Homework 3

Due: Wednesday, 22. Sept. 2021

Exercise 1. We say that a functor $G : \mathsf{C} \to \mathsf{D}$ *preserves limits* if, for any diagram $\phi : \mathsf{I} \to \mathsf{C}$, G sends limit cones over ϕ to limit cones over $G \circ \phi$. Equivalently, we say that G preserves limits if, for any I, there is a isomorphism

$$\mathrm{Nat}(\mathrm{const}_d, G \circ \phi) \cong \mathrm{Hom}_{\mathsf{D}}(d, G(\lim_{\mathsf{I}} \phi)).$$

natural in $\phi \in \mathsf{C}^{\mathsf{I}}$ and $d \in \mathsf{D}$.

Let (F, G, ϵ, η) be an adjunction between categories C and D.

1. Show that G preserves limits. (Hint: You can use the fact from HW2 that $\mathrm{Hom}_{\mathsf{C}}(c, -)$ preserves limits)

2. Give the dual definition to the above (preservation of colimits). Conclude by duality that F preserves colimits.

Exercise 2. Let $\phi : I \to J$ be a fully faithful functor, and let C be a cocomplete category. Denote by

$$\begin{aligned} \phi^* : \mathrm{Fun}(J, \mathsf{C}) &\longrightarrow \mathrm{Fun}(I, \mathsf{C}) \\ F &\longmapsto F \circ \phi \end{aligned}$$

the restriction functor. Denote by

$$\phi_! : \mathrm{Fun}(I, \mathsf{C}) \longrightarrow \mathrm{Fun}(J, \mathsf{C})$$

the left adjoint of ϕ^*, i.e. the functor which sends each $F : I \to \mathsf{C}$ to the left Kan extension of F along ϕ. Prove that the unit

$$\eta : \mathrm{id}_{\mathrm{Fun}(I,\mathsf{C})} \Longrightarrow \phi^* \circ \phi_!$$

is an isomorphism.

Exercise 3. Let C be a category, and denote by $よ : \mathsf{C} \to \mathsf{Set}_{\mathsf{C}}$ the Yoneda embedding.

1. Show that $\mathrm{Id}_{\mathsf{Set}_{\mathsf{C}}} : \mathsf{Set}_{\mathsf{C}} \to \mathsf{Set}_{\mathsf{C}}$ is a left Kan extension of $よ$ along $よ$. (Hint: Use the colimit characterization of the LKE, and analyze the representable functor $\mathrm{Hom}_{\mathsf{Set}_{\mathsf{C}}}(よ_! よ(X), -)$.)

2. Use part 1 to give an explicit formula for X as a colimit of representable functors h_c.

Exercise 4. Let C be a category, and denote by $よ : \mathsf{C} \to \mathsf{Set}_{\mathsf{C}}$ the Yoneda embedding. Suppose given a cocomplete category D and a functor $F : \mathsf{C} \to \mathsf{D}$. Denote by $よ_! F : \mathsf{Set}_{\mathsf{C}} \to \mathsf{D}$ the left Kan extension of F along $よ$.

1. Show that $\mathcal{Y}_! F$ has a right adjoint, given on objects by

$$\begin{aligned} R : \mathsf{D} &\longrightarrow \mathsf{Set}_{\mathsf{C}} \\ d &\longmapsto \mathrm{Hom}_{\mathsf{D}}(F(-), d) \end{aligned}$$

2. For a functor $G : \mathsf{Set}_{\mathsf{C}} \to \mathsf{D}$, show that the counit $\mathcal{Y}_!(\mathcal{Y}^*(G)) \Rightarrow G$ is an isomorphism if and only if G sends colimit cones in $\mathsf{Set}_{\mathsf{C}}$ to colimit cones in D.

3. Denote by $\mathrm{Fun}^{\mathrm{colim}}(\mathsf{Set}_{\mathsf{C}}, \mathsf{D})$ the full subcategory of $\mathrm{Fun}(\mathsf{Set}_{\mathsf{C}}, \mathsf{D})$ on the functors which preserve colimit cones. Conclude that

$$\mathcal{Y}_! : \mathrm{Fun}(\mathsf{C}, \mathsf{D}) \longrightarrow \mathrm{Fun}^{\mathrm{colim}}(\mathsf{Set}_{\mathsf{C}}, \mathsf{D})$$

is an equivalence of categories.