We can easily draw the first few standard n-simplices as subsets of $\mathbb{R}^{n+1}$:



The standard 3-simplex can be identified with a tetrahedron, as in the main text. However, for obvious reasons, we cannot draw it as a subset of $\mathbb{R}^4$.

1 *Topological spaces and simplices*

We will aim to build up our topological spaces out of simplices. Our building blocks on the topological side are the *geometric n-simplices*.

Definition 2.1. For any $n \geq 0$, the *geometrical n-simplex*, denoted $|\Delta^n|$, is the set (together with the subspace topology)

$$|\Delta^n| = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\} \subset \mathbb{R}^{n+1}.$$

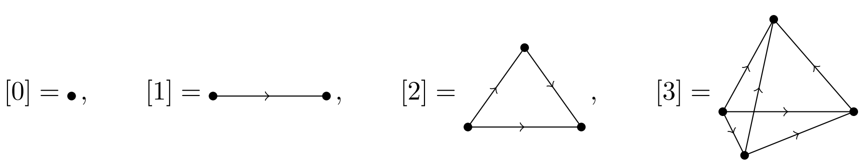
However, we won't work directly with the topological spaces in question. Rather, we want to define *combinatorial data* which tells us how to assemble a given space out of simplices. To this end, we define a category whose morphisms describe the combinatorics of the standard n-simplices:

Definition 2.2. Denote by Δ the category whose objects are linearly ordered sets

$$[n] = \{0, \dots, n\}$$

and whose morphisms $[n] \to [m]$ are weakly monotonic maps; i.e. maps such that if $i \leq j$ in $[n]$, then $f(i) \leq f(j)$ in $[m]$.

The objects $[n]$ of Δ are to be interpreted as simplices with ordered vertices. For example,



Before we can really relate the simplex category Δ to our geometric simplices, we need to understand its combinatorics a little better. In particular, we need to define some special morphisms.

Definition 2.3. We define two special classes of morphisms in Δ:[1]

- The *face maps* are the maps

$$\begin{aligned} \delta_i^n : [n-1] &\longrightarrow [n] \\ k &\longmapsto \begin{cases} k & k < i \\ k+1 & k \geq i \end{cases} \end{aligned}$$

  for any $n \geq 1$ and and any $0 \leq i \leq n$.

- The *degeneracy maps* are the maps

$$\begin{aligned} \sigma_i^n : [n+1] &\longrightarrow [n] \\ k &\longmapsto \begin{cases} k & k \leq i \\ k-1 & k \geq i \end{cases} \end{aligned}$$

  for any $n \geq 0$ and any $0 \leq i \leq n$.

The reason these are special is that they are, in some sense, all the information you need to define functors out of the simplex category. A functor

$$F : \Delta \longrightarrow \mathsf{C}$$

defines objects $F([n])$ and morphisms $F(\delta_i^n)$ and $F(\sigma_i^n)$. It turns out that if you only remember this data, you can get the functor F back. More generally, if you can define objects $F([n])$ and morphisms $F(\delta_i^n)$ and $F(\sigma_i^n)$ satisfying certain identities — the *simplicial identities* — you can construct a functor F as above.

Definition 2.4. The *simplicial identities* are the relations:

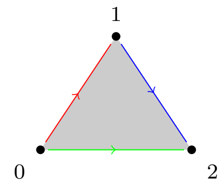
$$\begin{aligned} \delta_i^{n+1} \circ \delta_j^n &= \delta_{j+1}^{n+1} \circ \delta_i^n & i \leq j \\ \sigma_i^n \circ \sigma_j^{n+1} &= \sigma_j^n \circ \sigma_{i+1}^{n+1} & j \leq i \\ \sigma_i^n \circ \delta_j^{n+1} &= \begin{cases} \delta_j^n \circ \sigma_{i-1}^{n-1} & j < i \\ \mathrm{id}_{[n]} & j = i, i+1 \\ \delta_{j-1}^n \circ \sigma_i^{n-1} & j > i+1 \end{cases} \end{aligned}$$

Exercise 9. Show that the face and degeneracy maps satisfy the simplicial identities.

Proposition 2.5. *Let* C *be a category. The following data uniquely defines a functor* $F : \Delta \longrightarrow \mathsf{C}$.

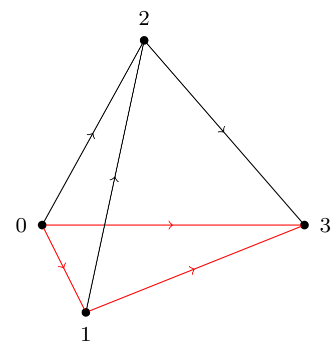
- *Objects* $F([n]) \in \mathsf{C}$.

[1] We draw the face maps δ_i^n for $n = 2, 3$. Let us first start with $[2]$:



The map δ_2^2 is the inclusion of the red 1-simplex, the map δ_1^2 is the inclusion of the green 1-simplex, and the map δ_0^2 is the inclusion of the red 1-simplex.

Now consider $[3]$. We will not draw all of the faces of $[3]$, instead highlighting in red the image of δ_2^3:



- *Morphisms which satisfy the simplicial identities:*
  - *for every face map δ_i^n, a morphism $F(\delta_i^n)$ from $F([n-1])$ to $F([n])$;*
  - *for every degeneracy map σ_i^n, a morphism $F(\sigma_i^n)$ from $F([n+1])$ to $F([n])$.*

Proof. This is tedious, but straightforward. □

Now that we have some basic idea of the combinatorics of Δ, let's try to connect Δ to topology. To get a better sense of the connection between the simplex category and our geometric n-simplices, we define a functor.

Construction 2.6. The *geometric realization functor* is a functor

$$\begin{aligned} |-|:\ \Delta &\longrightarrow \mathsf{Top} \\ [n] &\longmapsto |\Delta^n|. \end{aligned}$$

We define this functor on morphisms as follows

- Suppose $\delta_i^n : [n-1] \to [n]$ is a face map. We define its image in Top by

$$\begin{aligned} |\delta_i^n|:\qquad |\Delta^{n-1}| &\longrightarrow |\Delta^n| \\ (t_0, t_1, \ldots, t_{n-1}) &\longmapsto (t_0, \ldots, \underbrace{0}_{i^{th}}, \ldots, t_{n-1}) \end{aligned}$$

- Suppose $\sigma_i^n : [n+1] \to [n]$ is a degeneracy map. We define its image in Top by

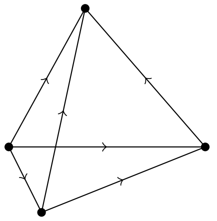
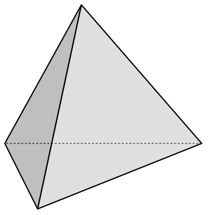
$$\begin{aligned} |\sigma_i^n|:\qquad |\Delta^{n+1}| &\longrightarrow |\Delta^n| \\ (t_0, t_1, \ldots, t_{n+1}) &\longmapsto (t_0, \cdots, t_{i-1}, t_i + t_{i+1}, \ldots, t_{n+1}) \end{aligned}$$

More generally, this gives rise to the functor which sends $\phi : [n] \to [m]$ to the map

$$\begin{aligned} |\phi|:\qquad |\Delta^n| &\longrightarrow |\Delta^m| \\ (t_0, t_1, \ldots, t_n) &\longmapsto (a_0, a_1, \ldots, a_m) \end{aligned}$$

where $a_j := \sum_{i \in \phi^{-1}(j)} t_i$.

That is, ρ assigns to each combinatorial n-simplex its topological counterpart.

$\rho :$  $\longmapsto$ 

Our aim going forward will be to work with topological spaces and categoires in terms of the combinatorics of simplices. In Algebraic Topology, you have likely already seen simplicial complexes — one way of describing topological spaces in terms of simplices. We will be using another, more flexible notion, that of *simplicial sets*.

The idea is the following: We want a collection of simplices in every dimension, together with compatibility data which tells us how to glue them together.

Definition 2.7. A **simplicial set** is a functor

$$X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}.$$

A simplicial set X consists of:

- For every $n \geq 0$, a set X_n of 0-**simplices**.
- **Face and degeneracy maps**

$$d_i^n : X_n \to X_{n-1}$$

  and

$$s_i^n : X_n \to X_{n+1}$$

  satisfying the duals of the simplicial identities of Definition 2.4.

We denote by Set_Δ the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$. A morphism $X \to Y$ in Set_Δ consist of a collection $f_n : X_n \to Y_n$ which commute with the face and degeneracy maps.

2 *Understanding simplicial sets*

Before we unwind what simplicial sets really are, lets consider something simpler: simplicial complexes. Actually, let's make things even simpler, and consider *finite* simplicial complexes.

Definition 2.8. A **finite simplicial complex** K consists of a set V_K — the **vertices** of K — and a subset $\mathrm{Sim}(K) \subset \mathbb{P}(V_K) \setminus \varnothing$ of the power set of V_K — the **simplices** of K. The set $\mathrm{Sim}(K)$ of simplices of K is required to satisfy the conditions

1. If $S \subset V_K$ is in $\mathrm{Sim}(K)$, and $\varnothing \neq U \subset S$, then $U \in \mathrm{Sim}(K)$.
2. Every singleton lies in $\mathrm{Sim}(K)$.

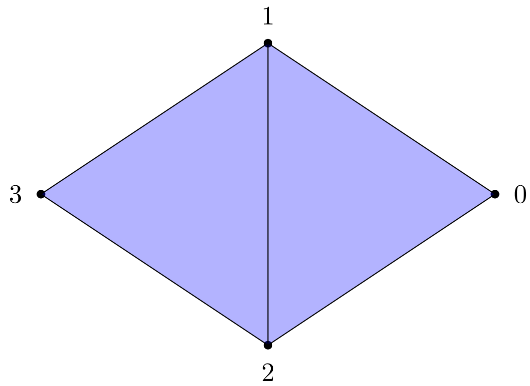
A simplicial complex represents a way of gluing together simplices to get a topological space. We want to think of each $S \in \mathrm{Sim}(K)$ with $|S| = n + 1$ as an n-simplex. If $U \subset S$ and $U \subset T$, then we identify the faces of S and T which correspond to U. Intuitively, this looks as follows.

Example 2.9.

1. Consider the complex K with vertices $V_K = \{0, 1, 2, 3\}$ and simplices

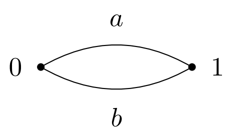
$$\mathrm{Sim}(K) := \{\{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{1, 3\}, \{2, 3\}, \{0, 1, 2\}, \{1, 2, 3\}\}$$

   There are precisely two 2-simplices, and they share a single face: the 1-simplex $\{1, 2\}$. We picture this simplicial set as



2. Define a complex K by setting $V_K := [n]$ and $\mathrm{Sim}(K) = \mathbb{P}([n]) \setminus \varnothing$. Every simplex of this complex can be viewed as a face of the n-simplex $\{0, 1, 2, \ldots, n\} = [n]$. This means we can simply visualize this complex as the n-simplex $|\Delta^n|$.

The problem with the framework of simplicial complexes is that it is quite rigid. For example, we might think of the space X:

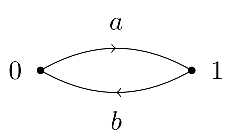


as being glued out of simplices — two 0-simplices labeled 0 and 1, and two 1-simplices labeled a and b. However, this is *not* a simplicial complex! In a simplicial complex, a simplex is uniquely determined by its set of vertices, but both one simplices in the above example have the same vertices.

To fix this, we can generalize a bit. Instead of having a simplex be a subset of the set of vertices, we can say that, for any n, we have a set X_n, which the set of n-simplices of X. We might then try to describe the simplicial set we drew above as

$$\begin{aligned} X_0 &= \{0, 1\} \\ X_1 &= \{a, b\} \\ X_n &= \varnothing \qquad n > 1 \end{aligned}$$

The problem is that we now have to remember which vertices are faces of a. To do this, it helps us to think of our n simplices as having an order. For instance, we can think of a and b as *arrows*



We can then remember some faces by defining a function

$$\begin{aligned} d_0^1 : X_1 &\longrightarrow X_0 \\ a &\longmapsto 1 \\ b &\longmapsto 0 \end{aligned}$$

which tells us the face obtained from each 1-simplex by forgetting the earliest vertex. We can remember the rest of the faces by defining a function

$$\begin{aligned} d_1^1 : X_1 &\longrightarrow X_0 \\ a &\longmapsto 0 \\ b &\longmapsto 1 \end{aligned}$$

which tells us the face obtained from each 1-simplex by forgetting the last vertex. From this data, we can then reconstruct the original picture.

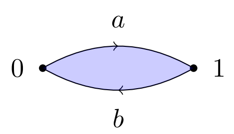
If these data look familiar, there's a good reason — this should look a bit like a simplicial set[2]. For each object $[n] \in \Delta$, we have defined a set X_n of n-simplices, and for each *face map* $\delta_i^n : [n-1] \to [n]$, we have defined a face map

$$d_i^n : X_n \to X_{n-1}$$

which takes an n-simplex, and spits out its i^{th} face.

[2] Actually, what we've defined is what is sometimes called a *semi-simplicial set*, but since we won't be using this notion in future, we omit further discussion of this concept.

But this is only really 'half' of the data of a simplicial set. What about the degeneracies? It turns out there is another problem with our current framework. Suppose we take our simplicial complex X, and try to 'fill' the gap between a and b with a 2-simplex. We can easily draw what we would expect to get out of this:

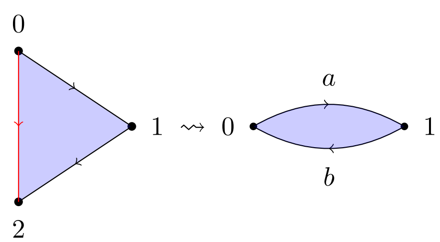


but there's a problem — a and b don't form a triangle!

The solution to this problem is subtle — we need to define a 'third leg' of the triangle, so we can glue a 2-simplex into the gap. To do this, we can define a new 1-simplex w_0, which we think of as being collapsed into the point 0. We can then define a 2-simplex σ whose faces are

$$d_0^2(\sigma) = b, \quad d_1^2(\sigma) = w_0, \quad d_2^2(\sigma) = a.$$

We can then think of the filled complex as being what we obtain when we take a 2-simplex and collapse its 1st face down to a point:



But how do we remember that w_0 isn't a normal one simplex? We need to encode the data that w_0 gets collapsed down to a point.

The solution to this problem is provided precisely by the degeneracy maps in a simplicial set. In a simplicial set, we get maps

$$s_i^n : X_n \to X_{n+1}$$

which send each simplex σ in X_n to a simplex $s_i^n(\sigma) \in X_{n+1}$, which is **degenerate** — we view this (geometrically) as being collapsed down to σ by collapsing the i^{th} and $(i+1)^{\text{st}}$ vertices to the same point, and extending this map linearly to the whole simplex.

There is a trade-off implicit in extending to simplicial sets. We gain flexibility in the ways we can glue things together, but we lose any hope of finiteness. Previously, with simplicial complexes, we had a finite set of simplices to work with. With simplicial sets, the only non-infinite example is the empty simplicial set. For instance, the simplicial set which consists of a single point x has simplices

$$X_0 = \{x\}, \quad X_1 = \{s_0^0(x)\}, \quad X_2 = \{s_0^1(s_0^0(x))\}, \ldots$$

(note that, by the simplicial identities $s_1^1(s_0^0(x)) = s_0^1(s_0^0(x))$).

Definition 2.10. Let X be a simplicial set. By the simplicial identities, all of the degeneracy maps are injective. We call a simplex $\sigma \in X_n$ **degenerate** if it is in the image of a degeneracy map. We call all other simplices **non-degenerate**.

The good news is that we don't really need to draw the degenerate simplices — all of them get collapsed to non-degenerate ones in our pictures. The other good news is that we can still manufacture simplicial sets from power sets.

Example 2.11 (Key examples)**.** Let $k \geq 0$ $\mathcal{K} \subset \mathbb{P}([k])$ be a subset of its power set. We can define a simplicial set $\Delta^{\mathcal{K}}$ as follows:

- The set $\left(\Delta^{\mathcal{K}}\right)_n$ of n-simplices of $\Delta^{\mathcal{K}}$ is the set of monotone maps $f : [n] \to [k]$ such that the image of f is contained in U for some $U \in \mathcal{K}$.
- Given a map $\psi : [m] \to [n]$, we define the corresponding map

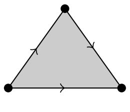
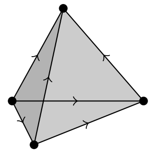
$$\begin{aligned} \psi^* : \left(\Delta^{\mathcal{K}}\right)_n &\longrightarrow \left(\Delta^{\mathcal{K}}\right)_m \\ f &\longmapsto f \circ \psi \end{aligned}$$

it is relatively easy to check that this does, indeed, define a functor $\Delta^{\text{op}} \to \text{Set}$ — i.e., a simplicial set.

We note that the non-degenerate simplices of $\Delta^{\mathcal{K}}$ correspond precisely to those $W \subset [k]$ such that $W \subset U$ for some $U \in \mathcal{K}$. This allows us to draw these simplicial sets (at least in low dimensions).

Three types of simplicial sets arising in this way are particularly useful to us. We now consider $S = \{0, 1, 2, \ldots, n\}$, without any order involved.

1. Suppose that we start with the whole power set, i.e., $\mathcal{K} = \mathbb{P}(\{0, 1, \ldots, k\})$. We get a simplicial set we call the **standard (combinatorial) k-simplex**, and denote by Δ^k. Every non-degenerate simplex of Δ^k is a face of the k-simplex $\text{id}_{[k]} : [k] \to [k]$. We can thus picture the simplicial set Δ^n as standard (ordered) simplices:

$\Delta^1 = \bullet$, $\quad \Delta^1 = \bullet \longrightarrow \bullet$, $\quad \Delta^2 =$  , $\quad \Delta^3 =$ 

2. Suppose instead, we take the set $\mathcal{K} \subset \mathbb{P}([k])$ defined by

$$\mathcal{K} = \{U \subsetneq [k]\}$$

   We now get much the same pictures as before, but we are missing the biggest simplex. The simplicial set we obtain is called the **boundary of** Δ^k, and is denoted $\partial\Delta^k$.

3. Finally, fix $i \in [k]$. Consider the subset $\mathcal{K} \subset \mathbb{P}([k])$ consisting of all subsets $U \subset [k]$ with $U \neq [k]$ and $U \neq [k] \setminus \{i\}$. We obtain a simplicial set which, geometrically, should look like Δ^k, with the i^{th} face and interior scooped out. We call this the i^{th} **horn of** Δ^k, and denote it by Λ^k_i.

   We can draw all of the horns for $k = 2$:

$$\Lambda^2_0 = \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}, \quad \Lambda^2_1 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}, \quad \Lambda^2_2 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

3 *Categories and simplicial sets*

Before moving on to the more familiar setting of spaces built out of simplices, let's consider how we might represent categories as a special kind of simplicial set.

Definition 2.12. Let Cat denote the 1-category of small 1-categories. We define a functor

$$\tau : \Delta \longrightarrow \mathsf{Cat} \qquad [n] \longmapsto \tau^n$$

Where τ^n is the category with objects $0, 1, \ldots, n$, and

$$\mathrm{Hom}_{\tau^n}(i, j) = \begin{cases} * & i \leq j \\ \emptyset & i > j. \end{cases}$$

Remark 2.13. We can visualize the category τ^n as

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

this is the category associated to the poset $[n]$.

The functor τ allows us to construct a simplicial set out of any category, and (in a much more convoluted fashion) a category out of any simplicial set. This construction falls into a general framework of "nerve and realization", which we will use repeatedly throughout the source. We will carefully work through the construction for τ, and in future make similar constructions with more minimal discussion.

Construction 2.14. Our first step is to consider the Yoneda embedding on Δ:

$$\begin{aligned} \text{よ} : \Delta &\longrightarrow \mathsf{Set}_\Delta \\ [n] &\longmapsto \mathrm{Hom}_\Delta(-, [n]). \end{aligned}$$

This functor actually relates quite simply to a previous construction: $\mathcal{Y}([n])$ is simply the standard combinatorial n-simplex Δ^n! Note that by Yoneda's lemma, there is a natural isomorphism $\mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^k, X) \cong X_n$ for any simplicial set $X \in \mathsf{Set}_\Delta$.

We can then consider the left Kan extension of the functor $\tau : \Delta \to \mathsf{Cat}$ along the functor $\mathcal{Y}$.[3] This will give us a functor, called the **fundamental category functor**

[3] This kind of Kan extension is sometimes called a *Yoneda extension.*

$$\begin{aligned} \tau : \mathsf{Set}_\Delta &\longrightarrow \mathsf{Cat} \\ X &\longmapsto \mathrm{colim}_{\Delta^k \to X}\, \Delta^k \end{aligned}$$

Of equal importance, the functor τ will have a right adjoint, the **nerve functor** given by

$$\begin{aligned} N : \mathsf{Cat} &\longrightarrow \mathsf{Set}_\Delta \\ \mathsf{C} &\longmapsto \mathrm{Hom}_{\mathsf{Cat}}(\tau^{(-)}, \mathsf{C}) \end{aligned}$$

To see that these two functors are adjoint, let $\mathsf{C} \in \mathsf{Cat}$ and $X \in \mathsf{Set}_\Delta$. We can then can write down a sequence of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Set}_\Delta}(X, N(\mathsf{C})) &\cong \mathrm{Hom}_{\mathsf{Set}_\Delta}(\operatorname*{colim}_{\Delta^k \to X} \Delta^k, N(\mathsf{C})) \\ &\cong \lim_{\Delta^k \to X} \mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^k, N(\mathsf{C})) \\ &\cong \lim_{\Delta^k \to X} \mathrm{Hom}_{\mathsf{Cat}}(\tau^k, \mathsf{C}) \\ &\cong \mathrm{Hom}_{\mathsf{Cat}}(\lim_{\Delta^k \to X} \tau^k, \mathsf{C}) \\ &= \mathrm{Hom}_{\mathsf{Cat}}(\tau(X), \mathsf{C}) \end{aligned}$$

We can also note that the nerve functor $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ will be fully faithful. This is a direct consequence of the fact that the canonical morphism

$$\operatorname*{colim}_{\tau^k \to \mathsf{C}} \tau^k \to \mathsf{C}$$

is an isomorphism[4]. This tells us that the counit $\epsilon : \tau \circ N \Rightarrow \mathrm{id}_{\mathsf{Cat}}$ is a natural isomorphism, and so N is fully faithful.

[4] If you are unconvinced, try proving this!

Of these two functors, the nerve N is the more useful and easier to understand. Let's try to unpack the data encoded in the nerve $N(\mathsf{C})$ of a category C:

- The 0-simplices are functors $\tau^0 \to \mathsf{C}$, i.e. objects of C.
- A 1-simplex of $N(\mathsf{C})$ is a functor $\tau^1 \to \mathsf{C}$. This yields two objects x_0 and x_1 in C, and a morphism $f : x_0 \to x_1$ in C: $x_0 \xrightarrow{f} x_1$.
- A 2-simplex of $N(\mathsf{C})$ is a functor $\tau^2 \to \mathsf{C}$. In principal, this is encoded of the data of three objects x_0, x_1, and x_2 of C, and three morphisms: $f_{01} : x_0 \to x_1$, $f_{12} : x_1 \to x_2$ and $f_{02} : x_0 \to x_2$. We could draw this as a diagram in C that looks like a 2-simplex:

$$\begin{tikzcd} & x_1 \arrow[dr, "f_{12}"] & \\ x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & x_2 \end{tikzcd}$$

However, the condition that these data form a functor $\tau^2 \to \mathsf{C}$ means that this diagram must commute, i.e. we must have $f_{02} = f_{12} \circ f_{01}$. Consequently, a 2-simplex in $N(\mathsf{C})$ uniquely corresponds to the data of two composable morphisms in C:

$$x_0 \xrightarrow[f_{01}]{} x_1 \xrightarrow{f_{12}} x_2$$

More generally, an n-simplex corresponds uniquely to a string of n composable morphisms in C:

$$x_0 \xrightarrow[f_{01}]{} x_1 \xrightarrow{f_{12}} x_2 \longrightarrow \cdots \xrightarrow{f_{(n-1)n}} x_n$$

Our aim is now to characterize the simplicial sets which are the nerves of categories. Recall the simplicial *horns* Λ^n_i from Example 2.11 (2).

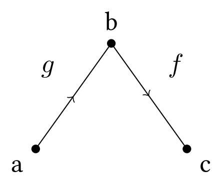
Definition 2.15. We call a horn Λ^n_i an **inner horn** if $n \geq 2$ and $0 < i < n$. We call Λ^n_i an **outer horn** if it is not an inner horn.

Note that there is a canonical inclusion $\Lambda^n_i \hookrightarrow \Delta^n$. We say that a simplicial set X **admits inner horn fillers** if, for every morphism $f : \Lambda^n_i \to X$ from an inner horn to X, there is a morphism $g : \Delta^n \to X$ such that the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, "g"'] & \end{tikzcd}$$

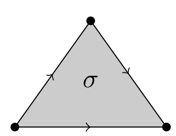
commutes. We say that X **has unique inner horn fillers** if there is precisely one g making this diagram commute.

Let's unpack what this means. In the lowest possible dimension, $n = 2$, we have precisely one inner horn: Λ^2_1. We can represent a map $\psi : \Lambda^2_1 \to X$ by labeling a drawing of Λ^2_1 by 0- and 1-simplices of X.



Here, for instance $a, b, c \in X_0$ and $f, g \in X_2$. Note that for these data to really give a map $\psi : \Lambda^2_1 \to X$, we need that $d^1_0(f) = c$, $d^1_1(f) = b$, $d^1_0(g) = b$, and $d^1_1(g) = a$.[5]

A filler for the horn ψ will then be a map $\phi : \Delta^2 \to X$, which we can draw as

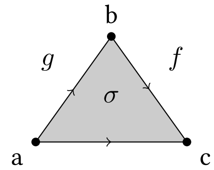


By Yoneda's Lemma, this map is uniquely specified by $\sigma \in X_2$. For σ to be a horn filler for ψ, we need that the two faces $d^2_0(\sigma)$ and $d^2_2(\sigma)$ of σ that would comprise the horn are

[5] To specify the map, all we really need are the 1-simplices f and g, together with the condition $d^1_0(g) = d^1_1(f)$. This is equivalent to noticing that we can write

$$\Lambda^2_1 \cong \Delta^1 \coprod_{\Delta^0} \Delta^1.$$

f and g, respectively. Pictorially, this means we are looking for a 2-simplex of X which looks like:



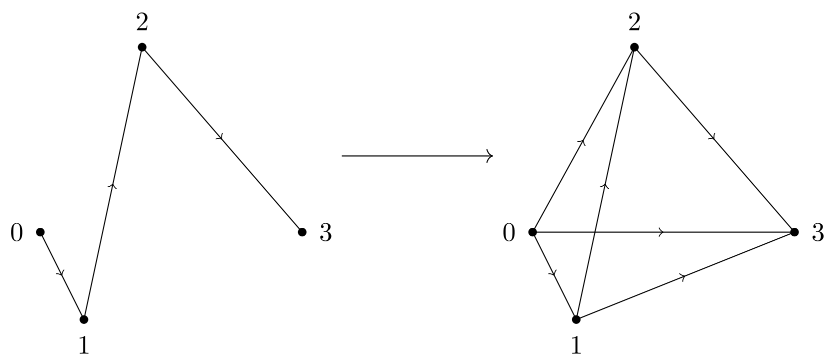
Warning: In general, this is not the same as simply specifying a 1-simplex from a to c. That would give rise to a map $\partial\Delta^2 \to X$ from the boundary of Δ^2.

We now claim that some of these horn filling properties characterize the nerves of categories up to isomorphism.

Definition 2.16. We define the *spine* of an n-simplex Δ^n to be the pushout

$$\mathrm{Sp}(\Delta^n) := \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subset \Delta^n$$

Graphically, this will look like:



Lemma 2.17. *Let* C *be a category. Then restricting along the spine inclusion* $\iota_n : \mathrm{Sp}(\Delta^n) \to \Delta^n$ *yields a bijection*

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, N(\mathsf{C})) \cong \mathrm{Hom}_{\mathsf{Set}_\Delta}(\mathrm{Sp}(\Delta^n), N(\mathsf{C})).$$

Proposition 2.18. *Let* $X \in \mathsf{Set}_\Delta$ *be a simplicial set. Then there is a category* C *and an isomorphism* $X \cong N(\mathsf{C})$ *if and only if* X *admits unique fillers for all inner horns.*

Proof. First, let $X = N(\mathsf{C})$ be the nerve of a category. For any $n \geq 2$ and any $0 < i < n$, the spine inclusion $\mathrm{Sp}(\Delta^n) \to \Delta^n$ factors through the horn inclusion $\Lambda^n_i \to \Delta^n$, i.e, we get

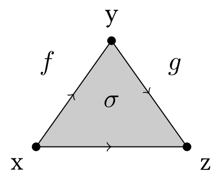
$$\mathrm{Sp}(\Delta^n) \hookrightarrow \Lambda^n_i \hookrightarrow \Delta^n.$$

Thus, the isomorphism of the previous lemma factors as

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, N(\mathsf{C})) \longrightarrow \mathrm{Hom}_{\mathsf{Set}_\Delta}(\Lambda^n_i, N(\mathcal{C})) \longrightarrow \mathrm{Hom}_{\mathsf{Set}_\Delta}(\mathrm{Sp}(\Delta^n), N(\mathsf{C}))$$

It thus suffices for us to show that the latter map is injective. However, by the previous lemma, given $\phi : \Lambda^n_i \to N(\mathsf{C})$, the image of each $(n-1)$-simplex in Λ^n_i is uniquely determined by its spine, so the map is, indeed, injective.

Now suppose that X admits unique fillers for every inner horn. Define a category C as follows: The objects of C are the 0-simplices of X. A morphism from x to y is a 1-simplex f of X with $d_0(f) = y$ and $d_1(f) = x$. The identity on $x \in X_0$ is $s_0(x) \in X_1$. The composition of $f : x \to y$ and $g : y \to z$ is given by forming the unique 2-simplex



filling the inner 2-horn $g \coprod_y f$, and then taking $g \circ f = d_1(\sigma)$.

We then must check unitality and associativity: !!!!!!!!!!!!!!! □

Corollary 2.19. *The functor $N : \mathsf{Cat} \to \mathsf{Set}_\Delta$ descends to an equivalence of categories between* Cat *and the full subcategory of* Set_Δ *on the simplicial sets which admit unique fillers for all inner horns.*

Exercise 10. Show the following: Let X in Set_Δ be a simplicial set. There is a groupoid G and an isomorphism $X \cong N(\mathrm{G})$ if and only if X admits unique fillers for *all* horns of .

4 *Simplicial sets as spaces: Kan Complexes*

We can now return to our geometric interpretation of simplicial sets. We begin by playing the same game with the geometric realization that we did with the nerve.

Definition 2.20. The **Geometric realization functor**

$$|-| : \mathsf{Set}_\Delta \longrightarrow \mathsf{Top}$$

is the left Kan extension of $|-| : \Delta \to \mathsf{Top}$ along the Yoneda embedding $\Delta \to \mathsf{Set}_\Delta$. It has a right adjoint, the **singular simplicial set functor** which is given by

$$\begin{aligned} \mathrm{Sing} : \mathsf{Top} &\longrightarrow \mathsf{Set}_\Delta \\ X &\longmapsto \mathrm{Hom}_{\mathsf{Top}}(|\Delta^{(}-)|, X) \end{aligned}$$

i.e., an n-simplex in $\mathrm{Sing}(X)$ is a continuous map $|\Delta^n| \to X$.

From the definition, we see that

$$|X| := \operatorname*{colim}_{\Delta^k \to X} \Delta^k.$$

However, we'd like a more intuitive description of this construction. Apply the general construction of colimits from coproducts and coequalizers, we see that we can write $|X|$ as the coequalizer of

$$\coprod_{\Delta^k \to \Delta^n \to X} |\Delta^k| \underset{f_*}{\overset{\mathrm{id}}{\rightrightarrows}} \coprod_{\Delta^m \to X} |\Delta^m|$$

Working out what this means, we see that this gives us the quotient of the space

$$\coprod_{n\geq 0} X_n \times |\Delta^n|$$

by the relations $(f^*(\sigma), x) \sim (\sigma, f_*(x))$, for $f : [n] \to [m]$, $\sigma \in X_m$, and $x \in |\Delta^n|$. Effectively, we obtain the geometric realization by realizing each n-simplex in X_n, then gluing these together in the way specified by the face and degeneracy maps of X.

The idea here is simply to make rigorous the pictures we have been drawing — we can draw a simplicial set as a collection of simplices, glued together along the face maps. The realization of a degenerate simplex $s_i^n(\sigma)$ just gets collapsed to the simplex σ, and so doesn't show up in the pictures.

The realization and singular simplicial set functor fit into the same framework as the nerve/fundamental category adjunction, where Sing plays the role of the nerve N, and the realization $|-|$ plays the role of τ. It is the same Yoneda extension-restriction construction we saw before, which will come up in various guises throughout this course.

4.1 Kan complexes

Previously, we were able to characterize the simplicial sets coming from categories and groupoids in terms of horn filling conditions. Our aim here is a bit more modest — we want to develop the metaphor from the overture in terms of horn fillers.

We expect a space to be something like a "higher category in which all morphisms are invertible," i.e., a higher groupoid. When we considered groupoids as simplicial sets, we noticed that they corresponded to precisely those simplicial sets which had **unique** fillers for **any** horn.

To try and find an analogous condition for Sing, we will have to weaken this condition slightly. In particular, we will have to lose uniqueness. This isn't surprising, given our metaphor — if paths in a space are like morphisms in a groupoid, we shouldn't expect that there is a *unique* way to compose them, or take the inverse. Rather, the composite (or inverse) should be *unique up to homotopy.*

Definition 2.21. We call a simplicial set $X \in \mathsf{Set}_\Delta$ a **Kan complex** if, for every map $f : \Lambda^n_i \to X$ from a horn into X, there exists a filler $g : \Delta^n \to X$ making the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

commute. Note that we do **not** require g to be unique.

Proposition 2.22. *Let* $X \in \mathsf{Top}$ *be a topological space. Then* $\mathrm{Sing}(X)$ *is a Kan complex.*

Proof. We begin by considering a a morphism $f : \Lambda^n_i \to \mathrm{Sing}(X)$. We would like to construct a dash arrow making the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

of simplicial sets commute. However, since Sing is right adjoint to $|-|$, giving the arrow $f : \Lambda^n_i \to \mathrm{Sing}(X)$ is equivalent to giving an arrow $\tilde{f} : |\Lambda^n_i| \to X$. Similarly, we can show that the desired map exists by showing that there is some $g : |\Delta^n| \to X$ making the diagram

$$\begin{tikzcd} |\Lambda^n_i| \arrow[r, "f"] \arrow[d, hook] & X \\ |\Delta^n| \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

commute.

Indeed, it will suffice for us to note that $|\Lambda^n_i|$ is a retract of $|\Delta^n|$, via a map $r : |\Delta^n| \to |\Lambda^n_i|$.[6] Since then, we can define $g = f \circ r$, and we will have $g \circ \iota = f \circ r \circ \iota = f$, so that the diagram commutes. □

[6] The retraction map can be computed out by taking projection along the vector from the center of the i^{th} face to $|\Delta^n|$ to the i^{th} vertex of $|\Delta^n|$.

Remark 2.23. Notice that we have not gone as far as before – we have not given an 'if and only if' statement characterizing singular simplicial sets of topological spaces. That said, it will turn out that Kan complexes are a good combinatorial analogue of topological spaces, though it will take us some time to make this precise.

4.2 Simplicial homotopies

We now want to see some of the ways in which Kan complexes behave like topological spaces.

Definition 2.24. Let X be a Kan complex, and let $a, b \in X_0$. A **path** from a to b in X is a map

$$f : \Delta^1 \longrightarrow X.$$

such that $f|_{\{0\}} = a$ and $f|_{\{1\}} = b$.

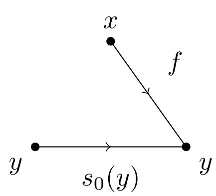
Proposition 2.25. *Let X be a Kan complex. The relation*

$$a \sim b \iff \begin{smallmatrix}\text{there is a path}\\ \text{from } a \text{ to } b\end{smallmatrix}$$

is an equivalence relation.

Proof. Since there is a degenerate 1-simplex from x to x for any $x \in X$, reflexivity is immediate.

To that the relation is symmetric, suppose we have an edge $f : x \to y$ in X. Then we can form a Λ^2_2-horn



in X. Since X is a Kan complex, we can fill this horn, obtaining in the process an edge $g : y \to x$ as desired.

A similar argument (now using a Λ^2_1-horn) yields transitivity. □

Definition 2.26. Let $f, g : X \to Y$ be maps of simplicial sets. A *simplicial homotopy from f to g* is a map

$$H : X \times \Delta^1 \longrightarrow Y$$

such that $H|_{X\times\{0\}} = f$ and $H|_{Y\times\{1\}} = g$.

Proposition 2.27. *Suppose Y is a Kan complex. Then for any simplicial set X, simplicial homotopy is an equivalence relation on the set $\mathrm{Hom}_{\mathsf{Set}_\Delta}(X, Y)$.*

Proof. Later. $\square$

3

Homotopy theory and model categories

We now want to study the homotopy theory of spaces in terms of Kan complexes. The hope that underpins this approach is that we can reduce knotty topological problems to straightforward (if lengthy) combinatorial ones.

But what does it mean to study 'homotopy theory'? There are many answers to this question, and we will here take the naïvest of these. We want to study topological spaces, treating *weak homotopy equivalences* as invertible morphisms. This may at first seem like an unremarkable approach, but a surprising amount of theory is needed to obtain a good way of doing this. Ideally, given a weak homotopy equivalence f, we'd like to be able to define an inverse f^{-1}.

As you likely know from algebraic topology, this isn't always the case, so we're stuck doing the next-best thing: *formally* inverting the weak homotopy equivalences.

Definition 3.1. Let C be a category. A **class of weak equivalences** in C is a collection $\mathcal{W}$ of morphisms in C such that

1. Every isomorphism in C is in $\mathcal{W}$.
2. If $g : a \to b$ and $f : b \to c$ are morphisms in $\mathcal{W}$, define the set $V := \{f, g, f \circ g\}$. If two of the morphisms of V belong to $\mathcal{W}$, then the other does as well.

We then call the pair $(\mathsf{C}, \mathcal{W})$ a **category with weak equivalences**.[1]

[1] To check your comprehension, it is a good exercise to show that

1. Let C be a category and $\mathcal{I}$ the set of isomorphisms in C, then $(\mathsf{C}, \mathcal{I})$ is a category with weak equivalences.
2. Let $\mathcal{W}_{HE}$ be the set of weak homotopy equivalences in Top. Then $(\mathsf{Top}, \mathcal{W}_{HE})$ is a category with weak equivalences.

Given a category with weak equivalences $(\mathsf{C}, \mathcal{W})$, we can define a new category $\mathsf{C}[\mathcal{W}^{-1}]$, which is obtained from C by 'forcing' the morphisms in $\mathcal{W}$ to be invertible. We can do this by noticing a useful property of functors — they always send isomorphisms to isomorphisms.

Definition 3.2. Let $(\mathsf{C}, \mathcal{W})$ be a category with weak equivalences. For any other category D, denote by $\mathrm{Fun}^{\mathcal{W}}(\mathsf{C}, \mathsf{D}) \subset \mathrm{Fun}(\mathsf{C}, \mathsf{D})$ the full subcategory on those functors which send all the the morphisms in $\mathcal{W}$ to isomorphisms in D.

A **localization of** C **at** $\mathcal{W}$ is a functor

$$L_{\mathcal{W}} : \mathsf{C} \longrightarrow \mathsf{C}[\mathcal{W}^{-1}]$$

such that the induced map

$$L_{\mathcal{W}}^{*} : \mathrm{Fun}(\mathsf{C}[\mathcal{W}^{-1}], \mathsf{D}) \longrightarrow \mathrm{Fun}(\mathsf{C}, \mathsf{D})$$

is fully faithful and has essential image $\mathrm{Fun}^{\mathcal{W}}(\mathsf{C},\mathsf{D})$.

Example 3.3.

1. ([4, Ex. 3.1.1]) Consider the category $\mathrm{B}\mathbb{N}$ with a single object $*$, and whose morphisms are precisely the natural numbers, with composition given by addition. The localization of $\mathrm{B}\mathbb{N}$ at the set of all of its morphisms is precisely $\mathrm{B}\mathbb{Z}$ — the groupoid associated to the integers.

2. Consider the category $[1]$, which consists of two objects 0 and 1, and a single morphism $f : 0 \to 1$. The localization of $[1]$ at the set of all of its morphisms is the *walking isomorphism*: the category with objects 0 and 1, and morphisms $f : 0 \to 1$ and $g : 1 \to 0$ such that $g \circ f = \mathrm{id}_0$ and $f \circ g = \mathrm{id}_1$.

3. (Homological algebra) Given an Abelian category[2] A, denote by $\mathcal{W}$ the set of quasi-isomorphisms in $\mathsf{Ch}(\mathsf{A})$. Then the derived category $\mathsf{D}(\mathsf{A})$ is the localization of $\mathsf{Ch}(\mathsf{A})$ at $\mathcal{W}$.

[2] If you don't know what this is, don't worry, but keep this example in mind if you take a homological algebra course.

Lemma 3.4. *Localizations are unique up to equivalence of categories commuting with the localization functor.*

Proof. Suppose given two localizations $L_{\mathcal{W}} : \mathsf{C} \to \mathsf{D}$ and $K_{\mathcal{W}} : \mathsf{C} \to \mathsf{E}$ Since $K_{\mathcal{W}}$ lies in $\mathrm{Fun}^{\mathcal{W}}(\mathsf{C},\mathsf{E})$, the fact that $L_{\mathcal{W}}$ is a localization means that there is a functor

$$F : \mathsf{D} \longrightarrow \mathsf{E}$$

such that $F \circ L_{\mathcal{W}} \cong_\eta K_{\mathcal{W}}$. By the same reasoning, there is a functor

$$G : \mathsf{E} \longrightarrow \mathsf{D}$$

such that $G \circ K_{\mathcal{W}} \cong_\nu L_{\mathcal{W}}$.

Two further applications of the definition show that $G \circ F \cong \mathrm{id}_{\mathsf{D}}$ and $F \circ G \cong \mathrm{id}_{\mathsf{C}}$. $\square$

Definition 3.5. Denote by $\mathcal{W}_{\mathrm{simp}}$ the collection of all morphisms $f : X \to Y$ in Set_Δ such that $|f| : |X| \to |Y|$ is a weak homotopy equivalence of spaces.

Our goal in this chapter will be to elucidate[3] the following claim:

[3] I'm being a little bit evasive with my wording here, because in the interest of saving time, we will not be proving this statement. Instead, we will be building up evidence for it, as well as some useful technology.

Theorem 3.6. *The realization functor descends to an equivalence*

$$|-| : \mathsf{Set}_\Delta[\mathcal{W}_{\mathrm{simp}}^{-1}] \longrightarrow \mathsf{Top}[\mathcal{W}_{HE}^{-1}]$$

1 Cofibrations and invertibility

Looking at the categories Top and Set_Δ, we can notice that in sufficiently nice cases, weak homotopy equivalences are 'almost' invertible — i.e., are actual homotopy equivalences.

1.1 Case study: CW complexes

We cite without proof a celebrated theorem of Whitehead:

Theorem 3.7 (Whitehead)**.** *Let $f : X \to Y$ be a continuous map between CW complexes. Then the following are equivalent.*

- *f is a weak homotopy equivalence.*
- *f is a homotopy equivalence.*

This tells us something very interesting about the localization map

$$L_{\mathcal{W}_{HE}} : \mathsf{Top} \longrightarrow \mathsf{Top}[\mathcal{W}_{HE}^{-1}],$$

Namely, if we restrict this map to the full subcategory $\mathsf{CW} \subset \mathsf{Top}$ of CW complexes, we get a functor that is defined by simply identifying homotopic maps! In some sense, this means that part of the localization $\mathsf{Top}[\mathcal{W}_{HE}^{-1}]$ already 'lives inside' CW.

Even better, there is another famous theorem, which lets us completely restrict our attention to CW.

Theorem 3.8. *Let X be a topological space. Then there is a CW complex Y, and a weak homotopy equivalence $f : Y \to X$.*

This tells us something remarkable: if we identify homotopic maps in CW, we get the *entire* homotopy category $\mathsf{Top}[\mathcal{W}_{HE}^{-1}]$!

It turns out that we can generalize this behavior. The category Top carries the structure of a **Model Category** — a structure which allows us to find 'good' objects like CW complexes for which the weak equivalences can be turn into real homotopy equivalences.

To try and understand this better, let's think about what a CW complex really is. Any CW complex X can be built as follows

Construction 3.9. We start with the empty space $\varnothing$. We can add a single point to $\varnothing$ by forming a pushout square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & * \arrow[d] \\ \varnothing \arrow[r] & * \end{tikzcd}$$

We keep doing this until we get a discrete space X_0, which consists of the 0-cells of X.[4]

Next we take each 1-cell $\alpha : D^1 \hookrightarrow X$ of X. This comes equipped with an *attaching map*

$$f_\alpha : \partial D^1 \to X_0.$$

We can then add the 1-cell α to X by forming a pushout square

$$\begin{tikzcd} \partial D^1 \arrow[r] \arrow[d, "f_\alpha"'] & D^1 \arrow[d] \\ X_0 \arrow[r] & Y \end{tikzcd}$$

We do this until we have added all the 1-cells of X to X_0, and obtain a space X_1 — the *1-skeleton* of X.

[4] Notice that $* = D^0$ is the 0-dimensional cell, and that $\varnothing = \partial D^0$ is its boundary.

We continue to play the same game, adding n-cells to X_{n-1} via pushouts of the form

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & D^n \arrow[d] \\ X_{n-1} \arrow[r] & Y \end{tikzcd}$$

until eventually we reach the space X (perhaps after infinitely many iterations).

This tells us that our CW complexes can be built from the empty set using nothing but the inclusions $\partial D^k \to D^k$. But what operations do we need to allow to be able to build any CW complex from these maps?

Definition 3.10. Suppose that C is a cocomplete category. A set $\mathcal{K}$ of morphisms in C is called **saturated** if

1. All isomorphisms are contained in $\mathcal{K}$.
2. If there is a pushout square
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$
such that $f \in \mathcal{K}$, then $g \in \mathcal{K}$. (We say $\mathcal{K}$ *is closed under pushouts.*)
3. If we have a countable sequence
$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \longrightarrow \cdots$$
of morphisms in $\mathcal{K}$, then the induced morphism $A_0 \to \operatorname{colim}_{i\in\mathbb{N}} A_i$ is in $\mathcal{K}$. (We say $\mathcal{K}$ is *closed under countable composition.*)
4. If there is a collection $\{f_i : A_i \to B_i\}_{i\in I}$ of morphisms in $\mathcal{K}$, then the induced morphism
$$\coprod_{i\in I} A_i \xrightarrow{\coprod f_i} \coprod_{i\in I} B_i$$
is in $\mathcal{K}$. (We say $\mathcal{K}$ is *closed under coproducts.*)
5. If there is a commutative diagram
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & D \arrow[r] & B \end{tikzcd}$$
such that $g \in \mathcal{K}$, then $f \in \mathcal{K}$. (We say $\mathcal{K}$ is *closed under retracts.*)

Definition 3.11. Let $\mathcal{J}$ be a set of morphisms in a cocomplete category $\mathcal{C}$. The smallest saturated class $\mathcal{K}$ containing $\mathcal{J}$ is called the **saturated hull** of $\mathcal{J}$. We will often denote the saturated hull of $\mathcal{J}$ by $\overline{\mathcal{J}}$.

Example 3.12. Consider the set

$$\mathcal{J} := \{\partial D^n \hookrightarrow D^n\}_{n\in\mathbb{N}}$$

of morphisms in Top. We will denote the saturated hull of $\mathcal{J}$ by $\mathcal{C}\mathrm{of}_{\mathrm{cl}}$, and call its elements **(classical) cofibrations**.

We can thus rewrite our description of CW complexes as follows.

Lemma 3.13. *Let X be a CW complex. Then the unique map $\varnothing \to X$ is a classical cofibration.*

Proof. We define a sequence of classical cofibrations

$$X_{-1} = \varnothing \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \tag{3.1}$$

iteratively as follows. Let C_n denote the set of n-cells of X. We define the map $X_{i-1} \to X_i$ to be the pushout

$$\begin{tikzcd} \coprod_{\alpha\in C_n} \partial D^n \arrow[r] \arrow[d, "\phi_\alpha"] & \coprod_{\alpha\in C_n} D^n \arrow[d] \\ X_{i-1} \arrow[r, "\phi_i"] & X_i \end{tikzcd}$$

Since $\mathcal{C}\mathrm{of}_{\mathrm{cl}}$ is closed under coproducts, the top arrow is in $\mathcal{C}\mathrm{of}_{\mathrm{cl}}$. Since $\mathcal{C}\mathrm{of}_{\mathrm{cl}}$ is closed under pushouts, the bottom arrow is as well.

The colimit of (3.1) is by definition the space X, and so since $\mathcal{C}\mathrm{of}_{\mathrm{cl}}$ is closed under countable composition, the map $\varnothing \to X$ is in $\mathcal{C}\mathrm{of}_{\mathrm{cl}}$. □

Remark 3.14. The reason for requiring that $\mathcal{C}\mathrm{of}_{\mathrm{cl}}$ be closed under retracts is not yet apparent. Our later discussion of Kan complexes will make clear why this axiom is of use.

Definition 3.15. We call a space $X \in \mathsf{Top}$ **(classically) cofibrant** if the unique map $\varnothing \to X$ is in $\mathcal{C}\mathrm{of}_{\mathrm{cl}}$.

The upshot of the preceding discussion (together with a slight strengthening of Whitehead's theorem to include all classically cofibrant objects), is the following.

Theorem 3.16. *Let $\mathsf{Top}^\circ \subset \mathsf{Top}$ denote the full subcategory on the classically cofibrant objects. Denote by $\mathrm{Ho}(\mathsf{Top})$ the category obtained from Top° by identifying homotopic morphisms. Then the localization map $L_{\mathcal{W}_{HE}}$ induces an equivalence of categories*

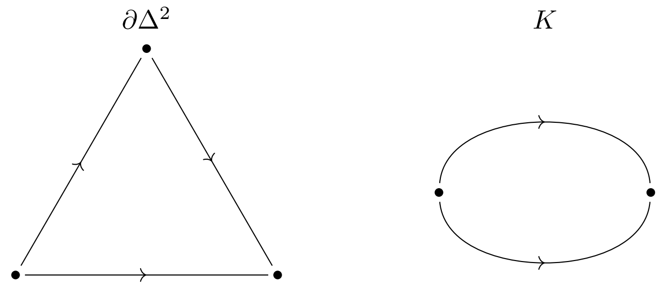
$$\mathrm{Ho}(\mathsf{Top}) \simeq \mathsf{Top}[\mathcal{W}_{HE}^{-1}].$$

1.2 Case study: Simplicial sets

Let's try the same approach with simplicial sets, to see if a similar notion of 'cofibrant' simplicial set exists, such that every weak homotopy equivalence of simplicial sets is a homotopy equivalence.

Before we define our cofibrations, though, let's examine whether we need them. Is every weak homotopy equivalence $f : X \to Y$ of simplicial sets already a homotopy equivalence?

Example 3.17. Consider $\partial\Delta^2$ and the simplicial set K with two vertices and two parallel edges, as pictured below.



Convince yourself that the realizations of these two spaces are homeomorphic to the circle, but every map $K \to \partial\Delta^2$ is nullhomotopic. Note that there is a map $\partial\Delta^2 \to K$ which induces a homotopy equivalence on realizations.

This example shows us that, in fact, not every weak homotopy equivalence of simplicial sets is a homotopy equivalence. Because of this, lets try to mimic our earlier proceedure for spaces.

Definition 3.18. Set

$$\mathcal{I} := \{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}.$$

We define the set of **(simplicial) cofibrations** to be the saturated hull $\mathcal{C}\mathrm{of}_\Delta$ of $\mathcal{I}$.[5]

Call a simplicial set X **(Kan-Quillen) cofibrant** if the unique map $\varnothing \to X$ is a simplicial cofibration.

[5] One might be tempted to ask why this is the natural analogue of the cofibrations in Top. The reason is that we can define homeomorphisms $|\Delta^n| \cong D^n$ which, on the boundary, restrict to homeomorphisms $|\partial\Delta^n| \cong \partial D^n$.

Unfortunately, unlike in the case of spaces, this does not solve our problem.

Lemma 3.19. *Every simplicial set is Kan-Quillen cofibrant.*

We will actually prove this lemma as a corollary of a more general proposition.

Proposition 3.20. *Let $f : X \to Y$ be a monomorphism[6] of simplicial sets. Then $f \in \mathcal{C}\mathrm{of}_\Delta$.*

[6] By a *monomorphism* in a category C, we mean a morphism $f : x \to y$ in C such that, for any $g, h : z \to x$ in C, if $f \circ g = f \circ h$, then $g = h$. This can be seen as a generalization of the notion of an injective map:

Exercise 11. Show that the monomorphisms in Set are precisely the injective maps. Show that the monomorphisms in Set_Δ are those morphisms $f : X \to Y$ such that, for every $n \geq 0$, the map $f : X_n \to Y_n$ on n-simplices is injective.

Proof. We first note that, given a pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "f'"'] & W \end{tikzcd}$$

in Set_Δ, if f is a monomorphism, then f' is also a monomorphism.

We will inductively construct a sequence of Kan-Quillen cofibrations

$$X =: X^0 \xrightarrow{f_1} X^1 \longrightarrow X^2 \longrightarrow \cdots$$

Such that Y is the colimit of the sequence.

To begin with, let $I_0 = Y_0 \setminus X_0$ be the set of 0-simplices of Y which are not 0-simplices of X. We can then define X^1 and f_1 as the pushout

$$\begin{tikzcd} \coprod_{\alpha\ inI_0} \partial\Delta^0 \arrow[r] \arrow[d] & \coprod_{\alpha\ inI_0} \Delta^0 \arrow[d] \\ X \arrow[r, "f_1"'] & X^1 \end{tikzcd}$$

so that X^1 contains all the 0-simplices of Y. By universal property, we get an induced map $\psi^1 : X^1 \to Y$ such that $\psi^1 \circ f_1 = f$.

Now suppose we have constructed X^k, and a monomorphism $\psi^k : X^k \to Y$ which is surjective all the n-simplices of Y for $n < k$. We then set $I_k = Y_k \setminus (X^k)_k$ to be the set of k-simplices of Y not already contained in X^k. There are canonical inclusions

$$\coprod_{\alpha \ inI_k} \partial\Delta^k \longrightarrow \coprod_{\alpha\in I_k} \Delta^k \longrightarrow Y.$$

Since all the $(k-1)$-simplices of Y are contained in X^k, the inclusion

$$\coprod_{\alpha \ inI_k} \partial\Delta^k \longrightarrow Y.$$

factors through ψ^k We thus get a commutative diagram

$$\begin{tikzcd}
\coprod_{\alpha \ inI_k} \partial\Delta^k \arrow[r] \arrow[d] & \coprod_{\alpha\in I_k} \Delta^k \arrow[ddr, bend left] & \\
X^k \arrow[drr, bend right, "\psi^k"'] & & \\
& & Y
\end{tikzcd}$$

Forming the pushout then yields a space X^{k+1} and a monomorphism $\psi^{k+1} : X^{k+1} \to Y$ which hits all the k-simplices of Y, and such that the diagram

$$\begin{tikzcd}
\coprod_{\alpha \ inI_k} \partial\Delta^k \arrow[r] \arrow[d] & \coprod_{\alpha\in I_k} \Delta^k \arrow[d] \arrow[ddr, bend left] & \\
X^k \arrow[r, "f_{k+1}"'] \arrow[drr, bend right, "\psi^k"'] & X^{k+1} \arrow[dr, "\psi^{k+1}"] & \\
& & Y
\end{tikzcd}$$

commutes.

By induction, we have constructed a sequence of Kan-Quillen cofibrations $f_i : X^{i-1} \to X^i$ whose countable composition is the map

$$f : X \to Y$$

Thus, this map is a Kan-Quillen cofibration, as desired. □

It thus becomes clear that we need more machinery if we want a nice description of the localization.

2 Anodyne maps

The big issue we run into when trying to turn weak homotopy equivalences of simplicial sets into homotopy equivalences is that simplicial homotopies are not usually reversable

— they have a direction specified by the direction of the corresponding 1-simplices. This means, in particular, that simplicial homotopy is not an equivalence relation. To rectify this, we again need to restrict our attention to a special class of objects. Fortunately, this is a class we have already encountered: Kan complexes.

Recall that a Kan complex X is defined by the property that, for any $0 \leq i \leq n$, and any map $\Lambda^n_i \to X$, there exists a dashed morphism as below making the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

commute. Our hope will be to relate this *lifting property* to the saturated classes of morphisms we introduced earlier.

Definition 3.21. We define the class $\mathcal{A}$ of **anodyne maps** to be the saturated hull of the set

$$\{\Lambda^n_i \hookrightarrow \Delta^n \mid 0 \leq i \leq n\}.$$

We will call a map $f : X \to Y$ of simplicial sets a **Kan fibration** if, for every $i : A \to B$ in $\mathcal{A}$ and every solid commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a dashed arrow as indicated making the diagram commute.

It will turn out that we don't actually need to test f against every anodyne map to see that it is a Kan fibration. It will suffice to check only the familiar horn inclusions. This follows from some quite general theory:

Definition 3.22. Let C be a category, and let $f : X \to Y$ and $i : A \to B$ be morphisms in C. We say that f **has the right lifting property (RLP) with respect to** i and that i **has the left lifting property (LLP) with respect to** f if, for every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in C, there exists a dashed arrow as above making the diagram commute. In this case we write $i \perp f$.

Let $\mathcal{B}$ be a set of morphisms in C. We denote

$${}_{\perp}\mathcal{B} := \{i \mid i \perp f \text{ for every } f \in \mathcal{B}\}$$

and

$$\mathcal{B}_{\perp} := \{f \mid i \perp f \text{ for every } i \in \mathcal{B}\}$$

Lemma 3.23. *Let* C *be a cocomplete category, and let* $\mathcal{B}$ *be a set of morphisms in* C*. Then* ${}_{\perp}\mathcal{B}$ *is saturated.*

Proof. This amounts to checking the conditions for saturation. It is immediate that isomorphisms are contained in ${}_\perp\mathcal{B}$. Closure under coproducts is a straightforward application of the universal property.

To check closure under pushouts, suppose that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[r] \arrow[d, "j"] & X \arrow[d] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

is a diagram in C. Suppose that the left-hand square is a pushout square with $i \in {}_\perp\mathcal{B}$, and suppose that $f \in \mathcal{B}$. Then by definition, we can solve the lifting problem defined by the outer square, to get a dashed arrow

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[r] \arrow[d, "j"] & X \arrow[d] \\ B \arrow[r] \arrow[urr, dashed] & D \arrow[r] & Y \end{tikzcd}$$

The universal property of the pushout then provides a map $D \to X$, which is easily checked to be a solution to the left-hand lifting problem. Hence $j \in {}_\perp\mathcal{B}$.

To see closure under countable composition, suppose that

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \cdots$$

is a sequence of morphisms in ${}_\perp\mathcal{B}$, and suppose we are given a lifting problem

$$\begin{tikzcd} B_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{colim}_{i\in\mathbb{N}} B_i \arrow[r] & Y \end{tikzcd}$$

where $f \in \mathcal{B}$. By universal property, this gives us a countable sequence of lifting problems

$$\begin{tikzcd} B_i \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B_{i+1} \arrow[r] & Y \end{tikzcd}$$

which each admit a solution by assumption. The solutions display X as a cone over the $\{B_i\}$, and thus yield a map $\operatorname{colim}_{i\in\mathbb{N}} B_i \to X$ by universal property, which solves the lifting problem.

To see closure under retracts, suppose we have a retract diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"] \arrow[rr, bend left, "\mathrm{id}"] & C \arrow[r] \arrow[d, "i"] & A \arrow[d, "j"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & D \arrow[r] & B \end{tikzcd}$$

with $i \in {}_\perp\mathcal{B}$. Then suppose we are given a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

with $f \in \mathcal{B}$. We can stick these diagrams together, and solve the middle lifting problem,

$$\begin{tikzcd} A \arrow[r, "i_A"'] \arrow[d, "j"] \arrow[rr, bend left, "\mathrm{id}"] & C \arrow[r, "r_A"] \arrow[d, "i"] & A \arrow[r] \arrow[d, "j"] & X \arrow[d, "f"] \\ B \arrow[r, "i_B"] \arrow[rr, bend right, "\mathrm{id}"'] & D \arrow[r, "r_B"] \arrow[urr, dashed, "h"] & B \arrow[r] & Y \end{tikzcd}$$

It is then easy to check that $h \circ i_B$ solves the initial lifting problem. □

Corollary 3.24. *Let* C *be a cocomplete category, and* $\mathcal{B}$ *a set of morphisms in* C*. Then* $\overline{\mathcal{B}} \subset {}_\perp(\mathcal{B}_\perp)$.

Corollary 3.25. *A map* $f : X \to Y$ *of simplicial sets is a Kan fibration if and only if it has the RLP with respect to the maps* $\Lambda^n_i \to \Delta^n$ *for all* $0 \leq i \leq n$.

Corollary 3.26. *A simplicial set* X *is a Kan complex if and only if the unique map* $X \to \Delta^0$ *is a Kan fibration.*

So now we have two ways of finding special types of objects, both related to saturated classes. Kan-Quillen cofibrant objects are the objects such that the unique map $\varnothing \to X$ is a Kan-Quillen cofibration; and **Kan-Quillen fibrant objects** (Kan complexes) are the objects such that the unique map $X \to \Delta^0$ has the RLP with respect to the Anodyne maps.

It will turn out that any weak homotopy equivalence between two Kan complexes is, in fact, a homotopy equivalence, so in some sense the *fibrant objects* in Set_Δ are good in the same way that the *cofibrant objects* in Top are.

3 *Model categories*

We now want to axiomatize the kind of structures we have been examining. In effect, we have a category C, and a set $\mathcal{W}$ of weak equivalences, and we want to see the localization $\mathrm{C}[\mathcal{W}^{-1}]$ as "living inside" C. We have noticed that we can do this for Top by defining a saturated set of morphisms $\mathcal{C}$of, and considering only the cofibrant objects, and we suggested that we can do this for Set_Δ by defining a saturated set of *anodyne* morphisms, and considering *fibrant objects* with respect to these. We will now introduce a structure which incorporates both of these ideas, and allows us to deal with localizations in a very organized manner.

Definition 3.27. Let C be a complete, cocomplete category. A **model structure** on C consists of three distinguished sets of morphisms: A set $\mathcal{W}$ called *weak equivalences*, a set $\mathcal{C}$of called *cofibrations*, and a set $\mathcal{F}$ib called *fibrations*. These sets are required to satisfy the axioms:

(M1) The pair $(\mathsf{C}, \mathcal{W})$ is a category with weak equivalences.

(M2) Each of the classes $\mathcal{W}$, $\mathcal{C}$of, and $\mathcal{F}$ib are stable under retracts.

(M3) A lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

in C has a solution, provided that at least one of the following holds:

- We have $i \in \mathcal{C}\mathrm{of} \cap \mathcal{W}$ and $f \in \mathcal{F}\mathrm{ib}$.
- We have $i \in \mathcal{C}\mathrm{of}$ and $f \in \mathcal{F}\mathrm{ib} \cap \mathcal{W}$.

(M4) Any morphism $X \to Y$ in C admits a factorization

$$X \xrightarrow{i} \widetilde{X} \xrightarrow{f} Y$$

Where $i \in \mathcal{C}\mathrm{of} \cap \mathcal{W}$, and $f \in \mathcal{F}\mathrm{ib}$.

(M5) Any morphism $X \to Y$ in C admits a factorization

$$X \xrightarrow{j} \hat{Y} \xrightarrow{g} Y$$

Where $j \in \mathcal{C}\mathrm{of}$, and $f \in \mathcal{F}\mathrm{ib} \cap \mathcal{W}$.

We will refer to the morphisms of $\mathcal{C}\mathrm{of} \cap \mathcal{W}$ as **trivial cofibrations** and the morphisms of $\mathcal{F}\mathrm{ib} \cap \mathcal{W}$ as **trivial fibrations**.

Remark 3.28. Notice that this means the set $\mathcal{C}$of of cofibrations and the set $\mathcal{C}\mathrm{of} \cap \mathcal{W}$ of trivial cofibrations are both saturated, by Lemma 3.23.

One important observation follows immediately from the definitions:

Lemma 3.29. *Let* $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ *be a model category. Then* $(\mathsf{C}^{\mathrm{op}}, \mathcal{F}\mathrm{ib}, \mathcal{C}\mathrm{of}, \mathcal{W})$ *is a model category.*

The use of this lemma is that, if we prove a statement about a model category, we immediately obtain its dual statement. For instance, if we notice that the set of cofibrations is stable under pushouts, we immediately obtain that the set of fibrations is stable under pullbacks.

Example 3.30.

1. Any complete and cocomplete category C carries a model structure in which $\mathcal{C}\mathrm{of} = \mathcal{F}\mathrm{ib}$ is the set of all morphisms in C, and $\mathcal{W}$ is the set of isomorphisms in C

2. The category Cat of small categories carries a model structure where $\mathcal{C}$of consists of those functors which are injective on objects, $\mathcal{F}$ib consists of the *isofibrations*[7] and $\mathcal{W}$ consists of the equivalences of categories.

[7] An *isofibration* $F : \mathsf{C} \to \mathsf{D}$ is a functor such that, for any $c \in \mathsf{C}$ and any isomorphism $f : F(c) \xrightarrow{\cong} d$ in D, there exists an isomorphism $g : c \xrightarrow{\cong} b$ in C such that $F(g) = f$.

3. There is a model structure on the category Top in which $\mathcal{C}\mathrm{of}$ is the set of classical cofibrations, $\mathcal{F}\mathrm{ib}$ is the set of *Serre fibrations*[8], and $\mathcal{W}$ is the set of weak homotopy equivalences.

[8] ....

Example 3.31. There is a model structure on Set_Δ with

$\mathcal{C}\mathrm{of}$: Cofibrations are monomorphisms.

$\mathcal{F}\mathrm{ib}$: Fibrations are Kan fibrations.

$\mathcal{W}$: Weak equivalences are weak homotopy equivalences.

Proposition 3.32. *Let* $(C, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ *be a model category. Then*

1. $\mathcal{C}\mathrm{of}_\perp = (\mathcal{F}\mathrm{ib} \cap \mathcal{W})$ *and* $\mathcal{C}\mathrm{of} = {}_\perp(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$

2. $(\mathcal{C}\mathrm{of} \cap \mathcal{W})_\perp = \mathcal{F}\mathrm{ib}$ *and* $(\mathcal{C}\mathrm{of} \cap \mathcal{W}) = {}_\perp\mathcal{F}\mathrm{ib}$.

Proof. It suffices for us to show $\mathcal{C}\mathrm{of} = {}_\perp(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$ and $(\mathcal{C}\mathrm{of} \cap \mathcal{W}) = {}_\perp\mathcal{F}\mathrm{ib}$. The other statements follow by duality. We prove one of these statements, and leave the other to the reader.

Note first that $\mathcal{C}\mathrm{of} \subset {}_\perp(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$ by (M3). Now suppose $i : A \to B$ is in ${}_\perp(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$. We can factor i as

$$A \xhookrightarrow{g} \widehat{B} \xrightarrow[\sim]{f} B$$

by (M5). We can then form the lifting problem

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook, "i"'] & \widehat{B} \arrow[d, two heads, "f", "\sim"'] \\ B \arrow[r, "\mathrm{id}_B"'] & B \end{tikzcd}$$

which has a solution $r : B \to \widehat{B}$ by (M3). This then yields a retract diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "i"'] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "g"] & A \arrow[d, "i"] \\ B \arrow[r, "r"'] & \widehat{B} \arrow[r, "\sim", "f"'] & B \end{tikzcd}$$

Thus, by (M2), $i \in \mathcal{C}\mathrm{of}$. $\square$

Remark 3.33. This implies that (1) $\mathcal{C}\mathrm{of}$ and $(\mathcal{C}\mathrm{of} \cap \mathcal{W})$ are saturated, and (2) a model structure is uniquely determined by $\mathcal{C}\mathrm{of}$ and $\mathcal{W}$ or by $\mathcal{F}\mathrm{ib}$ and $\mathcal{W}$.

4 Factorizations and the small object argument

We haven't yet discussed the factorization axioms (M4) and (M5) in our definition of model categories. There is a fairly miraculous trick for showing that such factorization axioms hold, due to Quillen. To make use of this argument, however, we will need some additional terminology.

Definition 3.34. A partial ordered set I is called *filtered* if every finite subset $U \subset I$ has an upper bound, i.e. there is an $x \in I$ such that, for every $y \in U$, $y \leq x$. We call a colimit of a diagram

$$F : I \longrightarrow \mathsf{C}$$

where I is a filtered poset a *filtered colimit*, and we call such a diagram a *filtered diagram*

Definition 3.35. Let C be a cocomplete category. We call an object $x \in \mathsf{C}$ *compact*[9] if the representable functor

$$\mathrm{Hom}_{\mathsf{C}}(x, -) : \mathsf{C} \longrightarrow \mathsf{Set}$$

commutes with filtered colimits.

Equivalently, we say x is compact if, for every filtered diagram

$$\begin{aligned} F : I &\longrightarrow \mathsf{C} \\ i &\longmapsto y_i \end{aligned}$$

with colimit cone $\eta : F \Rightarrow \mathrm{const}_c$ and every $f : x \to c$, the following conditions hold:

1. There exists $i \in I$ and $f_i : x \to c$ such that the diagram

$$\begin{tikzcd} x \arrow[r, "f_i"] \arrow[dr, "f"'] & y_i \arrow[d, "\eta_i"] \\ & c \end{tikzcd}$$

   commutes.

2. If $g_i : x \to y_i$ and $g_j : x \to y_j$ are two morphisms satisfying condition (1), then there is a $k \in I$ with $i \leq k$ and $j \leq k$ such that the diagram

$$\begin{tikzcd} x \arrow[r, "g_i"] \arrow[d, "g_j"'] & y_i \arrow[d] \\ y_j \arrow[r] & y_k \end{tikzcd}$$

   commutes.

Proposition 3.36 (Quillen's Small Object Argument)**.** *Let C be a locally small cocomplete category, and $\mathcal{A}$ a small set of morphisms of C such that the source of every morphism in $\mathcal{A}$ is compact, then every morphism $f : c \to d$ in C admits a factorization*

$$\begin{tikzcd} c \arrow[rr, "f"] \arrow[dr, "h"'] & & d \\ & e \arrow[ur, "g"'] & \end{tikzcd}$$

where h is in the saturated hull of $\mathcal{A}$, and g has the right lifting property with respect to every morphism of $\mathcal{A}$.

Proof. The key trick here is to push each lifting problem "one step further" in a colimit diagram: We want to construct a $\mathbb{N}$-index sequence

$$c \longrightarrow c_1 \longrightarrow c_2 \longrightarrow \cdots$$

[9] The original term used by Quillen was *small*, however, small is now often used for an unrelated set-theoretic notion.

and a cocone with tip d over this diagram:

$$\begin{tikzcd}
c \arrow[r] \arrow[drrr, "f"'] & c_1 \arrow[r] \arrow[drr, "f_1"'] & c_2 \arrow[r] \arrow[dr, "f_2"] & \cdots \\
& & & d
\end{tikzcd}$$

with the following properties:

- Each morphism $c_i \to c_{i+1}$ lies in $\mathcal{A}$.
- Given a lifting problem

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "s"'] & c_i \arrow[d] \\
b \arrow[r] & d
\end{tikzcd}$$

  with $s \in \mathcal{A}$, then there exists a morphism $g : b \to c_{i+1}$ such that the diagram

$$\begin{tikzcd}
a \arrow[rr] \arrow[dr] \arrow[dd, "s"'] & & c_i \arrow[dl] \arrow[dd] \\
& c_{i+1} \arrow[dr] & \\
b \arrow[ur, "g"] \arrow[rr] & & d
\end{tikzcd}$$

  commutes.

Let us first see why this would imply the proposition. We define $c_\infty = \operatorname{colim}_{\mathbb{N}} x_i$. Since each $c_i \to c_{i+1}$, the transfinite composite $f_\infty : c \to c_\infty$ lies in the saturated hull of $\mathcal{A}$. Moreover, the cocone with tip Y yields a factorization

$$\begin{tikzcd}
c \arrow[rr, "f"] \arrow[dr, "f_\infty"'] & & d \\
& c_\infty \arrow[ur, "p"'] &
\end{tikzcd}$$

of f. It thus remains only for us to check that p has the right lifting property with respect to all morphisms in $\mathcal{A}$.

Given a lifting problem

$$\begin{tikzcd}
a \arrow[r, "m"] \arrow[d, "s"'] & c_\infty \arrow[d, "p"] \\
b \arrow[r, "n"] & d
\end{tikzcd}$$

with s in $\mathcal{A}$, the fact that a is compact allows us to find a morphism $a \to c_i$ such that the diagram

$$\begin{tikzcd}
a \arrow[rr, "m"] \arrow[dr, "\tilde{m}"] \arrow[dd, "s"'] & & c_\infty \arrow[dd, "p"] \\
& c_i \arrow[ur] \arrow[dr] & \\
b \arrow[rr, "n"'] & & d
\end{tikzcd}$$

commutes. We can use our second property to find $g : b \to c_{i+1}$ such that the diagram

$$\begin{tikzcd} a \arrow[rrr, "m"] \arrow[dr, "\tilde{m}"] \arrow[ddd, "s"'] & & & c_\infty \arrow[ddd, "p"] \\ & c_i \arrow[urr] \arrow[d] & & \\ & c_{i+1} \arrow[uurr] \arrow[drr] & & \\ b \arrow[ur, "g"] \arrow[rrr, "n"'] & & & d \end{tikzcd}$$

commutes. The induced map $b \xrightarrow{g} c_{i+1} \longrightarrow c_\infty$ then solves the initial lifting problem.

We now only need to show that the desired sequence c_i of objects in C exists. We consider the set I of *all* diagrams of the form

$$\begin{tikzcd} a_i \arrow[r, "m_i"] \arrow[d, "s_i"] & c \arrow[d, "f"] \\ b_i \arrow[r, "n_i"] & d \end{tikzcd}$$

where $s_i \in \mathcal{A}$. Then we can form the coproduct over $i \in I$ of the morphisms $s_i : a_i \to b_i$:[10]

$$\coprod_{i \in I} a_i \longrightarrow \coprod_{i \in I} b_i$$

[10] note that morphisms in $\mathcal{A}$ can appear more than once in this coproduct.

We thus get a commutative square

$$\begin{tikzcd} \coprod_{i\in I} a_i \arrow[r] \arrow[d] & c \arrow[d, "f"] \\ \coprod_{i\in I} b_i \arrow[r] & d \end{tikzcd}$$

Forming the pushout then gives us an object c_1 and a commutative diagram

$$\begin{tikzcd} \coprod_{i\in I} a_i \arrow[r] \arrow[d] & c \arrow[d, "\tau_1"] \arrow[ddr, bend left, "f"] & \\ \coprod_{i\in I} b_i \arrow[r] \arrow[drr, bend right] & c_1 \arrow[dr, dashed, "f_1"] & \\ & & d \end{tikzcd}$$

Where, by definition, τ_1 is in $\overline{\mathcal{A}}$. Moreover, given a particular lifting problem

$$\begin{tikzcd} a \arrow[r, "m"] \arrow[d, "s"] & c \arrow[d, "f"] \\ b \arrow[r, "n"] & d \end{tikzcd}$$

with $s \in \mathcal{A}$, the summand inclusions

$$\begin{tikzcd} a \arrow[r] \arrow[d, "s"] & \coprod_{i\in I} a_i \arrow[r] \arrow[d] & c \arrow[d, "\tau_1"] \arrow[ddr, bend left, "f"] & \\ b \arrow[r] & \coprod_{i\in I} b_i \arrow[r] \arrow[drr, bend right] & c_1 \arrow[dr, dashed, "f_1"] & \\ & & & d \end{tikzcd}$$

give us the desired partial lift. Iterating this process gives us the desired sequence c_i. $\square$

Corollary 3.37. *Suppose that $\mathcal{A}$ and C satisfy the hypotheses of Proposition 3.36. Then*

$$\overline{\mathcal{A}} = {}_\perp(\overline{\mathcal{A}})_\perp$$

We now want to specialize this argument to the case of Set_Δ. The first step is to identify the compact objects in Set_Δ.

Definition 3.38. We call a simplicial set $X \in \mathrm{Set}_\Delta$ *finite* if X has only finitely many non-degenerate simplices.

Lemma 3.39. *Let $X \in \mathrm{Set}_\Delta$, and denote by $P^{\mathrm{fin}}(X)$ the poset of finite simplicial subsets $Y \subset X$. Then $P^{\mathrm{fin}}(X)$ is filtered, and*

$$X \cong \operatorname*{colim}_{Y \in P^{\mathrm{fin}}(X)} Y.$$

Proof. The fact that $P^{\mathrm{fin}}(X)$ is filtered follows from the fact that a finite union of finite simplicial sets is itself finite.

By the explicit description of colimits in Set, we see that $\mathrm{colim}_{Y\in P^{\mathrm{fin}}(X)} Y \subset X$. On the other hand, suppose that $\Delta^n \to X$ is a simplex. Then the image of Δ^n is a finite simplicial subset of X, and so $X \subset \mathrm{colim}_{Y\in P^{\mathrm{fin}}(X)} Y$. $\square$

Proposition 3.40. *A simplicial set $X \in \mathrm{Set}_\Delta$ is a compact object if and only if X is finite.*

Proof. Suppose that X is compact. Then since

$$X \cong \operatorname*{colim}_{Y \in P^{\mathrm{fin}}(X)} Y$$

the identity $\mathrm{id}_X : X \to X$ must factor through some $Y \in P^{\mathrm{fin}}(X)$, i.e., the identity must factor as $X \to Y \hookrightarrow X$. This, however implies $X = Y$, and so X is finite.

On the other hand, suppose that X is finite, let I be a filtered poset, and $F : I \to \mathrm{Set}_\Delta$ be a diagram with colimit cocone $\eta : F \Rightarrow \mathrm{const}_Y$. Suppose we are given a map $f : X \to Y$. For any non-degenerate simplex σ of X, there is then an $i \in I$ such that $f|_\sigma : \Delta^n \to Y$ factors through $F(i_\sigma)$.

For every non-degenerate simplex σ of X, we choose such an i_σ. This is a finite set by assumption, and since I is filtered, there is a $k \in I$ such that $i_\sigma \leq k$ for every non-degenerate simplex σ of X. This means that, for each such σ, $f|_\sigma$ factors through $F(k)$. We can thus, for any n, define maps from the non-degenerate n-simplices of X to the n-simplices of $F(k)$. We extend this map to degenerate simplices by forcing it to commute with degeneracy maps. Checking that this yields a well-defined map of simplicial sets $g : X \to F(k)$ shows that $f : X \to Y$ factors through $F(k)$, as desired.

Finally, suppose that $f : X \to Y$ factors through $F(j)$ as

$$X \xrightarrow{g_j} F(j) \longrightarrow Y$$

and $F(\ell)$ as

$$X \xrightarrow{g_\ell} F(\ell) \longrightarrow Y$$

For each non-degenerate simplex σ of X, we can find a $p_\sigma \in I$ such that the composites

$$X \xrightarrow{g_j} F(j) \longrightarrow F(p_\sigma)$$

and

$$X \xrightarrow{g_\ell} F(\ell) \longrightarrow F(p_\sigma)$$

agree on σ (by the construction of colimits in Set). Since I is filtered, We can thus find some $p \in I$ such that the composites

$$X \xrightarrow{g_j} F(j) \longrightarrow F(p)$$

and

$$X \xrightarrow{g_\ell} F(\ell) \longrightarrow F(p)$$

agree on all non-degenerate simplices. However, this implies that these composites agree on all simplices, and so we see that X is compact. □

Corollary 3.41. *For every $n \geq 0$ and every $\mathcal{K} \subset \mathbb{P}([n])$, the simplicial set $\Delta^{\mathcal{K}}$ is compact.*

Corollary 3.42. *Every morphism $f : X \to Y$ in Set_Δ factors as*

$$X \xrightarrow{s} Z \xrightarrow{p} Y$$

where s is an anodyne morphism, and p is a Kan fibration.

Proof. Apply the small object argument to the set

$$\mathcal{A} := \{\Lambda^n_i \to \Delta^n\}$$ □

Corollary 3.43. *Every morphism $f : X \to Y$ in Set_Δ factors as*

$$X \xrightarrow{s} Z \xrightarrow{p} Y$$

where s is in the saturated hull of $\{\partial\Delta^n \to \Delta^n\}$, and p has the RLP with respect to the morphisms in $\{\partial\Delta^n \to \Delta^n\}$.

5 Digression: Mapping spaces

We now pause in our contemplation of model categories to use some of the machinery we have developed. Viewing simplicial sets as an analogue of spaces, we want to define *mapping spaces* between two simplicial sets.

Definition 3.44. Let K and X be simplicial sets. The *simplicial mapping space* $\mathcal{M}\mathrm{ap}(K,X) \in \mathrm{Set}_\Delta$ is the simplicial set defined by the isomorphism

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathcal{M}\mathrm{ap}(K,X)) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(K \times \Delta^n, X).$$

The functoriality in Δ^{op} is given by the functoriality of $K \times \Delta^{(-)}$.

Proposition 3.45. *There is an adjunction*

$$K \times (-) : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Set}_\Delta : \mathcal{M}\mathrm{ap}(K,-).$$

Proof. We prove the proposition by providing a bijection

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(Z, \mathcal{M}\mathrm{ap}(K,X)) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(K \times Z, X).$$

natural in X and Z.

We first introduce the *evaluation map*:

$$\mathrm{ev} : K \times \mathcal{M}\mathrm{ap}(K,X) \longrightarrow X$$

which sends a k-simplex $(\sigma, \gamma) \in K \times \mathcal{M}\mathrm{ap}(K,X)$ to the simplex $\gamma(\sigma \times \mathrm{id}_{\Delta^n})$. We can then define a map

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(Z, \mathcal{M}\mathrm{ap}(K,X)) \longrightarrow \mathrm{Hom}_{\mathrm{Set}_\Delta}(K \times Z, X)$$

which sends $f : Z \to \mathcal{M}\mathrm{ap}(K,X)$ to the map

$$\begin{aligned} \tilde{f} : K \times Z &\longrightarrow X \\ (\sigma, \tau) &\longmapsto \mathrm{ev}(\sigma, f(\tau)). \end{aligned}$$

In the other direction, we can define a map

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(K \times Z, X) \longrightarrow \mathrm{Hom}_{\mathrm{Set}_\Delta}(Z, \mathcal{M}\mathrm{ap}(K,X))$$

which sends $f : K \times Z \to X$ to the map

$$\begin{aligned} \hat{f} : Z &\longrightarrow \mathcal{M}\mathrm{ap}(K,X) \\ \sigma &\longmapsto f|_{K \times \sigma} \end{aligned}$$

It is straightforward to check that these maps are mutually inverse. $\square$

We now want to explore the behavior of mapping spaces with respect to anodyne morphisms and Kan fibrations.

Definition 3.46. Let $f : A \to B$ and $g : C \to D$ be morphisms of simplicial sets. We define the *pushout-product* of f and g to be the canonical map

$$f \wedge g : A \times D \coprod_{A \times C} B \times C \longrightarrow B \times D.$$

Example 3.47. Let $f : \{0\} \to \Delta^1$, and $g : \Delta^{\{0,1\}} \to \Delta^2$. Then the source of the pushout-product $f \wedge g$ can be drawn schematically as

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & (0,2) \\ (1,0) \arrow[r] & (1,1) \end{tikzcd}$$

as a subset of $[1] \times [2]$.

One key facet of the interplay between the pushout-product and mapping spaces is the use of *adjoint lifting problems.*

Lemma 3.48. *Let $p : X \to Y$, $f : A \to B$, and $g : C \to D$ morphisms in* Set_Δ*. Then a lifting problem*

$$\begin{tikzcd} A \times D \coprod_{A \times C} B \times C \arrow[r] \arrow[d, "f \wedge g"'] & X \arrow[d, "p"'] \\ B \times D \arrow[r] & Y \end{tikzcd}$$

uniquely corresponds to a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & \mathcal{M}\mathrm{ap}(D, X) \arrow[d] \\ B \arrow[r] & \mathcal{M}\mathrm{ap}(C, X) \times_{\mathcal{M}\mathrm{ap}(C,Y)} \mathcal{M}\mathrm{ap}(D, Y) \end{tikzcd}$$

Moreover, the former lifting problem has a solution if and only if the latter does.

Exercise 12. Formulate and prove a version of 3.48 for a general pair of adjoint functors

$$L : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Set}_\Delta : R.$$

Exercise 13. Let $\mathcal{B}$ be a set of morphisms in Set_Δ, and let $\mathcal{C}$ be a saturated set of morphisms in Set_Δ. Let $\mathcal{Q}$ denote the set of morphisms f in Set_Δ such that $f \wedge g$ is in $\mathcal{C}$ for any $g \in \mathcal{B}$.

1. Show that $\mathcal{Q}$ is saturated.

Proposition 3.49. *Define*

$$\mathcal{B}_1 := \{\Lambda^n_i \to \Delta^n\}$$

$$\mathcal{B}_2 := \left\{ \{i\} \times \Delta^n \coprod_{\{i\} \times \partial\Delta^n} \Delta^1 \times \partial\Delta^n \to \Delta^1 \times \Delta^n \mid i \in \{0, 1\} \right\}$$

$$\mathcal{B}_3 := \left\{ \{i\} \times B \coprod_{\{i\} \times A} \Delta^1 \times A \to \Delta^1 \times B \mid i \in \{0, 1\}, \quad A \hookrightarrow B \text{ monic} \right\}$$

Then $\overline{\mathcal{B}_1} = \overline{\mathcal{B}_2} = \overline{\mathcal{B}_3}$ is the set of anodyne morphisms.

Proof. We first show that $\mathcal{B}_1 \subset \overline{\mathcal{B}_3}$. Fix Λ^n_k with $k < n$, and define morphisms

$$\Delta^n \xrightarrow{\{1\}\times(-)} \Delta^1 \times \Delta^n \xrightarrow{\quad q \quad} \Delta^n$$

where q is defined on the level of posets as

$$\begin{aligned} q : [1] \times [n] &\longrightarrow [n] \\ (i,j) &\longmapsto \begin{cases} j & i = 1 \\ j & i = 0,\ j \leq k \\ k & i = 0,\ j > k \end{cases} \end{aligned}$$

One then has a retract diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \{0\} \times \Delta^n \coprod_{\{0\}\times\Lambda^n_k} \Delta^1 \times \Lambda^n_k \arrow[r] \arrow[d] & \Lambda^n_k \arrow[d] \\ \Delta^n \arrow[r, "\{1\}\times(-)"'] & \Delta^1 \times \Delta^n \arrow[r, "q"'] & \Delta^n \end{tikzcd}$$

For the case $n = k$, take the retract obtained for $k = 0$, and reverse all arrows.

We next show that $\mathcal{B}_2 \subseteq \overline{\mathcal{B}_1}$. We will show one of the two cases — the other is completely analogous. Consider the morphism

$$A = \{0\} \times \Delta^n \coprod_{\{0\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \longrightarrow \Delta^1 \times \Delta^n = B$$

in Set_Δ. Every non-degenerate simplex of $\Delta^1 \times \Delta^n$ is contained one of the 'maximal' $(n+1)$-simplices given by

$$\begin{aligned} \sigma_i : [n+1] &\longrightarrow [1] \times [n] \\ j &\longmapsto \begin{cases} (0,j) & j \leq i \\ (1, j-1) & j > i \end{cases} \end{aligned}$$

for $0 \leq i \leq n$. We can define simplicial sets inductively by setting $A_{n+1} = A$, and

$$A_i = A_{i+1} \cup \sigma_i$$

to get a filtration

$$A = A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_0 = B$$

It will thus suffice for us to show that $A_{i+1} \to A_i$ is anodyne. However, we can notice that the intersection of σ_i and A_{i+1} is precisely a Λ^{n+1}_i-horn. Thus we have a pushout diagram

$$\begin{tikzcd} \Lambda^{n+1}_i \arrow[r, hook] \arrow[d] & \Delta^{n+1} \arrow[d, "\sigma_i"] \\ A^{i+1} \arrow[r, hook] & A_i \end{tikzcd}$$

and $A_{i+1} \to A_i$ is anodyne, as desired.

Finally, we show that $\overline{\mathcal{B}_3} = \overline{\mathcal{B}_2}$. It is immediate that $\overline{\mathcal{B}_2} \subseteq \overline{\mathcal{B}_3}$. Let I be the set of morphisms

$$I := \{\{0\} \to \Delta^1,\ \{1\} \to \Delta^1\}$$

Then by Exercise 13, we see that the set

$$\mathcal{Q} := \{f : A \to B \mid f \wedge g \in \overline{\mathcal{B}_2}\ \forall g \in I\}$$

is saturated. However, this means that $\mathcal{Q}$ is a saturated set containing all the simplex boundary inclusions, and thus contains the monomorphisms. Hence, $\mathcal{B}_3 \subset \overline{\mathcal{B}_2}$, and we are done. □

Corollary 3.50. *For any anodyne morphism f and any monomorphism g, then $f \wedge g$ is anodyne.*

Proof. The set $\{h \mid h \wedge g \text{ is anodyne}\}$ is saturated. It thus suffices to show that for any $h \in \mathcal{B}_3$, the map $h \wedge g$ is anodyne. However, any such h is itself of the form $\ell \wedge s$, where ℓ is $\{i\} \to \Delta^1$, and s is a monomorphism. We thus see that

$$h \wedge g = (\ell \wedge s) \wedge g = \ell \wedge (s \wedge g)$$

since $s \wedge g$ is a monomorphism, this lies in $\mathcal{M}_3$, and thus is anodyne. □

Corollary 3.51. *For any $i : A \to B$ a monomorphism, and any $p : X \to Y$ a Kan fibration, the morphism*

$$\mathcal{M}\mathrm{ap}(B, X) \to \mathcal{M}\mathrm{ap}(B, Y) \times_{\mathcal{M}\mathrm{ap}(A,Y)} \mathcal{M}\mathrm{ap}(A, X)$$

is a Kan fibration.

Proof. This follows from Lemma 3.48. □

Corollary 3.52. *For any Kan complex X and any simplicial set K, the mapping space $\mathcal{M}\mathrm{ap}(K, X)$ is a Kan complex.*

Proof. Apply the previous corollary to $\varnothing \to K$ and $X \to *$. □

We can now prove a statement we asserted at the end of last chapter, bolstering our case that Kan complexes are the right choice of 'good' objects in Set_Δ.

Corollary 3.53. *For X a Kan complex, homotopy of maps $B \to X$ is an equivalence relation on* $\mathrm{Hom}_{\mathrm{Set}_\Delta}(B, X)$.

Proof. A homotopy $H : \Delta^1 \times B \to X$ is, by definition, the same thing as a path $\Delta^1 \to \mathcal{M}\mathrm{ap}(B, X)$. Since paths define an equivalence relation on the vertices of any Kan complex, homotopy of maps is thus an equivalence relation. □

6 Cylinders

The trick here is that, with a model structure in place, we can define a notion of homotopy in the category in question, and then use this to greatly simplify the localization of the category at the weak equivalences.

Definition 3.54. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category, and let X be an object in C. A **cylinder object** $\mathrm{Cyl}(X) \in \mathsf{C}$ is an object equipped with a cofibration

$$X \amalg X \longrightarrow \mathrm{Cyl}(X)$$

and a weak equivalence

$$\mathrm{Cyl}(X) \longrightarrow X$$

such that the canonical map $X \amalg X \longrightarrow X$ factors as

$$X \amalg X \longrightarrow \mathrm{Cyl}(X) \xrightarrow{\simeq} X.$$

Dual Definition 3.54. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category, and let X be an object in C. A **path object** $\mathrm{Path}(X) \in \mathsf{C}$ is an object equipped with a fibration

$$\mathrm{Path}(X) \longrightarrow X \times X$$

and a weak equivalence

$$X \longrightarrow \mathrm{Path}(X)$$

Such that the canonical map $X \longrightarrow X \times X$ factors as

$$X \longrightarrow \mathrm{Path}(X) \longrightarrow X \times X.$$

Example 3.55.

1. In the model structure on Top described above, a cylinder object for X is the space $X \times I$, with structure maps
$$X \amalg X \cong X \times \{0,1\} \hookrightarrow X \times I \xrightarrow{\mathrm{proj}} X.$$
2. In our putative model structure on Set_Δ, a cylinder object for $X \in \mathsf{Set}_\Delta$ is the simplicial set $X \times \Delta^1$, with structure maps
$$X \amalg X \cong X \times \{0,1\} \hookrightarrow X \times \Delta^1 \xrightarrow{\mathrm{proj}} X.$$

Dual Example 3.55.

1. In the model structure on Top described above, a path object for X is the space $\mathcal{M}\mathrm{ap}(I, X)$ of continuous maps with the compact-open topology, equipped with the structure maps
$$X \xrightarrow{\mathrm{const}} \mathcal{M}\mathrm{ap}(I, X) \xrightarrow{\mathrm{ev}_0 \times \mathrm{ev}_1} X \times X.$$
2. In our putative model structure on Set_Δ, a path object for X is the simplicial set X^I, with structure maps
$$X \xrightarrow{\mathrm{const}} X^I \xrightarrow{\mathrm{ev}_0 \times \mathrm{ev}_1} X \times X.$$

Exercise 14. Show that, in any model category C, every object $X \in \mathsf{C}$ has both a path object and a cylinder object.

The main utility of cylinder and path objects is that they allow us to define a notion of homotopy in a model category.

Definition 3.56. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category, and let $f, g : X \to Y$ be a pair of morphisms in C. A **left homotopy** from f to g is a morphism

$$\mathrm{Cyl}(X) \xrightarrow{H} Y$$

from a cylinder object such that the composite map

$$X \amalg X \longrightarrow \mathrm{Cyl}(X) \xrightarrow{H} Y$$

is $f \amalg g : X \amalg X \to Y$.

Dual Definition 3.56. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category, and let $f, g : X \to Y$ be a pair of morphisms in C. A **right homotopy** from f to g is a morphism

$$X \xrightarrow{H} \mathrm{Path}(Y)$$

such that the composite

$$X \xrightarrow{H} \mathrm{Path}(Y) \longrightarrow Y \times Y$$

is $f \times g : X \to Y \times Y$.

Before we proceed, we briefly note that we can always obtain cylinder/path objects in a model category.

Lemma 3.57. *Let* $(\mathsf{C}, \mathcal{C}of, \mathcal{F}ib, \mathcal{W})$ *be a model category, and let* X *be an object of* C. *Then there is a cylinder object for* X.

Proof. We factor the map $\mathrm{id} \amalg \mathrm{id} : X \amalg X \to X$ using (M5) from 3.27. This yields a factorization

$$X \amalg X \xrightarrow{i} C \xrightarrow{s} X$$

where i is a cofibration and s is a trivial fibration (in particular, a weak equivalence). □

Ideally, we'd like our two notions of homotopy to be the same, i.e. that two maps are left-homotopic if and only if they are right-homotopic. This is not always the case, but often, it will be so.

Example 3.58. Let X be a topological space, and let $a, b : * \to X$ be the inclusions of two points. A left homotopy from a to b is a factorization

$$* \amalg * \longrightarrow I \times * \xrightarrow{H} X$$

of the map $a \amalg b : * \amalg * \to X$. In other words, this is a path from a to b in X.

On the other hand, a right homotopy from a to b is a factorization

$$* \longrightarrow X^I \longrightarrow X \times X$$

of the map $a \times b : * \to X \times X$. This is *also* a path from a to b. So, in Top, our two notions agree.

In general, left and right homotopy do not always agree. More importantly, left/right homotopy do not always define an equivalence relation on the set of maps $\mathrm{Hom}_{\mathsf{C}}(X, Y)$.

Example 3.59. In our proposed model structure on Set_Δ, a left homotopy from $f : X \to Y$ to $g : X \to Y$ can be written as a map $X \times \Delta^1 \to Y$.

Consider $\Lambda^2_1 \in \mathsf{Set}_\Delta$. Denote by $i_j : \Delta^0 \to \Lambda^2_1$ the inclusion of the j^{th} vertex. Then there is a homotopy from i_0 to i_1, and a homotopy from i_1 to i_2, but no homotopy from i_0 to i_2. The reason that left homotopy does not define an equivalence relation in this case is precisely because the target Λ^2_1 is not a fibrant object, i.e. a Kan complex.

To rectify these issues, we will need to introduce two special kinds of objects in a model category. We have already met both of these concepts in special cases.

Definition 3.60. Let $(\mathsf{C}, \mathcal{C}of, \mathcal{F}ib, \mathcal{W})$ be a model category. Denote the initial and terminal objects of C by $\varnothing$ and $*$, respectively. An object X in C is called **fibrant** if the unique map

$$X \longrightarrow *$$

is a fibration. Dually, X is called **cofibrant** if the unique map

$$\varnothing \longrightarrow X$$

is a cofibration.

Example 3.61. In our standard model structure on Top, every object is fibrant, and the cofibrant spaces are retracts of cell complexes. In our proposed model structure on Set_Δ, every object is cofibrant, and the fibrant objects are Kan complexes.

Lemma 3.62. *Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category and $X \in \mathsf{C}$ be a cofibrant object. Let $X \amalg X \xrightarrow{i_1 \amalg j_1} \mathrm{Cyl}_1(X) \xrightarrow{s_1} X$ and $X \amalg X \xrightarrow{i_2 \amalg j_2} \mathrm{Cyl}_2(X) \xrightarrow{s_2} X$ be two cylinder objects for X. Then*

1. *The morphisms $i_1 : X \to \mathrm{Cyl}_1(X)$ and $j_1 : X \to \mathrm{Cyl}_1(X)$ are trivial cofibrations.*
2. *The pushout K in the diagram*

$$\begin{tikzcd} X \arrow[r, "j_1"] \arrow[d, "i_2"] & \mathrm{Cyl}_1(X) \arrow[d, "p"] \\ \mathrm{Cyl}_2(X) \arrow[r, "q"] & K \end{tikzcd}$$

is a cylinder object for X when equipped with the maps $p \circ i_1$ and $q \circ j_2$ from $X \to K$ and the map $s : K \to X$ obtained by universal property.

Proof. To see (1), we note that by definition

$$X \xrightarrow{j_1} \mathrm{Cyl}_1(X) \xrightarrow{s_1} X$$

is a factorization of the identity, and $s_1 \in \mathcal{W}$. Since $\mathcal{W}$ satisfies 2-out-of-3, this means that $j_1 \in \mathcal{W}$. Moreover, since X is cofibrant, we can write the inclusion of the second factor $X \to X \amalg X$ as the coproduct of the identity on X with the cofibration $\varnothing \to X$. Since cofibrations are a saturated set, this implies this inclusion is a cofibration. Thus, j_1 is a composite of cofibrations, and so is itself a cofibration.

To see (2), we note that trivial cofibrations form a saturated class, and thus are closed under pushout. Since out argument for (1) shows that j_1 and i_2 are trivial cofibrations, we thus see that p and q must be trivial cofibrations as well. Universal property implies that the diagram

$$\begin{tikzcd} X \arrow[r, "j_1"] \arrow[d, "i_2"] & \mathrm{Cyl}_1(X) \arrow[d, "p"] \arrow[ddr, bend left, "s_1"] & \\ \mathrm{Cyl}_2(X) \arrow[r, "q"] \arrow[drr, bend right, "s_2"'] & K \arrow[dr, dashed, "s"] & \\ & & X \end{tikzcd}$$

induces a unique dashed map s. Since s_1 and p are weak equivalences, 2-out-of-3 implies that s must be as well.

The final verification — that $i_1 \amalg j_2 : X \amalg X \to K$ is a cofibration — is left to the reader. □

Lemma 3.63. *Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. If X is a cofibrant object and Y is any object, then left homotopy is an equivalence relation on $\mathrm{Hom}_{\mathsf{C}}(X, Y)$.*

Proof. First, let us show that f is left-homotopic to f. Let

$$X \amalg X \xrightarrow{i} \mathrm{Cyl}(X) \xrightarrow{s} X$$

be a cylinder object for X. It is then immediate that

$$X \amalg X \xrightarrow{i} \mathrm{Cyl}(X) \xrightarrow{f \circ s} Y$$

is a factorization of $f \amalg f : X \amalg X \to Y$. Thus $f \circ s$ is a homotopy from f to f.

Reflexivity of the relation follows immediately from the fact that if

$$X \amalg X \xrightarrow{i_1 \amalg i_2} \mathrm{Cyl}(X) \xrightarrow{s} X$$

is a cylinder object for X, then so is

$$X \amalg X \xrightarrow{i_2 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{s} X.$$

Transitivity follows immediately from Lemma 3.62, part (2). □

Dualizing, we immediately obtain

Dual Lemma 3.63. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. If Y is a fibrant object and X is any object, then right homotopy is an equivalence relation on $\mathrm{Hom}_{\mathsf{C}}(X, Y)$.

Finally, we aim to show that left and right homotopy describe the same equivalence relation on $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ when X is cofibrant and Y is fibrant.

Lemma 3.64. *Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category, and let X be a cofibrant object. Suppose that $f, g : X \to Y$ are left-homotopic. Then for any path object $\mathrm{Path}(Y)$ of Y, there is a right homotopy*

$$H : X \longrightarrow \mathrm{Path}(Y)$$

from f to g.

Proof. We begin with a cylinder object

$$X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{s} X$$

for X and a homotopy $G : \mathrm{Cyl}(X) \to Y$ from g to f. Additionally, we have fixed a path object

$$Y \xrightarrow{j} \mathrm{Path}(Y) \xrightarrow{p} Y \times Y.$$

We write a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "j \circ f"] \arrow[d, "i_0"] & \mathrm{Path}(Y) \arrow[d, "p"'] \\
\mathrm{Cyl}(X) \arrow[r, "(f \circ s) \times G"'] & Y \times Y
\end{tikzcd}$$

By the dual of Lemma 3.62 (1), we note that the map $\mathrm{Path}(Y) \to Y \times Y$ is a fibration. Additionally, i_0 is a cofibration. This means we can solve the lifting problem represented by this commutative diagram, to get a map

$$\begin{tikzcd} X \arrow[r, "j\circ f"] \arrow[d, "i_0"'] & \mathrm{Path}(Y) \arrow[d, "p"] \\ \mathrm{Cyl}(X) \arrow[r, "(f\circ s)\times G"'] \arrow[ur, "\psi"] & Y \times Y \end{tikzcd}$$

making the diagram commute. We can then check that $\psi \circ i_0 : X \to Y$ defines a homotopy from f to g.

To see this, let $\pi_1, \pi_2 : Y \times Y \to Y$ be the two projections. We then compute

$$\begin{aligned} \pi_1 \circ p \circ \psi \circ i_0 &= \pi_1 \circ ((f \circ s) \times G) \circ i_0 \\ &= f \circ s \circ i_0 \\ &= f \circ \mathrm{id}_X = f \end{aligned}$$

and

$$\begin{aligned} \pi_2 \circ p \circ \psi \circ i_0 &= G \circ i_0 \\ &= g. \end{aligned}$$

completing the proof. □

Corollary 3.65. *Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. Let X be a fibrant object, and Y a cofibrant object. Let $f, g : X \to Y$ be two morphisms. The following are equivalent.*

1. *For any cylinder object $\mathrm{Cyl}(X)$ of X, there is a left homotopy from f to g with cylinder object $\mathrm{Cyl}(X)$.*
2. *f and g are left-homotopic.*
3. *For any path object $\mathrm{Path}(Y)$ of Y, there is a right homotopy from f to g with path object $\mathrm{Path}(Y)$.*
4. *f and g are right-homotopic.*

Proof. Follows immediately from Lemma 3.64 and its dual. □

7 *Localizing model categories*

Our penultimate objective in this chapter will be to see how a model structure on C with weak equivalences $\mathcal{W}$ allows us to describe the homotopy category $\mathsf{C}[\mathcal{W}^{-1}]$ with a minimum of difficulty.

Definition 3.66. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. We call an object $X \in \mathsf{C}$ **fibrant-cofibrant** if it is both fibrant and cofibrant.

Let $X, Y \in \mathsf{C}$ be fibrant-cofibrant objects. We call a morphism $f : X \to Y$ a **homotopy equivalence** in C if there is a morphism $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

Example 3.67.

1. In the model structure on Top, the homotopy equivalences in the model-categorical sense are precisely the classical homotopy equivalences.

2. In our putative model structure on Set_Δ, the homotopy equivalences are the simplicial homotopy equivalences.

Proposition 3.68. *Let* $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ *be a model category, and let* X *and* Y *be fibrant-cofibrant objects in* C*. If a morphism* $f : X \to Y$ *is in* $\mathcal{W}$*, then it is a homotopy equivalence.*

Proof. We first notice that, given a weak equivalence $f : X \to Y$, we can factor it as

$$X \xhookrightarrow[i]{\sim} Z \overset{}{\underset{s}{\twoheadrightarrow}} Y$$

where i is a trivial cofibration. By 2-out-of-3, we see that s is also a weak equivalence, and thus a trivial fibration. It thus suffices to show the statement for trivial fibrations and trivial cofibrations. Since these two cases are dual, it will in fact suffice to prove one of them.

Suppose $f : X \to Y$ is a trivial cofibration. Fix a path object

$$Y \xrightarrow{i} \mathrm{Path}(Y) \xrightarrow{s} Y \times Y$$

for Y. Since f is a trivial cofibration and X is fibrant, we can solve the lifting problem

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "g"] & * \end{tikzcd}$$

to obtain a dashed morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. We then consider the lifting problem

$$\begin{tikzcd} X \arrow[r, "i \circ f"] \arrow[d, "f"'] & \mathrm{Path}(Y) \arrow[d, "s"] \\ Y \arrow[r, "(f \circ g) \times \mathrm{id}_Y"'] \arrow[ur, dashed, "\psi"] & Y \times Y \end{tikzcd}$$

since f is a trivial cofibration and Y is a fibration, we can solve this lifting problem to obtain a morphism $\psi : Y \to \mathrm{Path}(Y)$. The map ψ is, by construction, a right-homotopy from $f \circ g$ to id_Y, completing the proof. □

Remark 3.69. This proposition is sometimes known as *Whitehead's Theorem*, as the version of this statement for topological spaces was proven by Whitehead in the 1940's.

We want to compute the localization $\mathsf{C}[\mathcal{W}^{-1}]$ by restricting to fibrant-cofibrant objects, and taking homotopy classes of maps. The next step towards doing so is to show that every object is a fibrant-cofibrant object up to weak equivalence.

Definition 3.70. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. For X and Y fibrant-cofibrant objects, denote by $\pi_0\mathrm{Hom}_{\mathsf{C}}(X, Y)$ the set of homotopy classes of maps from X to Y in C. We define the **homotopy category** $\mathrm{Ho}(\mathsf{C})$ to have

- Objects the fibrant-cofibrant objects of C.
- Hom-sets given by

$$\mathrm{Hom}_{\mathrm{Ho(C)}}(X, Y) := \pi_0 \mathrm{Hom}_{\mathrm{C}}(X, Y).$$

Exercise 15. Show that Ho(C) is at well-defined category.

Construction 3.71. We aim to construct a canonical functor $\Pi_{\mathrm{C}} : \mathrm{C} \to \mathrm{Ho(C)}$.

We first define Π_{C} on objects. Let $X \in \mathrm{C}$. We can factor the unique morphism

$$\varnothing \longrightarrow X$$

into a cofibration and a trivial fibration.

$$\varnothing \hookrightarrow QX \xrightarrow[p_X]{\sim} X$$

The object QX is cofibrant, and weakly equivalent to X. We call QX a **cofibrant replacement** for X.

We then factor the unique morphism

$$QX \longrightarrow *$$

into a trivial cofibration and a fibration

$$QX \xrightarrow[i_X]{\sim} RQX \longrightarrow *$$

Note that RQX is fibrant, and since the morphism $\varnothing \to RQX$ factors as $\varnothing \hookrightarrow QX \hookrightarrow RQX$, we see that RQX is also cofibrant.

We also fix an important convention: if X is already cofibrant, we choose $QX = X$ and the trivial fibration $QX \twoheadrightarrow X$ to be id_X. Similarly, if X (and thus QX) is already fibrant, we choose $RQX = QX$, and the trivial cofibration $QX \hookrightarrow RQX$ to be id_{QX}. We then can define an assignment on objects to be

$$\Pi_{\mathrm{C}}(X) = RQX.$$

Now let $f : X \to Y$ be a morphism in C. We obtain a diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, hook] & QY \arrow[d, two heads, "p_Y"] \\ QX \arrow[r, "f \circ p_X"'] & Y \end{tikzcd}$$

Since p_Y is a trivial fibration and $\varnothing \to QX$ is a cofibration, we can choose a solution to the lifting problem, i.e., a map $Qf : QX \to QY$ such that the diagram

$$\begin{tikzcd} QX \arrow[r, "Qf"] \arrow[d, "p_X"'] & QY \arrow[d, "p_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

We need this map to be at least unique up to homotopy. Suppose $g : QX \to QY$ is another map making this diagram commute. Let

$$X \amalg X \xrightarrow{(j_0, j_1)} \mathrm{Cyl}(X) \xrightarrow{\ s\ } X$$

be a cylinder object for X. Then we can form a lifting problem

$$\begin{tikzcd} X \amalg X \arrow[r, "(Qf, g)"] \arrow[d, "(j_0, j_1)"'] & QY \arrow[d, two heads, "p_Y"] \\ \mathrm{Cyl}(X) \arrow[r, "f \circ p_X \circ s"'] & Y \end{tikzcd}$$

and solve it to obtain a left-homotopy H from Qf to g.

We now note that, given our choice of Qf, we can play the same game. Form the lifting problem

$$\begin{tikzcd} QX \arrow[r, "i_Y \circ Qf"] \arrow[d, hook, "i_X"'] & RQY \arrow[d, two heads] \\ RQX \arrow[r] & * \end{tikzcd}$$

Solving this yields a map $RQf : RQX \to RQY$ such that the diagram

$$\begin{tikzcd} QX \arrow[r, "Qf"] \arrow[d, "i_X"'] & QY \arrow[d, "i_Y"] \\ RQX \arrow[r, "RQf"] & RQY \end{tikzcd}$$

commutes. A dual argument to that given above shows that RQf is unique up to right-homotopy with this property.

We can thus define $\Pi_{\mathsf{C}}(f) := [RQf]$. Functoriality follows immediate from the uniqueness of RQf up to homotopy.

Theorem 3.72 (Quillen)**.** *Let* $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ *be a model category. The functor*

$$\Pi_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathrm{Ho}(\mathsf{C})$$

exhibits $\mathrm{Ho}(\mathsf{C})$ *as a localization of* C *at* $\mathcal{W}$*.*

Proof. Proposition 3.68 shows that Π_{C} sends morphisms in $\mathcal{W}$ to isomorphisms. This immediately tells us that the functor

$$\Pi_{\mathsf{C}}^* : \mathrm{Fun}(\mathrm{Ho}(\mathsf{C}), \mathsf{D}) \longrightarrow \mathrm{Fun}(\mathsf{C}, \mathsf{D})$$

has essential image contained in $\mathrm{Fun}^{\mathcal{W}}(\mathsf{C}, \mathsf{D})$. To see that this is *precisely* the essential image, suppose we are given a functor $F : \mathsf{C} \to \mathsf{D}$ which sends the morphisms in $\mathcal{W}$ to isomorphisms. Since the set of objects of $\mathrm{Ho}(\mathsf{C})$ is a subset of the objects of C, we can define $\widetilde{F} : \mathrm{Ho}(\mathsf{C}) \to \mathsf{D}$ on objects by $\widetilde{F}(X) = F(X)$.

Now suppose that $f, g : X \to Y$ are homotopic morphisms between fibrant-cofibrant objects of C. Choosing a cylinder object $(j_0, j_1) : X \amalg X \to \mathrm{Cyl}(X)$ with projection

$s : \mathrm{Cyl}(X) \to X$, we obtain a left homotopy $h : \mathrm{Cyl}(X) \to Y$ from f to g, i.e., the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, equal] \arrow[dr, "\sim", "j_0"'] \arrow[drr, bend left, "f"] & & \\ X & \mathrm{Cyl}(X) \arrow[l, "\sim"', "s"] \arrow[r, "h"] & Y \\ X \arrow[u, equal] \arrow[ur, "\sim"', "j_1"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

Applying F to this diagram shows us that $F(f) = F(g)$. Thus, for any homotopy class of morphisms $[f] : X \to Y$ between fibrant-cofibrant objects, there is a well-defined $\widetilde{F}([f]) : \widetilde{F}(X) \to \widetilde{F}(Y)$. We have thus constructed a functor $\widetilde{F} : \mathrm{Ho}(\mathsf{C}) \to \mathsf{D}$.

To see that $\widetilde{F} \circ \Pi_{\mathsf{C}} \cong F$, we note that for each object $X \in \mathsf{C}$ we have chosen a zig-zag of weak equivalences

$$X \xleftarrow[p_X]{\sim} QX \xrightarrow[i_X]{\sim} RQX$$

For any morphism $f : X \to Y$ in C, we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & QX \arrow[l, "\sim", "p_X"'] \arrow[r, "\sim", "i_X"'] \arrow[d, "Qf"] & RQX \arrow[d, "RQf"] \\ Y & QY \arrow[l, "\sim", "p_Y"'] \arrow[r, "\sim", "i_Y"'] & RQY \end{tikzcd}$$

If we apply F to this diagram, the elements of $\mathcal{W}$ become isomorphisms, and the right-hand column becomes $\widetilde{F} \circ \Pi_{\mathsf{C}}(X) \to \widetilde{F} \circ \Pi_{\mathsf{C}}(Y)$. We thus obtain a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(i_X)\circ F(p_X)^{-1}", "\sim"'] \arrow[d, "F(f)"'] & \widetilde{F}(\Pi_{\mathsf{C}}(X)) \arrow[d, "\widetilde{F}(\Pi_{\mathsf{C}}(f))"] \\ Y \arrow[r, "\sim", "F(i_Y)\circ F(p_Y)^{-1}"'] & \widetilde{F}(\Pi_{\mathsf{C}}(Y)) \end{tikzcd}$$

yielding a natural isomorphism $F \cong \tilde{F} \circ \Pi_{\mathsf{C}}$.

We now show that Π_{C}^* is fully faithful. Suppose that we are given $\mu, \nu : F \Rightarrow G$, where $F, G : \mathrm{Ho}(\mathsf{C}) \to \mathsf{D}$, and suppose that $\mu \circ \Pi_{\mathsf{C}} = \nu \circ \Pi_{\mathsf{C}}$. This implies, in particular, that for any fibrant-cofibrant object $X \in \mathsf{C}$

$$\mu_X = \mu_{\Pi_{\mathsf{C}}(X)} = \nu_{\Pi_{\mathsf{C}}(X)} = \nu_X.$$

So $\mu = \nu$, and Π_{C}^* is faithful.

Now suppose we are given two functors $F, G : \mathrm{Ho}(\mathsf{C}) \to \mathsf{D}$, and a natural transformation $\mu : F \circ \Pi_{\mathsf{C}} \Rightarrow G \circ \Pi_{\mathsf{C}}$. For any fibrant-cofibrant object $X \in \mathsf{C}$, we define $\eta_X : F(X) \to G(X)$ to be μ_X. For any $[f] : X \to Y$ in $\mathrm{Ho}(\mathsf{C})$, we then see that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\mu_X"] \arrow[d, "F({[f]})"'] & G(X) \arrow[d, "G({[f]})"] \\ F(Y) \arrow[r, "\mu_Y"'] & G(Y) \end{tikzcd}$$

commutes, so $\eta : F \Rightarrow G$ is a natural transformation.

The final check we must perform is to show $\eta \circ \Pi_{\mathsf{C}} = \mu$. For fibrant-cofibrant objects $X \in \mathsf{C}$, we already know that $\eta_{\Pi_{\mathsf{C}}(X)} = \mu_X$. However, for every $Y \in \mathsf{C}$, we have a commutative diagram

$$\begin{tikzcd} F(\Pi_C(Y)) \arrow[d, "\mu_Y"] & F(\Pi_C(QY)) \arrow[l, "\cong", "F(p_Y)"'] \arrow[r, "\cong", "F(i_Y)"'] \arrow[d, "\mu_{QY}"] & F(\Pi_{\mathsf{C}}(RQY)) \arrow[d, "\mu_{RQY}"] \\ G(\Pi_C(Y)) & G(\Pi_C(QY)) \arrow[l, "\cong", "G(p_Y)"'] \arrow[r, "\cong", "G(i_Y)"'] & G(\Pi_{\mathsf{C}}(RQY)) \end{tikzcd}$$

which shows that μ_Y is uniquely determined by μ_{RQY}. More generally, a natural transformation $F \circ \Pi_{\mathsf{C}} \Rightarrow G \circ \Pi_{\mathsf{C}}$ is uniquely determined by its components at fibrant-cofibrant objects of C. Thus $\Pi_{\mathsf{C}} \circ \eta = \mu$, and the proof is complete. □

8 *Quillen adjunctions*

Our final consideration in the study of model categories is how to relate two model categories to each other. Given any two categories with weak equivalences $(\mathsf{C}, \mathcal{W}_{\mathsf{C}})$ and $(\mathsf{D}, \mathcal{W}_{\mathsf{D}})$, any functor

$$F : \mathsf{C} \longrightarrow \mathsf{D}$$

such that $F(\mathcal{W}_{\mathsf{C}}) \subset \mathcal{W}_{\mathsf{D}}$ will induce a functor on localizations $\mathsf{C}[\mathcal{W}_{\mathsf{C}}^{-1}] \to \mathsf{D}[\mathcal{W}_{\mathsf{D}}^{-1}]$. However, this makes no use of the model-categorical techniques we have developed over the course of this chapter, and thus will likely be quite complicated to work with.

For the remainder of this section, we will fix two model categories $(\mathsf{C}, \mathcal{C}\mathrm{of}_{\mathsf{C}}, \mathcal{F}\mathrm{ib}_{\mathsf{C}}, \mathcal{W}_{\mathsf{C}})$ and $(\mathsf{D}, \mathcal{C}\mathrm{of}_{\mathsf{D}}, \mathcal{F}\mathrm{ib}_{\mathsf{D}}, \mathcal{W}_{\mathsf{D}})$. The basic idea is that, to relate model structures, we will need *two* functors: a left adjoint which respects cofibrations, and a right adjoint which respects fibrations. This is not as strange as it seems — cofibrations are stable under various kinds of colimits, and fibrations under various kinds of limits, so it makes sense to associate left adjoints with cofibrations, and right adjoints with fibrations.

Lemma 3.73. *Let*

$$F : \mathsf{C} \rightleftarrows \mathsf{D} : G$$

be an adjunction. The following are equivalent.

1. *The functor F preserves cofibrations and trivial cofibrations.*

2. *The functor G preserves fibrations and trivial fibrations.*

3. *The functor G preserves fibrations, and the functor F preserves cofibrations.*

4. *The functor G preserves trivial fibrations, and the functor F preserves trivial cofibrations.*

Proof. By adjointness, we see that, for any $i : A \to B$ in C and $p : X \to Y$ in D, there is a correspondence between lifting problems

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & G(X) \arrow[d, "G(p)"] & & F(A) \arrow[r] \arrow[d, "F(i)"'] & X \arrow[d, "p"] \\ B \arrow[r] & G(Y) & \longleftrightarrow & F(B) \arrow[r] & Y \end{tikzcd}$$

Suppose F preserves trivial cofibrations, and let $p \in \mathcal{F}\mathrm{ib}_{\mathsf{D}}$. Then for any $i \in \mathcal{C}\mathrm{of}_{\mathsf{C}} \cap \mathcal{W}_{\mathsf{C}}$, the right-hand lifting problem has a solution, so the left-hand lifting problem also has a solution. Thus $G(p) \in (\mathcal{C}\mathrm{of}_{\mathsf{C}} \cap \mathcal{W}_{\mathsf{C}})_{\perp}$, and so $G(p) \in \mathcal{F}\mathrm{ib}$. Thus G preserves fibrations. Dually, we see that if G preserves fibrations, then F preserves trivial cofibrations.

An completely analogous argument shows that F preserves cofibrations if and only if G preserves fibrations. □

Remark 3.74. As usual, take a moment to convince yourself that the definition of a Quillen adjunction is self-dual. More precisely, check that if $F : \mathsf{C} \leftrightarrow \mathsf{D} : G$ is a Quillen adjunction, then the dual adjunction $G^{\mathrm{op}} : \mathsf{D}^{\mathrm{op}} \leftrightarrow \mathsf{C}^{\mathrm{op}} : F^{\mathrm{op}}$ is a Quillen adjunction between opposite model categories.

Definition 3.75. We call the adjunction $F \dashv G$ a **Quillen adjunction** if any of the equivalent conditions of Lemma 3.73 are satisfied.

Lemma 3.76 (Ken Brown's lemma)**.** *Let $F \dashv G$ be a Quillen adjunction. Then F preserves weak equivalences between cofibration objects. Dually, G preserves weak equivalences between fibrant objects.*

Proof. We show the statement for cofibrant objects. Let $f : A \to B$ be a weak equivalence between cofibrant objects of C. Form the pushout

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & A \arrow[d, "i_A"] \\ B \arrow[r, "i_B"] & A \amalg B \end{tikzcd}$$

and note that i_A and i_B are cofibrations.

Consider the map $(f, \mathrm{id}_B) : A \amalg B \to B$. We can factor this map as a cofibration q and a trivial fibration p:

$$\begin{tikzcd} A \amalg B \arrow[r, "{(f,\mathrm{id}_B)}"] \arrow[d, hook, "q"'] & B \\ C \arrow[ur, "\sim", "p"'] & \end{tikzcd}$$

Since f, id_B, and p are weak equivalences, it follows by 2-out-of-3 that $q \circ i_A$ and $q \circ i_B$ are weak equivalences, and hence trivial cofibrations.

Applying F, we obtain a diagram

$$\begin{tikzcd} F(B) \arrow[r, "\mathrm{id}_{F(B)}"] \arrow[d, "F(q \circ i_B)"'] & F(B) \\ F(C) \arrow[ur, "F(p)"'] & \end{tikzcd}$$

Since F preserves trivial cofibrations, $F(q \circ i_B)$ is a weak equivalence, and thus, by 2-out-of-3, so is $F(p)$. We also obtain a diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "F(q \circ i_A)"'] & F(B) \\ F(C) \arrow[ur, "F(p)"'] & \end{tikzcd}$$

As before, $F(q \circ i_A)$ is a trivial cofibration, and we already know $F(p)$ is a weak equivalence. Thus $F(f)$ is a weak equivalence, as desired. □

Definition 3.77. Let

$$F : \mathsf{C} \rightleftarrows \mathsf{D} : G$$

be a Quillen adjunction. We denote by C_c the full subcategory of C on the cofibrant objects. Note that the cofibrant replace Q constructed above defines a functor

$$Q : \mathsf{C}[\mathcal{W}_{\mathsf{C}}^{-1}] \longrightarrow \mathsf{C}_c[\mathcal{W}^{-1}]$$

We define the **left-derived functor** of F to be the composite

$$\mathbb{L}F : \mathsf{C}[\mathcal{W}_{\mathsf{C}}^{-1}] \xrightarrow{Q} \mathsf{C}_c[\mathcal{W}_{\mathsf{C}}^{-1}] \xrightarrow{F} \mathsf{D}[\mathcal{W}_{\mathsf{D}}^{-1}]$$

Dual Definition 3.77. Let

$$F : \mathsf{C} \rightleftarrows \mathsf{D} : G$$

be a Quillen adjunction. We denote by D_f the full subcategory of D on the fibrant objects. Note that the fibrant replacement R constructed above defines a functor

$$R : \mathsf{D}[\mathcal{W}_{\mathsf{C}}^{-1}] \longrightarrow \mathsf{D}_f[\mathcal{W}^{-1}]$$

We define the **right-derived functor** of G to be the composite

$$\mathbb{R}G : \mathsf{D}[\mathcal{W}_{\mathsf{D}}^{-1}] \xrightarrow{Q} \mathsf{D}_f[\mathcal{W}_{\mathsf{D}}^{-1}] \xrightarrow{F} \mathsf{C}[\mathcal{W}_{\mathsf{C}}^{-1}]$$

Proposition 3.78. *A Quillen adjunction*

$$F : \mathsf{C} \rightleftarrows \mathsf{D} : G$$

induces an adjunction between derived functors.

$$\mathbb{L}F : \mathsf{C}[\mathcal{W}_{\mathsf{C}}^{-1}] \rightleftarrows \mathsf{D}[\mathcal{W}_{\mathsf{D}}^{-1}] : \mathbb{R}G$$

Moreover, if $\epsilon : \mathrm{id} \Rightarrow G \circ F$ *and* $\eta : F \circ G \Rightarrow \mathrm{id}$ *are the unit and counit of the Quillen adjunction, then the composite*

$$Q(X) \xrightarrow{\epsilon_{Q(X)}} G(F(Q(X))) \xrightarrow{G(i_{F(Q(X))})} G(R(F(Q(X))))$$

is the unit of the derived adjunction.

Proof. We show adjointness. See [7, Proposition 1.3.13] for the characterization of the derived unit. Let

$$\phi_{X,Y} : \mathrm{Hom}_{\mathsf{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathsf{C}}(X, G(Y))$$

be the natural isomorphism of our (underived) adjunction.

We begin with two natural isomorphisms, which are a consequence of our characterization of $\mathsf{C}[\mathcal{W}_{\mathsf{C}}^{-1}] \cong \mathrm{Ho}(\mathsf{C})$ from Theorem 3.72. There is an isomorphism natural in X and Y:

$$\mathrm{Hom}_{\mathsf{D}[\mathcal{W}_{\mathsf{D}}^{-1}]}(F(Q(X)), Y) \cong \mathrm{Hom}_{\mathsf{D}}(F(Q(X)), R(Y))_{/\sim}$$

where the latter denotes the quotient by the relation of left/right homotopy. Similarly, there is an isomorphism natural in X and Y

$$\mathrm{Hom}_{\mathsf{C}[\mathcal{W}_{\mathsf{C}}^{-1}]}(X, G(R(Y))) \cong \mathrm{Hom}_{\mathsf{C}}(Q(X), G(R(Y)))_{/\sim}$$

We want to define a natural isomorphism

$$\mathrm{Hom}_{\mathsf{D}[\mathcal{W}_{\mathsf{D}}^{-1}]}(F(Q(X)), Y) \cong \mathrm{Hom}_{\mathsf{D}}(F(Q(X)), R(Y))_{/\sim} \xrightarrow{\phi_{Q(X),R(Y)}} \mathrm{Hom}_{\mathsf{C}}(Q(X), G(R(Y)))_{/\sim} \cong \mathrm{Hom}_{\mathsf{C}[\mathcal{W}_{\mathsf{C}}^{-1}]}(X, G(R(Y)))$$

To do this requires only that we check that $\phi_{Q(X),R(Y)}$ and its inverse respect homotopy.

We will show the first of these claims. Let

$$R(Y) \xrightarrow{\sim} \mathrm{Path}(R(Y)) \twoheadrightarrow R(Y) \times R(Y)$$

be a path object for $R(Y)$. And let $H : F(Q(X)) \to \text{Path}(R(Y))$ be a homotopy from $f : F(Q(X)) \to R(Y)$ to $g : F(Q(X)) \to R(Y)$.

Since G preserves products, fibrations, fibrant objects, and weak equivalences between fibrant objects, we see that $G(\text{Path}(R(Y)))$ is a path object for $G(R(Y))$. By the naturality of ϕ, we then see that

$$\phi(H) : Q(X) \to G(\text{Path}(R(Y)))$$

is a right homotopy between $\phi(f)$ and $\phi(g)$. □

Definition 3.79. We call a Quillen adjunction

$$F : \mathsf{C} \rightleftarrows \mathsf{D} : G$$

a **Quillen equivalence** if the left- and right-derived functors are equivalences of localized categories.

Theorem 3.80 (Quillen). *The adjunction*

$$|-| : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Top} : \mathrm{Sing}$$

is a Quillen equivalence between the Kan-Quillen model structure on Set_Δ *and the classical model structure on topological spaces.*

Proof. See [10], [7], or [5]. □

.

4

Quasi-categories

We now have all the pieces in place to make the key definition of this course. Let us recall several ideas we have developed to date:

1. An ∞-category in which all morphisms are invertible (i.e., and ∞-groupoid) is the same thing as a topological space.

2. A topological space can be modeled as a *Kan complex* — a simplicial set X such that $X \to \Delta^0$ has the right lifting property with respect to all horn inclusions.

3. Every 1-category is an ∞-category.

4. Categories can be identified with those simplicial sets X such that every lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] & \Delta^0 \end{tikzcd}$$

with $n \geq 2$ and $0 < i < n$ admits a *unique* solution.

Motivated by these facts, we make the following definition.

Definition 4.1. An **inner horn inclusion** is a horn inclusion $\Lambda^n_i \to \Delta^n$ with $n \geq 2$ and $0 < i < n$.

We call $X \in \mathrm{Set}_\Delta$ a **quasi-category** if the unique morphism $X \to \Delta^0$ in Set_Δ has the right lifting property with respect to every inner horn inclusion.

Quasi-categories — first introduced by Boardman and Vogt, and developed further by Joyal, Lurie, and many others — will be our chosen model for ∞-categories.

We now aim to develop some technology for quasi-categories, in analogy with our development of homotopy theory for spaces.

Definition 4.2. We will denote by $\mathcal{IA}$ the saturated hull of the inner horn inclusions

$$\{\Lambda^n_i \to \Delta^n\}_{0<i<n}.$$

We call $\mathcal{IA}$ the **inner anodyne** morphisms in Set_Δ. [1]

We call the set $\mathcal{IA}_\perp$ the **inner fibrations**. Note that X is a quasi-category precisely when $X \to \Delta^0$ is an inner fibration.

[1] **Warning.** *In the Kan-Quillen model structure on* Set_Δ*, the anodyne morphisms were precisely the trivial cofibrations. However, while we will eventually define another model structure on* Set_Δ *— the* Joyal *model structure — which models ∞-categories, the inner anodyne maps will not be precisely the trivial cofibrations in this model structure. Rather, we will have an inclusion $\mathcal{IA} \subset \mathcal{C}\mathrm{of} \cap \mathcal{W}$.*

Consequently, the inner fibrations are not *the fibrations in the Joyal model structure. However, the quasi-categories* are *the fibrant objects.*

Exercise 16. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor of 1-categories. Show that $N(F) : N(\mathsf{C}) \to N(\mathsf{D})$ is an inner fibration of ∞-categories.

When discussing quasi-categories, we will use some different notation for familiar objects.

Notation 4.3. Let X and Y be quasi-categories. We denote by $\mathrm{Fun}(X, Y)$ the simplicial mapping space $\mathcal{M}\mathrm{ap}(X, Y)$, and we call it the **functor quasi-category**.

We aim to show that, this is, indeed, a quasi-category.

Definition 4.4. We define the following sets of morphisms in Set_Δ:

$$\mathcal{I}_1 := \{\Lambda^n_i \to \Delta^n \mid 0 < i < n\}$$

$$\mathcal{I}_2 := \left\{ \Delta^m \times \Lambda^2_1 \coprod_{\partial\Delta^m \times \Lambda^2_1} \partial\Delta^m \times \Delta^2 \to \Delta^m \times \Delta^2 \mid m \geq 0 \right\}$$

$$\mathcal{I}_3 := \left\{ S \times \Lambda^2_1 \coprod_{\partial K \times \Lambda^2_1} K \times \Delta^2 \to \Delta^m \times \Delta^2 \mid K \to S \text{ monomorphism} \right\}$$

Proposition 4.5. $\overline{\mathcal{I}_1} = \overline{\mathcal{I}_2} = \overline{\mathcal{I}_3}$.

Proof. The proof is analogous to that of Proposition 3.49, though somewhat more technical, and is left as an exercise to the reader. □

Corollary 4.6. *Let $X \in \mathrm{Set}_\Delta$ be a quasi-category. Then* $\mathrm{Fun}(Y, X)$ *is a quasi-category for every $Y \in \mathrm{Set}_\Delta$.*

1 *The Bergner & Joyal model structures*

Intuitively, an ∞-category should be something like a category $\mathcal{C}$ with spaces $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y)$ instead of sets. However, the composition, associativity, and unitality of this category should not be strict, but rather should only hold up to a coherently chosen set of homotopies.[2]

There is, however, a stricter notion of a category with mapping *spaces* of some kind instead of hom-sets: an *enriched category*. We will use a particular kind of enriched category in our discussion of quasi-categories.

[2] We leave aside, for the moment, what precisely we mean by "coherent". A quasi-categorical avatar of the non-uniqueness of composition is the fact that, in a quasi-category $\mathcal{C}$, each horn Λ^2_1 representing a pair of composable morphisms may have multiple different fillings to a simplex Δ^2, which yields multiple possible composites.

Definition 4.7. A *simplicially enriched category* $\mathcal{C}$ — or *simplicial category* for short — consists of

- A set $\mathrm{Ob}(\mathcal{C})$, called the *objects* of $\mathcal{C}$.
- For each $x, y \in \mathrm{Ob}(\mathcal{C})$, a simplicial set $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y) \in \mathrm{Set}_\Delta$ called the *mapping space* from x to y.
- For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$, a morphism

$$- \circ - : \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y) \times \mathcal{M}\mathrm{ap}_{\mathcal{C}}(y, z) \longrightarrow \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, z)$$

  of simplicial sets called *composition*.

- For each $x \in \mathrm{Ob}(\mathcal{C})$, an element $\mathrm{id}_x \in \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,x)_0$, called the *identity on* x.

These data are then subject to the usual unitality and associativity conditions, i.e. for $x, y, z, w\, \mathrm{Ob}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,y) \times \mathcal{M}\mathrm{ap}_{\mathcal{C}}(y,z) \times \mathcal{M}\mathrm{ap}_{\mathcal{C}}(z,w) \arrow[r, "\mathrm{id}\times(-\circ-)"] \arrow[d, "(-\circ-)\times\mathrm{id}"'] & \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,y) \times \mathcal{M}\mathrm{ap}_{\mathcal{C}}(y,w) \arrow[d, "-\circ-"] \\
\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,z) \times \mathcal{M}\mathrm{ap}_{\mathcal{C}}(z,w) \arrow[r, "-\circ-"'] & \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,w)
\end{tikzcd}$$

and, for any $x, y \in \mathrm{Ob}(\mathcal{C})$, the diagrams

$$\begin{tikzcd}
\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,y) \times \Delta^0 \arrow[r, "\mathrm{id}\times\{\mathrm{id}_y\}"] \arrow[dr, "\cong"'] & \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,y) \times \mathcal{M}\mathrm{ap}_{\mathcal{C}}(y,y) \arrow[d, "-\circ-"] \\
& \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Delta^0 \times \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,y) \arrow[r, "\{\mathrm{id}_x\}\times\mathrm{id}"] \arrow[dr, "\cong"'] & \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,x) \times \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,y) \arrow[d, "-\circ-"] \\
& \mathcal{M}\mathrm{ap}_{\mathcal{C}}(x,y)
\end{tikzcd}$$

commute.

Definition 4.8. Given a simplicially enriched category $\mathcal{C}$, we can define the *homotopy category* $h\mathcal{C}$ of $\mathcal{C}$ to have $\mathrm{Ob}(h\mathcal{C})$ the objects of $\mathcal{C}$, and hom-sets given by path components of the mapping spaces.

$$h\mathcal{C}(x,y) := \pi_0\left(\mathcal{C}(x,y)\right).$$

As in the world of 2-categories and bicategories, we might hope that we can ”strictify” an infinity category $\mathcal{C}$ so as to give a strict composition operation which is coherently homotopy equivalent to the non-strict composition operation. More precisely, we would like to define a “strictification functor”

$$\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta.$$

Which we will call the *rigidification*[3]. In fact, we are going to find that there is a Quillen adjunction

$$\mathfrak{C} : \mathsf{Set}_\Delta \leftrightarrow \mathsf{Cat}_\Delta : N_\Delta$$

Fortunately for us, we already have a very good technique for generating left-adjoint functors out of Set_Δ — cosimplicial objects! So we want to describe a cosimplicial object

$$\mathfrak{C} : \Delta \to \mathsf{Cat}_\Delta.$$

To this end, lets think a little bit about how we interpret a simplicial set as a category, and what we would want from a strictification. Fairly obviously we want to think of the

[3] Following the terminology of [3]. Unfortunately, the functor $\mathfrak{C}$ is still fairly commonly referred to as “The left adjoint to the homotopy coherent nerve” in the literature. Cumbersome, to say the least.

0-simplices as objects, and the 1-simplices as morphisms. In this light, how do we think of a 2-simplex

$$\sigma = \begin{tikzcd} & y \arrow[dr, "f"] & \\ x \arrow[ur, "g"] \arrow[rr, "h"'] & & z \end{tikzcd} \qquad ?$$

Following our discussion above, we would generically say that σ "displays h as a composite of f with g." However, this is no good for rigidification. When we rigidify, we want to define a *strict* composite of f with g, rather than many possible, equivalent composites.

To make this possible, we simply define the strict composite $f \circ g = fg$ to be the *orieted path* of 1-simplices which first traces g and then f. We can then interpret our 2-simplex σ as displaying a *specific* 2-isomorphism $h \Rightarrow fg$. Our mapping space can thus be seen as the simplicial set

$$\Delta^1 = \begin{tikzcd} \overset{h}{\bullet} \arrow[r] & \overset{fg}{\bullet} \end{tikzcd} \cong \mathcal{M}\mathrm{ap}(x, z)$$

So how do we interpret a 3-simplex

$$\sigma = \begin{tikzcd} & & 2 \arrow[dr, "f"] & \\ 0 \arrow[urr, "v"] \arrow[rrr, "w"] \arrow[dr, "h"'] & & & 3 \\ & 1 \arrow[uur, "g"] \arrow[urr, "u"'] & & \end{tikzcd} \qquad ?$$

Well, let's think about the mapping space from 0 to 3. Counting the "strict composites" we've added, we have four morphisms from 0 to 3: w, uh, fv, and fgh. The 2-simplices give us 2-isomorphisms

$$\begin{aligned} \alpha &: w \Rightarrow fv \\ \beta &: w \Rightarrow uh \\ \gamma &: uh \Rightarrow fgh \\ \delta &: fv \Rightarrow fgh \end{aligned}$$

As with our interpretation of the 2-simplex, the 3-simplex itself is then interpreted as displaying a specific 3-isomorphism

$$\gamma \circ \beta \Rrightarrow \delta \circ \alpha$$

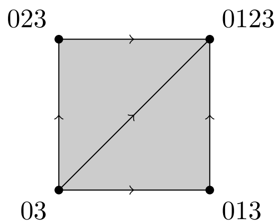
Viewed as a simplicial set, this is then

$$\begin{tikzcd} \overset{fv}{\bullet} \arrow[r] & \overset{fgh}{\bullet} \\ \underset{w}{\bullet} \arrow[u] \arrow[r] \arrow[ur] & \underset{uh}{\bullet} \arrow[u] \end{tikzcd} = \mathcal{M}\mathrm{ap}(0, 3)$$

Notice that, to express the isomorphism between $\delta \circ \alpha$ and $\gamma \circ \beta$, we had to define a new 2-morphism $w \Rightarrow fgh$. So we really interpret the 3-simplex as this 2-isomorphism $\mu : w \Rightarrow fgh$ together with two 2-isomorphisms $\mu \Rrightarrow \delta \circ \alpha$ and $\mu \Rrightarrow \gamma \circ \beta$.

We then notice that we can do this more cleverly. Instead of giving the 1-simplices names, we can identify each string of 1-simplices in Δ^n with the set of vertices it passes through.[4] Relabelling our mapping space, we get

[4] Note that this doesn't work in a generic simplicial set, but does work in the nerve of a poset.



$= \mathfrak{Map}(0,3)$

But this is just the nerve of the poset $P_{0,3}$ whose objects are subsets of $[3]$ containing both 1 and 3. With this in mind, we can make a preliminary definition our cosimplicial object $\mathfrak{C}$.

It's worth commenting here on why we don't simply use the cosimplicial object $[n] \mapsto [n]$, where $[n]$ is viewed as a simplicial category with discrete mapping spaces. The reason is that this collapses precisely the coherence data that we described above, and in more complicated infinity categories, will destroy much of the coherent data encoded in quasi-categories. There is, however, a morphism

$$\mathfrak{C}[\Delta^n] \to [n]$$

which is an isomorphism on objects, and induces a weak homotopy equivalence on mapping spaces. We will later see that these functors are weak equivalences in the relevant model structure on Cat_Δ.

Definition 4.9 (Rigidification)**.** We define a cosimplicial object called the **rigidification** to be the functor

$$\mathfrak{C} \colon \Delta \to \mathsf{Cat}_\Delta$$

defined as follows.

It sends objects $[n]$ to the simplicially enriched category $\mathfrak{C}[\Delta^n]$ defined as follows.

- $\mathrm{Ob}(\mathfrak{C}[\Delta^n]) = \{0, \dots, n\}$.
- For $i, j \in \{0, \dots, n\}$
$$\mathfrak{C}[\Delta^n](i,j) = N(P_{i,j}),$$
where $P_{i,j}$ is the poset of subsets of $\{i, \dots, j\}$ containing i and j.
- To define composition, we need a morphism of simplicial sets
$$N(P_{i,j}) \times N(P_{j,k}) \to N(P_{i,k}).$$
We have a multiplication $P_{i,j} \times P_{j,k} \to P_{i,k}$ given by $(I, J) \mapsto I \cup J$. This induces the composition map on nerves.

The poset $P_{i,j}$ can be equivalently described as the power set of the set $L_{i,j} := \{k \in [n] \mid i < k < j\}$, and is thus, geometrically, a cube.

A morphism $f \colon [m] \to [n]$ induces a functor $\mathfrak{C}[\Delta^m] \to \mathfrak{C}[\Delta^n]$ on objects by $i \mapsto f(i)$, and on hom-simplicial-sets $N(P_{i,j})$ as follows. On the underlying posets, we send

$$P_{i,j} \ni \{i, k_1, \dots, k_{r-1}, j\} \mapsto \{f(i), f(k_1), \dots, f(k_{r-1}), f(j)\} \in P_{f(i),f(j)}.$$

This extends to a functor of poset categories, hence a morphism of nerves.

Taking the Yoneda extension of the rigidification, we find a functor $\mathfrak{C} \colon \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$, also called the **rigidification**.

By the usual theory of cosimplicial objects (explained in Section **??**) we have the following results.

Lemma 4.10. *The rigidification $\mathfrak{C}$ preserves colimits.*

Definition 4.11 (homotopy-coherent nerve)**.** Let $\mathcal{C}$ be a simplicial category. The **homotopy-coherent nerve** of $\mathcal{C}$, denoted N_Δ, is the functor

$$N_\Delta(\mathcal{C}) = \mathsf{Cat}_\Delta(\mathfrak{C}(-), \mathcal{C}) \colon \Delta\mathrm{op} \to \mathsf{Set}_\Delta.$$

We thus have an adjunction

$$\mathfrak{C} : \mathsf{Set}_\Delta \leftrightarrow \mathsf{Cat}_\Delta : N_\Delta.$$

The right adjoint of our adjunction fits into a commutative triangle of right Quillen functors

$$\begin{tikzcd} & \mathsf{Cat}_\Delta \arrow[dr, "N_\Delta"] & \\ \mathsf{Cat} \arrow[ur, "\mathrm{disc}"] \arrow[rr, "N"'] & & \mathsf{Set}_\Delta \end{tikzcd}$$

where disc includes categories into Cat_Δ as simplicial categories with discrete mapping spaces.

While it would go far beyond the scope of this document to prove them, we will state and use the following theorems.

Theorem 4.12 (Bergner)**.** *There is a model structure on the category* Cat_Δ *with*

(WE) weak equivalences given by those functors $F : \mathcal{C} \to \mathcal{D}$ such that $hf : h\mathcal{C} \to h\mathcal{D}$ is essentation surjective and which induce weak homotopy equivalences of simplicial sets

$$F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(x, y)$$

for every $x, y \in \mathcal{C}$.[5]

(F) Fibrations given by those functors $F : \mathcal{C} \to \mathcal{D}$ such that $hF : h\mathcal{C} \to h\mathcal{D}$ is an isofibration and which induce Kan fibrations

$$F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(x, y)$$

for every $x, y \in \mathcal{C}$.

In particular, the fibrant objects are precisely those $\mathcal{C} \in \mathsf{Cat}_\Delta$ such that each $\mathcal{M}\mathrm{ap}_{\mathcal{C}}(x, y)$ is a Kan complex for every $x, y \in \mathrm{Ob}(\mathcal{C})$.

[5] This can be viewed as an ∞-categorical version of a very familiar theorem: A functor between 1-categories is an equivalence of categories if and only if it is essentially surjective and fully faithful (induces bijections on mapping spaces). Our version says that a functor $F : \mathcal{C} \to \mathcal{D}$ of simplicial categories is an equivalence if and only if it is essentially surjective and *homotopy fully faithful*, i.e. induces homotopy equivalences on all mapping spaces.

Using simplicially-enriched categories, we can also now define the weak equivalences of quasi-categories, and thus the model structure on quasi-categories.

Theorem 4.13 (Joyal)**.** *There is model structure on the category* Set_Δ *with*

(C) The cofibrations are the monomorphisms

(WE) A functor $f : X \to Y$ is a weak equivalence if and only if the induced functor

$$\mathfrak{C}[f] : \mathfrak{C}[X] \to \mathfrak{C}[Y]$$

is an equivalence in the Bergner model structure.

(FO) The fibrant objects are precisely the quasi-categories.

Remark 4.14 (Key observation). It is very important to notice that, while the Joyal and Kan-Quillen model structures are not the same, they have the same cofibrations. As a result, the have the same trivial fibrations: those $X \to Y$ which have the RLP against every simplex boundary inclusion.

Theorem 4.15 (Joyal)**.** *The adjoint functors*

$$\mathfrak{C} : \mathsf{Set}_\Delta \leftrightarrow \mathsf{Cat}_\Delta : N_\Delta$$

define a Quillen equivalence between the Joyal model structure and the Bergner model structure.

We will later see some exemplar computations to show that N_Δ preserves, e.g, fibrant objects.

2 Computing rigidifications

Let's go back and formalize some of our intuition about mapping spaces in $\mathfrak{C}[X]$. The k-simplices in $\mathfrak{C}[\Delta^n](i,j)$ have the form

$$\sigma := S_0 \subset S_1 \subset \cdots \subset S_k$$

where each $S_\ell \subset [n]$ such that $i, j \in S_\ell$ and, for all $s \in S_\ell$, $i \leq s \leq j$. We will reformulate the data involved in σ to obtain a new interpretation of the mapping spaces. We first write

$$S_0 = \{i = a_1 \leq a_2 \leq \cdots \leq a_r = j\}$$

and then define

$$S_k^\ell := \{s \in S_k \mid a_\ell \leq a_{\ell+1}\}.$$

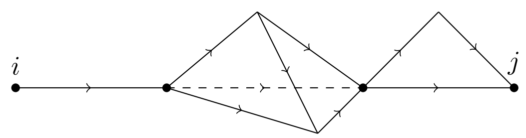
We can think of S_k^ℓ as an n-simplex in Δ^n with initial vertex a_ℓ and final vertex $a_{\ell+1}$. We can thus completely encode the information of S_0 and S_k in the string of simplices

$$S_k^1, S_k^2, \cdots, S_k^{r-1}$$

in Δ^n. Note that

- the final vertex of S_k^ℓ is the intitial vertex of $S_k^{\ell+1}$
- the initial vertex of S_k^1 is i, and
- the final vertex of S_k^{r-1} is j.

We can visualize this information as a *necklace* of n-simplices:



Definition 4.16. The **necklace** of shape $k_1, \ldots, k_r$ is the simplicial set

$$\mathcal{N} = \Delta^{k_1} \coprod_{\Delta^0} \Delta^{k_2} \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^{k_r} =: \Delta^{k_1} \vee \Delta^{k_2} \vee \cdots \vee \Delta^{k_r}$$

The images in $\mathcal{N}$ of the vertices $0, k_\ell \in \Delta^{k_\ell}$ are called the *joints* of $\mathcal{N}$ and the set of joints of $\mathcal{N}$ is denoted by $J_\mathcal{N}$. The set of vertices of $\mathcal{N}$ is denoted by $V_\mathcal{N}$.

The category Nec of necklaces is the subcategory of Set_Δ whose objects are necklaces, and whose morphisms are maps of simplicial sets $f : \mathcal{N} \to \mathcal{M}$ preserving the minimal and maximal joints.

We have thus shown

Lemma 4.17. *A k-simplex in $\mathfrak{C}[\Delta^n](i,j)$ is equivalently the following data:*

- *A necklace $\mathcal{N}$.*
- *A collection of subsets*
$$J_{\mathcal{N}} = S_0 \subset S_1 \subset \cdots \subset S_k = V_{\mathcal{N}}$$
- *A map $f : \mathcal{N}_{\vec{k}} \to \Delta^n$ sending the lowest joint to i and the highest joint to j.*

the simplex is degenerate if and only if there exists $0 \leq \ell \leq k$ such that $S_\ell = S_{\ell+1}$.

This may seem like we have unnecessarily complicated the definition of the rigidification. However this reformulation will allow us to give a generic description of mapping spaces in the rigidification of a simplicial sets.

Given a simplicial set K and vertices $x, y \in K$, we define $(\mathsf{Nec}_{/K})_{x,y}$ as the full subcategory of $\mathsf{Nec}_{/K}$ on those maps $\mathcal{N} \to K$ which send the initial joint to x and the final joint to y. By the functoriality of the rigidification, given a map $\mathcal{N} \to K$ in $(\mathsf{Nec}_{/K})_{x,y}$, we get a map
$$\mathfrak{C}[\mathcal{N}](x,y) \to \mathfrak{C}[K](x,y)$$
where we abuse notation by denoting by x the lowest joint of $\mathcal{N}$ and by y the highest joint of $\mathcal{N}$. These piece together to form a cone, so that we have a canonical map
$$\operatorname*{colim}_{(\mathsf{Nec}_{/K})_{x,y}} \mathfrak{C}[\mathcal{N}](x,y) \to \mathfrak{C}[K](x,y)$$
We can define a simplicial category E_K whose objects are the vertices of K, and with
$$E_K(x,y) := \operatorname*{colim}_{(\mathsf{Nec}_{/K})_{x,y}} \mathfrak{C}[\mathcal{N}](x,y).$$
The composition operation is given by "gluing necklaces", i.e. given $f : \mathcal{N} \to K$ in $(\mathsf{Nec}_{/K})_{a,b}$ and $g : \mathcal{M} \to K$ in $(\mathsf{Nec}_{/K})_{b,c}$ we can construct a new map
$$f \vee g : \mathcal{N} \vee \mathcal{M} := \mathcal{N} \coprod_{\Delta^{\{b\}}} \mathcal{M} \to K$$
in $(\mathsf{Nec}_{/K})_{a,c}$. It is a matter of unwinding the definitions to show that the canonical maps above piece together into a canonical functor
$$E_K \to \mathfrak{C}[K].$$
Note 4.18. The construction $K \mapsto E_K$ extends to a functor
$$E_{(-)} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta.$$
The simplicial functors $E_K \to \mathfrak{C}[K]$ assemble to a natural transformation $\mu : E_{(-)} \Rightarrow [C]$. Each of the functors μ_K is bijective on objects.

Before characterizing mapping spaces in general, we need some more facts about necklaces.

Definition 4.19. We define a functor

$$C_{(-)} : \mathsf{Nec} \to \mathsf{Cat}_\Delta$$

Which sends $\mathcal{N}$ to the simplicially enriched category $C_\mathcal{N}$ with $C_\mathcal{N}(x,y) = N(P_{x,y})$ where $P_{x,y}$ is the poset whose objects are totally ordered subsets of $V_\mathcal{N}$ starting at x and ending at y, and containing $J_\mathcal{N}$.

Lemma 4.20. *Let $\mathcal{N}$ be a necklace. Then there is an isomorphism*

$$\mathfrak{C}[\mathcal{N}] \cong C_\mathcal{N}$$

of simplicial categories.

Proof. This is definitional if $\mathcal{N}$ is a simplex. We now show that, for two necklaces $\mathcal{N}$ and $\mathcal{M}$, we have

$$C_{\mathcal{N}\vee\mathcal{M}} \cong C_\mathcal{N} \coprod_{C_{\Delta^0}} C_\mathcal{M}.$$

The functor in question is clearly bijective on objects (i.e., that $C_{(-)}$ preserves pushouts). On mapping spaces, This follows by careful examination of the universal property of pushout.

It then follows that, for any necklace

$$\mathcal{N} = \Delta^{k_1} \vee \cdots \vee \Delta^{k_r}.$$

we have

$$\begin{aligned}
\mathfrak{C}[\mathcal{N}] &\cong \mathfrak{C}\Delta^{k_1} \coprod_{\mathfrak{C}[\Delta^0]} \cdots \coprod_{\mathfrak{C}[\Delta^0]} \mathfrak{C}\Delta^{k_r} \\
&\cong C_{\Delta^{k_1}} \coprod_{C_{\Delta^0}} \cdots \coprod_{C_{\Delta^0}} C_{\Delta^{k_r}} \\
&\cong C_\mathcal{N},
\end{aligned}$$

proving the lemma. □

Corollary 4.21. *Suppose K is a necklace. Then the functor*

$$\mu_K : E_K \to \mathfrak{C}[K]$$

is an isomorphism of simplicial categories.

Proof. Since μ_K is bijective on objects, it will suffice to check that it is bijective on mapping spaces. Given $x, y \in K$, let $K_{x,y}$ be the full simplicial subset on those vertices $s \in K$ with $x \leq s \leq y$. Then $K_{x,y}$ is also necklace, and the inclusion $K_{x,y} \to K$ is a terminal object in $(\mathsf{Nec}_{/K})_{x,y}$, so that $E_K(x,y) = \mathfrak{C}[K_{x,y}](x,y)$ and

$$\mu_K : \mathfrak{C}[K_{x,y}](x,y) \to \mathfrak{C}[K](x,y)$$

is the inclusion. However, it follows immediately from 4.20 that this is an isomorphism of simplicial sets. □

Now, given a simplicial set K, the colimit cone exhibiting K as the colimit over its category of simplices $\Delta_{/K}$ gives rise to a canonical functor

$$\operatorname*{colim}_{\Delta_{/K}} E_{(\Delta^n)} \to E_K.$$

Lemma 4.22. *Let K be a simplicial set, and $x, y \in K$. Then the map*

$$\phi : \left(\operatorname*{colim}_{\Delta_{/K}} E_{(\Delta^k)}\right)(x,y) \to E_K(x,y)$$

is surjective.

Proof. Given an n-simplex $\sigma : \Delta^n \to E_K(x,y)$, there is by definition a necklace $\mathcal{N}$, a map $f : \mathcal{N} \to K$, and an n-simplex $\gamma : \Delta^n \to \mathfrak{C}[\mathcal{N}](x,y) \cong E_{\mathcal{N}}(x,y)$ representing σ in the colimit.

We thus have a commutative diagram of simplicial sets

$$\begin{tikzcd} \left(\operatorname{colim}_{\Delta_{/\mathcal{N}}} E_{\Delta^k}\right)(x,y) \arrow[r, "\cong"] \arrow[d, "f"'] & E_{\mathcal{N}}(x,y) \arrow[d, "f"] & \gamma \arrow[d, mapsto] \\ \left(\operatorname{colim}_{\Delta_{/K}} E_{(\Delta^k)}\right)(x,y) \arrow[r, "\phi"'] & E_K(x,y) & \sigma \end{tikzcd}$$

where the top map is an isomorphism by 4.21. We thus see that, by commutativity, there exists an n-simplex in

$$\left(\operatorname*{colim}_{\Delta_{/K}} E_{(\Delta^n)}\right)$$

which is sent to σ under ϕ. $\square$

Proposition 4.23 (Dugger-Spivak)**.** *For any simplicial set K, the canonical functor*

$$E_K \to \mathfrak{C}[K]$$

is an isomorphism of simplicial categories.

Proof. The functor in question is clearly bijective on objects, so it will suffices to show that it is an isomorphism on mapping spaces. For $x, y \in K$, we obtain a commutative diagram

$$\begin{tikzcd} \left(\operatorname{colim}_{\Delta_{/K}} E_{(\Delta^l)}\right)(x,y) \arrow[r, "\phi"] \arrow[d, "\cong"] & E_K(x,y) \arrow[d, "\mu_K"] \\ \left(\operatorname{colim}_{\Delta_{/K}} \mathfrak{C}[\Delta^k]\right)(x,y) \arrow[r, equal] & \mathfrak{C}[K](x,y) \end{tikzcd}$$

where the left-hand map is an isomorphism by 4.21. This thus implies that ϕ is injective. However, by 4.22, ϕ is also surjective, and thus an isomorphism, and thus

$$\mu_K : E_K(x,y) \to \mathfrak{C}[K](x,y)$$

is an isomorphism. $\square$

Note 4.24. This proposition thus allows us to represent simplices of the mapping spaces $\mathfrak{C}[K](x,y)$ explicitly in terms of

- an element $f : \mathcal{N} \to K$ of $(\mathsf{Nec}_{/K})_{x,y}$
- an n-simplex in $\mathfrak{C}[\mathcal{N}](x, y)$

modulo equivalence relations. In the particular case where K is a simplicial subset of the nerve of a poset, this simplifies substantially.

Corollary 4.25 (Dugger-Spivak)**.** *Let $X = N(Q)$ be the nerve of a poset Q. For $x, y \in Q$ define a poset $P_{x,y}$ whose objects are totally ordered chains*

$$x \leq z_1 \leq z_2 \leq \ldots \leq z_k \leq y$$

ordered by inclusion. Then

$$\mathfrak{C}[X](x, y) \cong N(P_{x,y})$$

Proof. This follows from unravelling the relevant colimit. □

Note 4.26. For a simplicial subset $Y \subset N(Q)$ of the nerve of a poset, the mapping space $\mathfrak{C}[Y](x, y) \subset \mathfrak{C}[N(Q)](x, y)$ consists of precisely those simplices such that the corresponding necklace in $N(Q)$ factors through Y.

Using this characterization, we can prove that certain morphisms of simplicial sets induce Bergner equivalences, and thus are themselves equivalences of simplicial sets.

Definition 4.27. Let Δ^n be an n-simplex. The simplicial subset

$$Z_n := \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subset \Delta^n$$

is referred to as the spine of Δ^n.

Lemma 4.28. *The inclusion $i_n : Z_n \to \Delta^n$ is a Joyal equivalence of simplicial sets.*

Proof. It suffices to show that $\mathfrak{C}[i_n] : \mathfrak{C}[Z_n] \to \mathfrak{C}[\Delta^n]$ is a Bergner equivalence. It is clearly essentially surjective (indeed, surjective on vertices).

However, we know that $\mathfrak{C}[\Delta^n](i, j)$ is contractible if $i \leq j$ and empty otherwise. It is immediate from the above theorem that $\mathfrak{C}[Z_n](i, j)$ is a 1-point space if $i \leq j$ and empty otherwise. Thus $\mathfrak{C}[i_n]$ induces homotopy equivalences on mapping spaces. □

We can also use this characterization to prove that N_Δ preserves fibrant objects (which is also an implication of the Quillen equivalence above).

Proposition 4.29. *Let $\mathcal{C}$ be a simplicial category such that for all x, y in $\mathcal{C}$, $\mathcal{C}(x, y)$ is a Kan complex. Then $N_\Delta(\mathcal{C})$ is an ∞-category.*

Proof. We need to show that we can solve the following lifting problem.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & N_\Delta(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Passing to adjuncts, we find the following lifting problem.

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \mathfrak{C}[\Delta^n] \arrow[ur, dashed] & \end{tikzcd}$$

We can view the simplicial category $\mathfrak{C}[\Lambda^n_i]$ as a subcategory of $\mathfrak{C}[\Delta^n]$, where the only data missing is the mapping space

$$\mathfrak{C}[\Delta^n](0,n).$$

Therefore, a solution to the above lifting problem need only fill in this information. This means we need only solve the corresponding lifting problem.

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i](0,n) \arrow[r] \arrow[d, hook] & \mathcal{C}(0,n) \\ \mathfrak{C}[\Delta^n](0,n) \arrow[ur, dashed] & \end{tikzcd}$$

A k-simplex of $\mathfrak{C}[\Lambda^n_i](0,n)$ is a chain

$$S_0 \subset S_1 \subset \ldots \subset S_k$$

of subsets of $[n]$ containing 0 and n, such that, for each consecutive $s,t \in S_0$, the subset $S_k^{s,t} := \{q \in S_k \mid s \leq q \leq t\}$ defines a simplex in Λ^n_i. More precise, there exists $0 \leq j \leq n$ with $j \neq i$ such that $S_k^{s,t} \subset \Delta^{\{0,1,\ldots,\hat{j},\ldots,n\}}$.

Let $\mathfrak{C}[\Lambda^n_i](0,n)^i$ and $\mathfrak{C}[\Delta^n](0,n)^i$ be the full sub-simplicial sets (in the latter case, subposet) on those objects which contain i. We can define a map of posets

$$\mathfrak{C}[\Delta^n](0,n) \to \mathfrak{C}[\Delta^n](0,n)^i$$

sending $S \mapsto S \cup \{i\}$. This is clearly homotopic to the identity, and one can check that both the map and the homotopy descend to

$$\mathfrak{C}[\Lambda^n_i](0,n) \to \mathfrak{C}[\Lambda^n_i](0,n)^i$$

We thus have a commutative diagram

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i](0,n) \arrow[r, "\simeq"] \arrow[d] & \mathfrak{C}[\Lambda^n_i](0,n)^i \arrow[d] \\ \mathfrak{C}[\Delta^n](0,n) \arrow[r, "\simeq"] & \mathfrak{C}[\Delta^n](0,n)^i \end{tikzcd}$$

However, by definition, $\mathfrak{C}[\Delta^n](0,n)^i$ is the image of the composition map

$$\mathfrak{C}[\Delta^n](0,i) \times \mathfrak{C}[\Delta^n](i,n) \to \mathfrak{C}[\Delta^n](0,n).$$

We thus can extend the commutative diagram above to

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i](0,n) \arrow[r, "\simeq"] \arrow[d] & \mathfrak{C}[\Lambda^n_i](0,n)^i \arrow[d] & \mathfrak{C}[\Lambda^n_i](0,i) \times \mathfrak{C}[\Lambda^n_i](i,n) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\ \mathfrak{C}[\Delta^n](0,n) \arrow[r, "\simeq"] & \mathfrak{C}[\Delta^n](0,n)^i & \mathfrak{C}[\Delta^n](0,i) \times \mathfrak{C}[\Delta^n](i,n) \arrow[l, "\cong"'] \end{tikzcd}$$

By 2-out-of-3, $\mathfrak{C}[\Lambda^n_i](0,n) \to \mathfrak{C}[\Delta^n](0,n)$ is a weak homotopy equivalence. $\square$

Notice that we have not only proved the given proposition, we have also shown

Lemma 4.30. *Every inner anodyne morphism is a trivial cofibration in the Joyal model structure.*

Proof. We know that the inner anodyne morphisms are monomorphisms, and the above proof shows that, for any $0 < i < n$, the morphism

$$\mathfrak{C}[\Lambda^n_i] \to \mathfrak{C}[\Delta^n]$$

is a weak equivalence in the Bergner model structure. □

Example 4.31. Let $\mathsf{Kan} \subset \mathrm{Set}_\Delta$ be the full subcategory on Kan complexes. We may consider Kan to be simplicially enriched by taking $\mathsf{Kan}(K, S) = \mathcal{M}\mathrm{ap}(K, S)$. We have seen that each mapping space is a Kan complex. Thus, the simplicial nerve $N_\Delta(\mathsf{Kan})$ is an ∞-category.

Definition 4.32 (∞-category of spaces)**.** The ∞-**category of spaces** is the category

$$\mathcal{S} = N_\Delta(\mathsf{Kan}).$$

3 *Opposites, joins, and overcategories*

Perhaps the most basic construction in category theory is the opposite category. Since we do not want to perform two versions of every construction we give for quasi-categories, we therefore need to give a good higher-categorical notion of an 'opposite quasi-category'.

Definition 4.33. Notice that we have a (unique) isomorphism of categories/posets $[n] \cong [n]^{\mathrm{op}}$. Define a functor

$$\mathrm{op} : \Delta \longrightarrow \Delta$$

which sends each object $[n]$ to the same object $[n]$, and sends a morphism $f : [n] \to [m]$ to the composite morphism

$$[n] \cong [n]^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} [m]^{\mathrm{op}} \cong [m]$$

It is not hard to see that this defines an *isomorphism of categories* from Δ to itself.

Let's unwind what this means, practically, on coface and codegeneracy maps. First note that the isomorphism $[n] \cong [n]^{\mathrm{op}}$ sends $0 \mapsto n$, $1 \mapsto n-1$, or, more generally,

$$i \longmapsto n - i$$

If we start with the coface map $\delta_i : [n-1] \to [n]$ which skips i, we notice that the corresponding map $\mathrm{op}(\delta_i) : [n-1] \to [n]$ is a composite of injective maps of posets, and thus injective. As a result, it must be a coface map itself. Moreover, the only element not in the image is $n - i$, so we see that $\mathrm{op}(\delta^n_i) = \delta^n_{n-i}$. Similarly, we conclude that $\mathrm{op}(\sigma^n_i) = \sigma^n_{n-i}$. The basic idea here is that we "reverse the order" of the elements of Δ. We will often schematically write $[n] \mapsto [n]^{\mathrm{op}}$ to describe this functor.

Definition 4.34. We obtain a cosimplicial object in Set_Δ as the composite functor

$$(\Delta^{(-)})^{\mathrm{op}} : \Delta \xrightarrow{\mathrm{op}} \Delta \xrightarrow{\Delta^{(-)}} \mathsf{Set}_\Delta$$

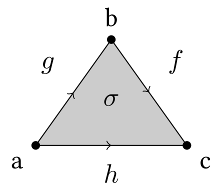
Notice that $(\Delta^n)^{\mathrm{op}} \cong \Delta^n$, and that the only difference is the action on morphisms.

Given a simplicial set X, we define a new simplicial set X^{op} by defining

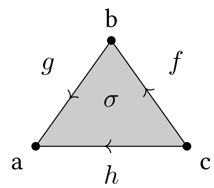
$$(X^{\mathrm{op}})_n := \mathrm{Hom}_{\mathsf{Set}_\Delta}((\Delta^n)^{\mathrm{op}}, X).$$

Notice that the only thing that changes are the simplicial maps (e.g. face and degeneracy maps), we still have $(X^{\mathrm{op}})_n \cong X_n$.

So how do we think about this construction? Let's consider a 2-simplex σ of X, which we picture, together with its faces, as



By construction, we get a corresponding simplex σ^{op} in X^{op}. But now, we have that $d_0(\sigma^{\mathrm{op}}) = d_2(\sigma) = g^{\mathrm{op}}$, and $d_2(\sigma^{\mathrm{op}}) = d_0(\sigma) = f^{\mathrm{op}}$. Similarly, $d_0(f^{\mathrm{op}}) = d_1(f) = b$, and $d_1(f^{\mathrm{op}}) = d_0(f) = c$. We can thus draw the 2-simplex σ^{op} in X^{op} as



i.e., we are simply reversing the directions of all of the arrows.

Exercise 17. Show that, for a 1-category $\mathsf{C} \in \mathsf{Cat}$, there is an isomorphism $N(\mathsf{C}^{\mathrm{op}}) \cong N(\mathsf{C})^{\mathrm{op}}$ of simplicial sets.

We now want to study slice quasi-categories, as a prelude to defining limits and colimits in quasi-categories. We will begin by defining an 'adjoint' construction, which will allow us to easily construct the necessary simplicial sets. Throughout this section, we will explore the parallels to 1-categories in the sidenotes.

Definition 4.35. The **extended simplex category** Δ_+ is the full subcategory of Cat on the object $[n]$ for $n \geq 0$, together with the object $[-1] := \varnothing$.

The **ordinal sum** is the functor

$$\oplus : \Delta_+ \times \Delta_+ \longrightarrow \Delta_+$$

which sends $([i],[j])$ to the linearly ordered set $[i] \oplus [j] := [i+j+1]$ which we view as the set

$$\{0, 1, \dots, i-1, i, 0', 1', \dots, (j-1)', j'\}.$$

Definition 4.36. Let X and Y be simplicial sets, we define the **join** of X and Y to be the simplicial set $X \star Y$ defined by

$$(X \star Y)_n = \coprod_{[i]\oplus[j]=n} X_i \times Y_j$$

where we define $X_{-1} := *$ is the singleton.

Exercise 18. Show that there are canonical inclusions $X \hookrightarrow X \star Y$ and $Y \hookrightarrow X \star Y$. Show that the assignment

$$\begin{aligned} (-) \star K : \mathsf{Set}_\Delta &\longrightarrow (\mathsf{Set}_\Delta)_{K/} \\ X &\longmapsto (K \hookrightarrow X \star K) \end{aligned}$$

extends to a functor. Show that this functor preserves colimits. Do the same for $K \star (-)$.

Definition 4.37. Let C and D be ordinary categories. The **join** of C and D, denoted $\mathsf{C} \star \mathsf{D}$, is the category with objects

$$\mathrm{Ob}(\mathsf{C} \star \mathsf{D}) = \mathrm{Ob}(\mathsf{C}) \amalg \mathrm{Ob}(\mathsf{D})$$

and morphisms

$$(\mathsf{C} \star \mathsf{D})(x, y) = \begin{cases} \mathsf{C}(x, y), & x, y \in \mathsf{C} \\ \mathsf{D}(x, y), & x, y \in \mathsf{D} \\ \{*\}, & x \in \mathsf{C}, y \in \mathsf{D} \\ \varnothing, & x \in \mathsf{D}, y \in \mathsf{C} \end{cases}.$$

Exercise 19. Show that, for 1-categories C and D, there is an isomorphism of simplicial sets

$$N(\mathsf{C} \star \mathsf{D}) \cong N(\mathsf{C}) \star N(\mathsf{D}).$$

As a corollary, show that $\Delta^n \star \Delta^k \cong \Delta^{n+k+1}$.

Proposition 4.38. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are quasi-categories. Then $\mathcal{C} \star \mathcal{D}$ is a quasi-category.*

Proof. Given a morphism $f : \Lambda^n_i \to \mathcal{C} \star \mathcal{D}$ where Λ^n_i is an inner horn, we consider three cases:

1. If the image of Λ^n_i is contained in $\mathcal{C} \subset \mathcal{C} \star \mathcal{D}$, then we can extend f to a morphism $\Delta^n \to \mathcal{C} \subset \mathcal{C} \star \mathcal{D}$.

2. If the image of Λ^n_i is contained in $\mathcal{D} \subset \mathcal{C} \star \mathcal{D}$, then we can extend f to a morphism $\Delta^n \to \mathcal{D} \to \mathcal{C}\mathcal{D}$

3. If neither of the previous cases holds, let j be the last vertex of Λ^n_i which is sent into X. Then f determines morphisms

$$\Delta^{\{0,\ldots,j\}} \to \mathcal{C}$$

and

$$\Delta^{\{j+1,\ldots,n\}} \to \mathcal{D}$$

which, together, define an n-simplex in $\mathcal{C} \star \mathcal{D}$, which extends f. $\square$

The main motivation for our definition of slice quasi-categories is the following description. Suppose that C is a 1-category, and $c \in \mathsf{C}$ is an object. We define the slice $\mathsf{C}_{/c}$, and let D be another category. Given a functor

$$F : \mathsf{D} \longrightarrow \mathsf{C}_{/c}$$

we can construct a functor

$$\tilde{F} : \mathsf{D} \star [0] \longrightarrow \mathsf{C}$$

which sends the unique element of $[0]$ to c. We define $\tilde{F}$ on D as the composite

$$\mathsf{D} \longrightarrow \mathsf{C}_{/c} \longrightarrow \mathsf{C}$$

For each (unique) morphism $d \to [0]$ in $\mathsf{D} \star [0]$, we define $\tilde{F}$ to be the morphism $F(d)$ in C.

It is not hard to check that, given a functor $G : \mathsf{D} \star [0] \to \mathsf{C}$ which sends 0 to c, we can define a corresponding functor $\overline{G} : \mathsf{D} \to \mathsf{C}_{/c}$, and that these two constructions define a bijection

$$\mathrm{Fun}(\mathsf{D}, \mathsf{C}_{/c}) \cong \mathrm{Fun}^c(\mathsf{D} \star [0], \mathsf{C})$$

where $\mathrm{Fun}^c(\mathsf{D} \star [0])$ denotes the subcategory on those functors which send $0 \mapsto c$, and those natural transformations whose component at 0 is id_c.

Using the join construction, we can now define a general version of over- and under-quasicategories.

Definition 4.39. Let $f : K \to X$ be a morphism of simplicial sets. We define a simplicial set $X_{/f}$ by the universal property

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(S, X_{/f}) \cong \mathrm{Hom}^f_{\mathsf{Set}_\Delta}(S \star K, X)$$

where $\mathrm{Hom}^f_{\mathsf{Set}_\Delta}(S \star K, X)$ denotes the subset consisting of maps $S \star K \to X$ which restrict to f on $K \subset S \star K$.

Dual Definition 4.39. Let $f : K \to X$ be a morphism of simplicial sets. We define a simplicial set $X_{f/}$ by the universal property

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(S, X_{f/}) \cong \mathrm{Hom}^f_{\mathsf{Set}_\Delta}(K \star S, X)$$

where $\mathrm{Hom}^f_{\mathsf{Set}_\Delta}(K \star S, X)$ denotes the subset consisting of maps $K \star S \to X$ which restrict to f on $K \subset K \star S$.

Notice that there is a natural map of simplicial sets

$$X_{/f} \longrightarrow X$$

which sends $\sigma : \Delta^n \star K \to X$ to $\sigma|_{\Delta^n}$.

Exercise 20. Let C be a category, and $x \in \mathsf{C}$ an object. Show that there is an isomorphism of simplicial sets $N(\mathsf{C}_{/x}) \cong N(\mathsf{C})_{/x}$.

4 *Left & right fibrations*

Our aim is to prove the following statements:

Theorem 4.40. *Let $\mathcal{C}$ be a quasi-category, and $f : K \to X$ a map of simplicial sets. Then*

1. *$\mathcal{C}_{/f}$ is an ∞-category.*
2. *$\mathcal{C}_{/f} \to \mathcal{C}$ is a Joyal equivalence if and only if it is a trivial Kan fibration.*

We will do this by exploring two new notions of fibrations of simplicial sets.

Definition 4.41. We denote by $\mathcal{RA}$ the set of *right anodyne morphisms*, which are defined to be

$$\mathcal{RA} := \overline{\{\Lambda^n_i \to \Delta^n \mid 0 < i \le n\}}.$$

Dually, we define the set of *left anodyne morphisms* to be

$$\mathcal{LA} := \overline{\{\Lambda^n_i \to \Delta^n \mid 0 \le i \le n\}}.$$

We call a morphism $p : X \to Y$ of simplicial sets a *right fibration* if it has the right lifting property with respect to all right anodyne morphisms. Dually, we call p a *left fibration* if it has the right lifting property with respect to all left anodyne morphisms.

The *fibre* of a fibration

$$f : X \to Y$$

of simplicial sets over a vertex $y \in Y$ is the pullback

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^0 \arrow[r, "\{y\}"'] & Y \end{tikzcd}$$

While it is immediate from the definitions that the fibres of a left or right fibration are quasi-categories, we will actually show more, namely that the fibres are Kan complexes. For this reason, one sometimes sees left/right fibrations referred to as *fibrations in ∞-groupoids*.

Remark 4.42. One can immediately notice that any Kan fibration is both a left and a right fibration. Similarly, any left or right fibration is an inner fibration.

We will prove our first claim with a familiar style of result:

Lemma 4.43. *Let $f : A \hookrightarrow B$ and $g : C \hookrightarrow D$ be monomorphisms of simplicial sets. Then the induced morphism*

$$A \star D \coprod_{A \star C} B \star C \to B \star D$$

is a monomorphism. If f is right anodyne or if g is left anodyne, then this morphism is inner anodyne.

Proof. We will show that if f is right anodyne, the pushout-product is inner anodyne. The other case is dual.

By the usual saturation arguments, it suffices to show that the lemma holds when $f : \Lambda^n_i \to \Delta^n$ is a right anodyne horn inclusion, and $g : \partial\Delta^m \to \Delta^m$ is a simplex boundary inclusion. We note that in this case, we can identify $\Delta^m \star \Delta^n \cong \Delta^{n+m+1}$.

We now identify which $n + m$-dimensional faces of this simplex lie in $\Lambda^n_i \star \Delta^m$, and which lie in $\Delta^n \star \partial\Delta^m$. Since the pushout-product, by definition, cannot contain the unique non-degenerate $n + m + 1$ simplex, this will completely determine the pushout-product. Note that the non-degerate $n + m$-simplices uniquely correspond to either (1) pairs consisting of an $n - 1$-simplex in the first factor and an m-simplex in the second, or (2) pairs consisting of an n-simplex in the first factor, and an $m - 1$-simplex in the second.

- In $\Lambda^n_i \star \Delta^m$, we have precisely those pairs of non-degenerate simplices (σ, γ) where σ is $(n - 1)$ dimensional, γ is the unique m-dimensional simplex, and σ is not the i^{th} face of Λ^n_i. This corresponds to the faces

$$d_0(\Delta^{m+n+1}), d_1(\Delta^{n+m+1}), \ldots, d_i(\widehat{\Delta^{n+m+1}}), \ldots, d_n(\Delta^{n+m+1})$$

  where the hat denotes omission.

- In $\Delta^n \star \partial\Delta^m$, we have every pair (σ, γ) of non-degenerate simplices such that γ is $(n - 1)$-dimensional.

We thus see that the pushout-product morphism can be identified with the inclusion

$$\Lambda^{n+m+1}_k \to \Delta^{n+m+1}$$

Since $0 < k \leq n$, this is inner anodyne, and the lemma is proved. □

Remark 4.44. Unlike in the case of the pushout-product using "$\times$", the pushout-product using the join is clearly not commutative even up to isomorphism. This accounts for the pair of hypotheses in the lemma above.

A virtually identical argument shows:

Lemma 4.45. *Let $f : A \hookrightarrow B$ and $g : C \hookrightarrow D$ be monomorphisms of simplicial sets. If f is left anodyne, the morphism*

$$A \star D \coprod_{A \star C} B \star C \to B \star D$$

is left anodyne. If g is right anodyne, the induced morphism is right anodyne.

Corollary 4.46. *Consider maps of simplicial sets*

$$A \xhookrightarrow{i} B \xrightarrow{p} X \xrightarrow{q} Y$$

Denote by $r = q \circ p$, and denote by p_0 and r_0 the composites $p \circ i$ and $r \circ i$, respectively. If q is an inner fibration, then the induced map

$$X_{p/} \to X_{p_0/} \times_{Y_{r_0/}} Y_{r/}$$

is a left fibration. If q is a left fibration, then the map

$$X_{p/} \to X_{p_0/} \times_{Y_{r_0/}} Y_{r/}$$

is a trivial Kan fibration as well.

One immediate corollary is the following

Corollary 4.47. *Let $\mathcal{C}$ be a quasi-category, and $f : K \to \mathcal{C}$ a map of simplicial sets. Then $\mathcal{C}_{f/} \to \mathcal{C}$ is a left fibration. In particular, $\mathcal{C}_{f/}$ is a quasi-category.*

Proof. We apply the previous corollary to the case where $A = \emptyset$, $X = \mathcal{C}$, $Y = \Delta^0$, and $B = K$. □

5 *Equivalences and left fibrations*

We now turn to discussing equivalences in quasi-categories.

Definition 4.48. Let $\mathcal{C}$ be a quasi-category. A morphism[6] $f : x \to y$ in $\mathcal{C}$ is called an *equivalence* if the corresponding morphism $[f]$ in the homotopy category $h\mathfrak{C}[\mathcal{C}]$ is an isomorphism.

[6] i.e. a 1-simplex.

As it turns out, there are quite a number of ways of characterizing these equivalences.

Definition 4.49. Given a quasi-category $\mathcal{C}$, define an equivalence relation define an equivalence relation on the 1-simplices of $\mathcal{C}$ by saying that, for $f, g \in \mathcal{C}_1$, $f \sim g$ if and only if there is a 2-simplex $\sigma \in \mathcal{C}_2$ such that $d_2(\sigma) = f$, $d_1(\sigma) = g$, and $d_0(\sigma)$ is degenerate.

Denote by $\mathrm{qCat} \subset \mathrm{Set}_\Delta$ the full subcategory on the quasi-categories. To each $\mathcal{C} \in \mathrm{qCat}$ associate the category $\gamma(\mathcal{C})$ whose objects are the 0-simplices of $\mathcal{C}$, and whose morphisms are equivalence classes of 1-simplices of $\mathcal{C}$ under the relation defined above.

Exercise 21. Show that this construction yields a well-defined functor $\gamma : \mathrm{qCat} \to \mathrm{Cat}$. Show that γ is left adjoint to the nerve functor $N : \mathrm{Cat} \to \mathrm{qCat}$. In particular, note that for $X \in \mathrm{qCat}$ there is a natural isomorphism $\gamma(X) \cong \tau_1(X)$, where τ_1 denotes the *fundamental category functor* left adjoint to the nerve $N : \mathrm{Cat} \to \mathrm{Set}_\Delta$.

Lemma 4.50. *A morphism $f : x \to y$ in a quasi-category $\mathcal{C}$ is an equivalence if and only if the induced morphism $[f]$ is an equivalence in $\gamma(X)$.*

Proof. There is a commutative diagram

$$\begin{tikzcd} & \mathrm{Cat}_\Delta \arrow[dr, "N_{HC}"] & \\ \mathrm{Cat} \arrow[ur, hook, "\iota"] \arrow[rr, "N"] & & \mathrm{Set}_\Delta \end{tikzcd}$$

The left adjoints to ι, N_{HC}, and N are, respectively, the homotopy category functor $h : \mathrm{Cat}_\Delta \to \mathrm{Cat}$, the rigidification $\mathfrak{C} : \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$, and the fundamental category functor $\tau_1 : \mathrm{Set}_\Delta \to \mathrm{Cat}$. Consequently, we obtain a natural isomorphism $\tau_1 \cong h \circ \mathfrak{C}$. Applied to a quasi-category $\mathcal{C}$ this, together with the exercise above, proves the lemma. □

A key characterization links equivalences in a quasi-category to left fibrations.

Lemma 4.51. *Suppose $p : \mathcal{C} \to \mathcal{D}$ is a left fibration of quasi-categories and $f : x \to y$ in $\mathcal{C}$ is a morphism such that $p(f)$ is an equivalence in $\mathcal{D}$. Then f is an equivalence.*

Proof. We will show that if $p(f)$ has a left inverse in $h\mathcal{D}$, then f has a left inverse in $h\mathcal{C}$. Applying this fact twice then yields the statement.

If $p(f)$ has a left inverse g, then we can form the lifting problem

$$\begin{tikzcd} & y & & & p(y) \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_x"'] & & x & \mapsto \; p(x) \arrow[ur, "p(f)"] \arrow[rr, "\mathrm{id}_{p(x)}"'] & & p(x) \end{tikzcd}$$

since this is a left horn inclusion, we can provide a lift, which encodes a left inverse to f in $h\mathcal{C}$. ☐

Lemma 4.52. *Suppose $p : \mathcal{C} \to \mathcal{D}$ is a right fibration, and let $x \in \mathcal{C}$. If $f : p(x) \to y$ is an equivalence in $\mathcal{D}$, then there is an equivalence $\tilde{f} : x \to \tilde{y}$ such that $p(\tilde{f}) = f$.*

Proof. Let $g : y \to p(x)$ be a homotopy inverse to f. Since p is a right fibration, there is a morphism $\tilde{g} : \tilde{y} \to x$ such that $p(\tilde{g}) = g$. By Lemma 4.51, $\tilde{g}$ is an equivalence. We then form the lifting problem

$$\begin{tikzcd} & \tilde{y} \arrow[dr, "\tilde{g}"] & & & y \arrow[dr, "g"] & \\ x \arrow[rr, "\mathrm{id}_x"'] & & x & \mapsto \; p(x) \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_{p(x)}"'] & & p(x) \end{tikzcd}$$

Since this is a right horn, we can fill it, yielding $\tilde{f} : x \to \tilde{y}$, which is left inverse to $\tilde{g}$.

Since $\tilde{g}$ is an equivalence, $\tilde{f}$ must be as well, completing the proof. ☐

Proposition 4.53. *Suppose that $\mathcal{C}$ is a quasi-category, and that $f : x \to y$ is a morphism in $\mathcal{C}$. Then f is an equivalence if and only if every lifting problem of the form*

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "\{f\}"] & \\ \Lambda^n_0 \arrow[r, "g"] \arrow[d, hook] & \mathcal{C} \\ \Delta^n & \end{tikzcd}$$

admits a solution.

Proof. Suppose that f is an equivalence and consider the lifting problem in the proposition. Notice that the inclusion $\Lambda^n_0 \to \Delta^n$ can be rewritten as

$$\Delta^1 \star \partial\Delta^{n-2} \coprod_{\{0\}\star\partial\Delta^{n-2}} \{0\} \star \Delta^{n-2} \to \Delta^1 \star \Delta^{n-2}$$

We can therefore pass to the adjoint lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathcal{C}_{/g|_{\Delta^{n-2}}} \arrow[d, "p"] \\
\Delta^1 \arrow[r, "\{\tilde{f}\}"] \arrow[dr, "\{f\}"'] & \mathcal{C}_{/g|_{\partial\Delta^{n-2}}} \arrow[d, "k"] \\
& \mathcal{C}
\end{tikzcd}$$

by (the dual of) Corollary 4.46, p and k are right fibrations. By Lemma 4.51, the morphism $\tilde{f}$ is an equivalence in $\mathcal{C}_{/g|_{\partial\Delta^{n-2}}}$. Applying Lemma 4.52, we see that this lifting problem has a solution.

Since the inclusion $\{0\} \to \Delta^1$ is a right horn, this lifting problem has a solution, so the original problem does as well.

On the other hand, suppose that each such lifting problem has a solution. We can form the lifting problem

$$\begin{tikzcd}
& y \\
x \arrow[ur, "f"] \arrow[r, "s_0(x)"'] & x
\end{tikzcd}$$

in $\mathcal{C}$ and see that this has a solution, yielding a morphism $h : y \to x$ which is left inverse to f in $\tau_1(\mathcal{C})$. A similar argument shows that h itself has a left inverse in $\tau_1(\mathcal{C})$, and thus is an isomorphism in $\tau_1(\mathcal{C})$. Thus, f is also an isomorphism in $\tau_1(\mathcal{C})$, and so is an equivalence. $\square$

Corollary 4.54. *A quasi-category $\mathcal{C}$ is a Kan complex if and only if every morphism of $\mathcal{C}$ is an equivalence.*

Proof. By definition, $p : \mathcal{C} \to \Delta^0$ has the right lifting property with respect to every inner horn inclusion. Proposition 4.53 shows that p has the right lifting property with respect to left horn inclusions, and the dual of Proposition 4.53 shows the lifting property with respect to right horn inclusions. $\square$

The import of this proposition is that our definitions satisfy the homotopy hypothesis: if we *define* ∞-groupoids as quasi-categories in which every 1-morphism is invertible, then we see that these are the same thing as Kan complexes!

We conclude this section with the final result we need for our discussion of limits and colimits.

Lemma 4.55. *The following three sets of morphisms generate the same saturated class in simplicial sets*

1. *The left horn inclusions*

$$\{\Lambda^n_i \to \Delta^n \mid 0 \leq i < n\}$$

2. *The pushout-products*

$$\left\{\Delta^m \times \{0\} \coprod_{\partial\Delta^m \times \{0\}} \partial\Delta^m \times \Delta^1 \to \Delta^m \times \Delta^1 \mid m \geq 0\right\}$$

3. *The pushout-products*

$$\left\{ K \times \{0\} \coprod_{S \times \{0\}} S \times \Delta^1 \mid S \hookrightarrow K \textit{ mono} \right\}$$

Proposition 4.56. *Suppose that $p : X \to Y$ is a left fibration and each fibre of p is a contractible Kan complex. Then p is a trivial Kan fibration.*

Proof. We show that p has the right lifting property against the morphisms $\partial\Delta^n \to \Delta^n$. Since every fibre of p is non-empty, we immediately have the LLP against $\varnothing = \partial\Delta^0 \to \Delta^0$.

Suppose $n > 0$, and consider a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow[g]{} & Y \end{array}$$

By considering the induced diagram

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f} & X \times_Y \Delta^n \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow[\mathrm{id}]{} & \Delta^n \end{array}$$

we can assume without loss of generality that $Y = \Delta^n$ and $g = \mathrm{id}_{\Delta^n}$. We thus need to solve the lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[\mathrm{id}]{} & \Delta^n \end{array}$$

To do this, we define a map

$$\begin{aligned} \gamma : \Delta^n \times \Delta^1 &\longrightarrow \Delta^n \\ (i, j) &\longmapsto \begin{cases} i & j = 0 \\ j & j = 1 \end{cases} \end{aligned}$$

Note that $\gamma|_{\Delta^n \times \{0\}} = \mathrm{id}_{\Delta^n}$. We can thus form a lifting problem

$$\begin{array}{ccc} \partial\Delta^n \times \{0\} & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \partial\Delta^n \times \Delta^1 & \xrightarrow[\gamma|_{\partial\Delta^n \times \Delta^1}]{} & \Delta^n \end{array}$$

By Lemma 4.55, the morphism $\partial\Delta^n \times \{0\} \to \partial\Delta^n \times \{1\}$ is left anodyne, and thus, we obtain a lift $\overline{\phi} : \partial\Delta^n \times \Delta^1 \to X$.

By definition, the restriction $\gamma|_{\Delta^n \times \{1\}}$ is constant on $n \in \Delta^n$, so we get a lifting problem in the fibre X_n:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\overline{\phi}|_{\partial\Delta^n \times \{1\}}"] \arrow[d] & X_n \arrow[d, "p_n"] \\
\Delta^n \arrow[r, "\gamma|^n_\Delta \times \Delta^1"'] & \Delta^0
\end{tikzcd}$$

Since X_n is a contractible Kan complex, p_n is a trivial Kan fibration, and thus the lifting problem has a solution

$$\tau : \Delta^n \longrightarrow X_n$$

We can then define a map

$$\phi : \partial\Delta^n \times \Delta^1 \coprod_{\partial\Delta^n \times \{1\}} \Delta^n \times \{1\} \longrightarrow X$$

using $\overline{\phi}$ and τ. This yields a lifting problem

$$\begin{tikzcd}
\partial\Delta^n \times \Delta^1 \coprod_{\partial\Delta^n \times \{1\}} \Delta^n \times \{1\} \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "\gamma"'] & \Delta^n
\end{tikzcd}$$

To solve this lifting problem, consider the maximal simplices in $\Delta^n \times \Delta^1$:

$$\begin{aligned}
\sigma_j : \Delta^{n+1} &\longrightarrow \Delta^n \times \Delta^1 \\
i &\longmapsto \begin{cases} (0, i) & i \leq j \\ (1, i-) & i > j \end{cases}
\end{aligned}$$

for $0 \leq j \leq n$. We define simplicial subsets of $\Delta^n \times \Delta^1$ by setting

$$U_0 := \partial\Delta^n \times \Delta^1 \coprod_{\partial\Delta^n \times \{1\}} \Delta^n \times \{1\},$$

and then defining

$$U_{j+1} = U_j \cup \sigma_j,$$

so that, in particular $U_{n+1} = \Delta^n \times \Delta^1$.

We notice that, for $j < n$, we have that

$$U_j \cap \sigma_j = \Lambda^{n+1}_{j+1}$$

is an inner horn. Since p is, in particular, an inner fibration, we can fill this inner horn to extend a map $\phi_j : U_j \to X$ to a map $\phi_{j+1} : U_{j+1} \to X$ in a manner compatible with γ. When $j = n$, The intersection

$$U_n \cap \sigma_n = \Lambda^{n+1}_{n+1}$$

is a *right* horn. By construction, however, the image of the edge $n \to n+1$ in Λ^{n+1}_{n+1} under ϕ_n lies in the fibre X_n, and thus is an equivalence. Consequently, this horn can be filled by (the dual of) Proposition 4.53. We have thus constructed a lift

$$\overline{\psi} : \Delta^n \times \Delta^1 \longrightarrow X$$

We then see that, by construction the restriction $\psi := \overline{\psi}|^n_\Delta \times \{0\}$ makes the diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\mathrm{id}"'] \arrow[ur, "\psi"] & \Delta^n \end{tikzcd}$$

commute, completing the proof. □

Corollary 4.57. *A left fibration $p : X \to Y$ is a trivial fibration if and only if every fibre of p is a contractible Kan complex.*

Proof. One direction is Proposition 4.56. For the other direction, we need only note that trivial Kan fibrations are stable under pullback. □

6 *Limits and colimits*

With these preliminaries out of the way, we can now move on to defining limits and colimits in a quasi-category. As in the 1-categorical case, we first need the notion of initial and terminal objects in a quasi-category.

The intuitive definition of initial/terminal objects is quite easy to grasp: we want to say that an object $y \in \mathcal{C}$ is terminal when, for every $x \in \mathcal{C}$, the mapping space from x to y is *contractible*.[7] However, this definition would require us to pass through simplicially-enriched categories every time we wanted to work with initial and terminal objects. To get around this issue, we will give a different model for mapping spaces in a quasi-category.

[7] This is a general feature one encounters moving from 1-category theory to higher category theory. If we require a morphism ϕ in a 1-categorical definition to be *unique*, then we require the *space* of such morphisms in be *contractible* in the corresponding ∞-categorical definition.

Definition 4.58. Let $\mathcal{C} \in \mathrm{qCat}$, and let $x, y \in \mathcal{C}$ be two objects. We define the *left mapping space* $\mathcal{M}\mathrm{ap}^L_{\mathcal{C}}(x, y)$ in $\mathcal{C}$ to be the pullback

$$\begin{tikzcd} \mathcal{M}\mathrm{ap}^L_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^0 \arrow[r, "\{y\}"'] & \mathcal{C} \end{tikzcd}$$

Dually, we define the *right mapping space* to be the pullback

$$\begin{tikzcd} \mathcal{M}\mathrm{ap}^R_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] & \mathcal{C}_{/y} \arrow[d] \\ \Delta^0 \arrow[r, "\{x\}"'] & \mathcal{C} \end{tikzcd}$$

The proof that these agree, at least up to homotopy, with the mapping spaces we already have requires more technology than we have time to develop. For reference, we state it as a proposition without proof

Proposition 4.59. *Let $\mathcal{C} \in \mathrm{qCat}$, and $x, y \in \mathcal{C}$. Then there are weak homotopy equivalences*

$$\mathcal{M}\mathrm{ap}^L_{\mathcal{C}}(x,y) \simeq \mathfrak{C}[\mathcal{C}](x,y) \simeq \mathcal{M}\mathrm{ap}^R_{\mathcal{C}}(x,y)$$

of simplicial sets.

Definition 4.60. Let $\mathcal{C}$ be a quasi-category, and x a vertex of $\mathcal{C}$. We say that x is an *terminal object* in $\mathcal{C}$ if

$$\mathcal{C}_{/x} \to \mathcal{C}$$

is a trivial Kan fibration. Dually, we say x is an *initial object* if

$$\mathcal{C}_{x/} \to \mathcal{C}$$

is a trivial Kan fibration.

Proposition 4.61. *Let $\mathcal{C} \in \mathrm{qCat}$. An object $x \in \mathcal{C}$ is initial if and only if $\mathfrak{C}[\mathcal{C}](x,y) \simeq \Delta^0$ for every $y \in \mathcal{C}$.*

Proof. By Corollary 4.57, we see that x is initial if and only if all the fibres of $\mathcal{C}_{x/} \to \mathcal{C}$ are contractible Kan complexes. These fibres are, by definition, the mapping spaces $\mathcal{M}\mathrm{ap}^L_{\mathcal{C}}(x,y)$, and so, by Proposition 4.59, are homotopy equivalent to $\mathfrak{C}[\mathcal{C}](x,y)$. □

Example 4.62. Let C be a 1-category with an initial object x. Then x is an initial object in $N(\mathsf{C})$.

Remark 4.63 (Important observation). By Proposition 4.61, every initial object in a quasi-category $\mathcal{C}$ is initial in the homotopy category $\tau_1(\mathcal{C}) = h\mathfrak{C}[\mathcal{C}]$. However, the converse *does not* hold. For example, let K be any Kan complex which is path-connected, but not contractible (e.g., $K = \mathrm{Sing}(S^1)$). We can define a simplicially-enriched category $\mathcal{C}_K$ with two objects, 0 and 1, and mapping spaces

$$\mathcal{C}_K(0,0) = \mathcal{C}_K(1,1) = \Delta^0$$

and

$$\mathcal{C}_K(0,1) = K \qquad \mathcal{C}_K(1,0) = \varnothing.$$

Since K is path connected, we see that $h\mathfrak{C}[\mathcal{C}]$ is isomorphic to the poset $[1]$, and thus has initial object 0. However, $\mathfrak{C}[\mathcal{C}](0,1)$ is *not* contractible, so 0 is not an initial object in the corresponding quasi-category.

Lemma 4.64. *Let $\mathcal{C}$ be a quasi-category. An object $x \in \mathcal{C}$ is initial if and only if every lifting problem*

$$\begin{tikzcd} \Delta^{\{0\}} \arrow[d] \arrow[dr, "\{x\}"] & \\ \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n & \end{tikzcd}$$

where $n \geq 1$ has a solution.

Proof. By definition, x is initial if and only if every lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^n \arrow[r] & \mathcal{C} \end{tikzcd}$$

has a solution (for $n \geq 0$). Each such lifting problem is equivalent to its adjoint lifting problem

$$\begin{tikzcd} \Delta^0 \star \partial\Delta^n \coprod_{\varnothing\star\partial\Delta^n} \varnothing \star \Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^0 \star \Delta^n & \end{tikzcd}$$

has a solution. However the morphism on the left is isomorphic to

$$\partial\Delta^{n+1} \to \Delta^{n+1},$$

proving the lemma. □

In standard 1-category theory, we see that initial objects are unique up to unique isomorphism. In higher category theory, such statements about uniqueness tend to be replaced by statements about having a *contractible space of choices*. The following is thus a higher-categorical analogue of the uniqueness of initial objects

Lemma 4.65. *Let $\mathcal{C}$ be a quasi-category with an inital object x. Denote by $\mathcal{D} \subset \mathcal{C}$ the full simplicial subset on the initial objects. Then $\mathcal{D}$ is a contractible Kan complex.*

Proof. Since $\mathcal{D}$ is non-empty, the map

$$q : \mathcal{D} \to \Delta^0$$

has the LLP against $\partial\Delta^0 \to \Delta^0$.

We then note that, for a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \Delta^n \arrow[r] & \Delta^0 \end{tikzcd}$$

Lemma 4.64 yields a solution. □

Lemma 4.66. *Let $\mathcal{C}$ be a quasi-category, and $f : x \to y$ an equivalence in $\mathcal{C}$. Then x is initial if and only if y is initial.*

Proof. It will suffice to show that, if x is initial, then so is y. Suppose that x is initial, and consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\phi"] \arrow[d] & \mathcal{C} \\ \Delta^n & \end{tikzcd}$$

where $\phi(0) = y$ and $n \geq 1$.

Since the map $\mathcal{C}_{x/} \to \mathcal{C}$ is a trivial Kan fibration, we can solve the lifting problem

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \partial\Delta^n \arrow[r, "\phi"] & \mathcal{C} \end{tikzcd}$$

to get a map $\overline{\psi} : \partial\Delta^n \to \mathcal{C}_{x/}$. This corresponds to a map

$$\psi : \Lambda^{n+1}_0 \to \mathcal{C}$$

such that $\psi(0) = x$, and $\psi|_{\partial\Delta^{\{1,\ldots,n+\}}} = \phi$. In particular $\psi(0) = x$ and $\psi(1) = \phi(0) = y$.

Since x is initial, the morphism $\psi(\Delta^{\{0,1\}}$ from x to y must represent the same morphism as f in the homotopy category, and thus must be an equivalence. As a result, we can solve the lifting problem

$$\begin{tikzcd} \Lambda^{n+1}_0 \arrow[r, "\psi"] \arrow[d] & \mathcal{C} \\ \Delta^{n+1} & \end{tikzcd}$$

by Proposition 4.53. Restricting the solution to $\Delta^{\{1,2,\ldots,n+1\}}$ yields a solution to the original lifting problem, and we thus see that y is initial by Lemma 4.64. □

We now have all the pieces in place to define limits and colimits in quasi-categories.

Definition 4.67. Let $\mathcal{C}$ be a quasi-category, and $f : K \to \mathcal{C}$ a morphism of simplicial sets. The *quasi-category of cones over* f is the quasi-category $\mathcal{C}_{/f}$. The *quasi-category of cocones under* f is the quasi-category $\mathcal{C}_{f/}$.

A *limit of* f *in* $\mathcal{C}$ is a terminal object of $\mathcal{C}_{/f}$. A *colimit of* f *in* $\mathcal{C}$ is an initial object of $\mathcal{C}_{f/}$.

Remark 4.68. Notice that, in our definition above, a limit of $f : K \to \mathcal{C}$ is not simply an object in $\mathcal{C}$, but rather a choice of a functor

$$\tilde{f} : \Delta^0 \star K \longrightarrow \mathcal{C}$$

extending f. This is precisely the data of a limit cone over f, as the next exercise helps illustrate.

Exercise 22. Show that, if I and C are 1-categories, and $F : \mathsf{I} \to \mathsf{C}$ is a functor, then there is an isomorphism of simplicial sets.

$$N(\text{cone}(F)) \cong N(\mathcal{C})_{/N(F)}.$$

Conclude that a limit cone of F is the same data as a limit of $N(F)$.

Definition 4.69. For a simplicial set K, we fix the notation $K^{\triangleleft} := \Delta^0 \star K$, and $K^{\triangleright} = K \star \Delta^0$.

Exercise 23. Let $\mathcal{C}$ be a quasi-category.

1. Prove that a limit over the empty diagram $\varnothing \to \mathcal{C}$ is a terminal object of $\mathcal{C}$.
2. If $\mathcal{C}$ is a Kan complex, show that $\varnothing \to \mathcal{C}$ has a limit if and only if $\mathcal{C}$ is contractible.

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