Homework 5

Due: Wednesday, 6. Oct. 2021

Exercise 1. Let C be a category. Show that $N(\mathsf{C})$ admits *unique* fillers for horns of types Λ^2_0 and Λ^2_1 if and only if C is a groupoid.

Definition. Denote by qCat the full subcategory of Set_Δ on the simplicial sets which have all *inner horn fillers* (i.e. all horn fillers for $\Lambda^n_i \to \Delta^n$ where $n \geq 2$ and $0 < i < n$). Denote by Kan the full subcategory of Set_Δ on the Kan complexes. Note that in neither case are the horn fillers required to be unique.

Exercise 2. For $X \in \mathsf{qCat}$, define an equivalence relation on the 1-simplices of X by saying that, for $f, g \in X_1$, $f \sim g$ if and only if there is a 2-simplex $\sigma \in X_2$ such that $d_2(\sigma) = f$, $d_1(\sigma) = g$, and $d_0(\sigma)$ is degenerate.

To each $X \in \mathsf{qCat}$ associate the category $\gamma(X)$ whose objects are the 0-simplices of X, and whose morphisms are equivalence classes of 1-simplices of X under the relation defined above. Show that this construction yields a well-defined functor $\gamma : \mathsf{qCat} \to \mathsf{Cat}$.

Exercise 3. Show that γ is left adjoint to the nerve functor $N : \mathsf{Cat} \to \mathsf{qCat}$. In particular, note that for $X \in \mathsf{qCat}$ there is a natural isomorphism $\gamma(X) \cong \tau_1(X)$.

Exercise 4. Prove that if $X \in \mathsf{Kan}$, $\gamma(X)$ is a groupoid.