Math 8850, Fall 2021
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Homework 6

Due: Wednesday, 13. Oct. 2021

Definition. Let Cat be the category of small categories. Let $\mathcal{I}$ be the category with two objects, 0 and 1, and a unique isomorphism between them. Define $\mathcal{C}$ to be the collection of morphisms in Cat which are injective on objects. Define $\mathcal{F}$ to be collection of morphisms in Cat which have the right lifting property with respect to the inclusion $[0] \to \mathcal{I}$. We call the elements of $\mathcal{F}$ the *isofibrations*. Let $\mathcal{W}$ be the collection of all equivalences of categories.

Exercise 1. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between small categories.

1. Define a category L whose objects are tuples (c, d, ϕ) where $c \in \mathsf{C}$, $d \in \mathsf{D}$, and $\phi : F(c) \xrightarrow{\cong} d$ is an isomorphism in D, and whose morphisms are given by

   $$\mathrm{Hom}_{\mathsf{L}}((c, d, \phi), (a, b, \psi)) := \mathrm{Hom}_{\mathsf{C}}(c, a).$$

   Show that F factors as

   $$\mathsf{C} \xrightarrow{G} \mathsf{L} \xrightarrow{H} \mathsf{D}$$

   such that G is an equivalence of categories, $G \in \mathcal{C}$, and $F \in \mathcal{F}$.

2. Define a category R with $\mathrm{Ob}(\mathsf{R}) := \mathrm{Ob}(\mathsf{C}) \coprod \mathrm{Ob}(\mathsf{D})$. For $c_1, c_2 \in \mathsf{C}$ and $d_1, d_2 \in \mathsf{D}$, define

   $$\begin{aligned} \mathrm{Hom}_{\mathsf{R}}(c_1, c_2) &:= \mathrm{Hom}_{\mathsf{D}}(F(c_1), F(c_2)) \\ \mathrm{Hom}_{\mathsf{R}}(d_1, d_2) &:= \mathrm{Hom}_{\mathsf{D}}(d_1, d_2) \\ \mathrm{Hom}_{\mathsf{R}}(c_1, d_1) &:= \mathrm{Hom}_{\mathsf{D}}(F(c_1), d_1) \\ \mathrm{Hom}_{\mathsf{R}}(d_1, c_1) &:= \mathrm{Hom}_{\mathsf{D}}(d_1, F(c_1)) \end{aligned}$$

   Show that F factors as

   $$\mathsf{C} \xrightarrow{G} \mathsf{R} \xrightarrow{H} \mathsf{D}$$

   where H is an equivalence of categories, $H \in \mathcal{F}$, and $G \in \mathcal{C}$.

Exercise 2. Show that $\mathcal{F}$ and $\mathcal{C}$, and $\mathcal{W}$ are stable under retracts.

Exercise 3. Suppose given a diagram of small categories

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{F} & \mathsf{C} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle P} \\ \mathsf{B} & \xrightarrow[G]{} & \mathsf{D} \end{array}$$

where $J \in \mathcal{C}$ and $P \in \mathcal{F}$.

1. Show that if J is an equivalence of categories, there exists a lift $L : \mathsf{B} \to \mathsf{C}$.

2. Show that if P is an equivalence of categories, there exists a lift $L : \mathsf{B} \to \mathsf{C}$.

Exercise 4. Conclude using the previous exercises that $(\mathsf{Cat}, \mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model category. Identify the fibrant objects and the cofibrant objects in this model structure.