Homework 4

Due: Wednesday, 29. Sept. 2021

Reading: Before next Monday, please read the section "Understanding simplicial sets" from the notes.

Exercise 1. Recall that a groupoid is a category in which all morphisms are invertible. We denote by $\mathsf{Grpd} \subset \mathsf{Cat}$ the full subcategory whose objects are the groupoids.

1. Given a groupoid $\mathcal{G}$, let
$$\pi_0(\mathcal{G}) := \mathrm{ob}(\mathcal{G})/_{\sim}$$
be the set of equivalence classes of objects under the relation $x \sim y$ if there exists $g : x \to y$ in $\mathcal{G}$. Show that the assignment
$$\pi_0 : \mathsf{Grpd} \to \mathsf{Set}; \quad \mathcal{G} \mapsto \pi_0(\mathcal{G})$$
on objects defines a functor.

2. We call a groupoid $\mathcal{G}$ *path-connected* if $\pi_0(\mathcal{G})$ is a singleton. Show that, for any path-connected groupoid $\mathcal{G}$, there is a group G and an equivalence of categories
$$\mathsf{BG} \simeq \mathcal{G}.$$

Exercise 2. A $(2,1)$-*category* is a 2-category $\mathbb{C}$ such that, for all $x, y \in \mathbb{C}$, the category $\mathbb{C}(x,y)$ is a groupoid.

1. Show that, given a (2,1)-category C, there is a 1-category $\mathrm{h}\mathbb{C}$ such that
   - $\mathrm{ob}(\mathrm{h}\mathbb{C}) = \mathrm{ob}(\mathbb{C})$
   - for all $x, y \in \mathbb{C}$, $\mathrm{h}\mathbb{C}(x,y) = \pi_0(\mathbb{C}(x,y))$.

Exercise 3. Denote by $\iota : \Delta \to \mathsf{Cat}$ the functor that sends each totally ordered set $[n]$ to the associated category, and define the *nerve functor* by
$$N : \mathsf{Cat} \to \mathsf{Set}_\Delta; \quad \mathsf{C} \mapsto \mathsf{Cat}(\iota(-), \mathsf{C}).$$
Let $\tau_1 : \mathsf{Set}_\Delta \to \mathsf{Cat}$ be the left Kan extension of ι along the Yoneda embedding $\Delta \to \mathsf{Set}_\Delta$.

1. Argue that τ_1 is left adjoint to N.

2. Consider the functor $E : \Delta \to \mathbf{Grpd}$ which sends $[n]$ to the groupoid $E([n])$ with objects $0, 1, \ldots n$ and a unique isomorphism between every pair of objects. Show that the functor
$$M : \mathsf{Grpd} \to \mathsf{Set}_\Delta; \quad \mathsf{C} \mapsto \mathsf{Grpd}\left(E(-), \mathsf{C}\right)$$
is naturally isomorphic to the restriction of N to Grpd.

3. Show that M admits a left adjoint. Is this left adjoint the same as τ_1?