Math 8850, Fall 2021
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Homework 2

Due: Wednesday, 15. Sept. 2021

Exercise 1. The dual notion to that of a colimit is a *limit.* Formally, a limit of a functor $F : \mathsf{I} \to \mathsf{C}$ is a colimit of the induced functor $F^{\mathrm{op}} : \mathsf{I}^{\mathrm{op}} \to \mathsf{C}^{\mathrm{op}}$.

1. Give a definition for a limit cone with is dual to the definition of colimit cocone given in class.

2. Without invoking duality, show that if for every diagram $F : \mathsf{I} \to \mathsf{C}$ there exists a limit cone (c, ρ) for F, then the functor

$$\mathrm{const} : \mathsf{C} \longrightarrow \mathsf{C}^{\mathsf{I}}$$

has a right adjoint $\lim_{\mathsf{I}}$.

Exercise 2. As discussed in class, a *coequalizer* is a colimit over a diagram of the form

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y.$$

1. Suppose a category C admits all coproducts and all coequalizers, and let $F : \mathsf{I} \to \mathsf{C}$ be a functor. Consider the diagram

$$\coprod_{\substack{f:i\to j \\ \text{morph of } \mathsf{I}}} F(i) \underset{t}{\overset{s}{\rightrightarrows}} \coprod_{i\in\mathsf{I}} F(i) \qquad (1)$$

Where $s : F(i) \to \coprod_{i\in\mathsf{I}} F(i)$ is the inclusion, and the component of t corresponding to $f : i \to j$ is the map

$$F(i) \xrightarrow{F(f)} F(j) \longrightarrow \coprod_{k\in\mathsf{I}} F(k).$$

Show that the coequalizer of (1) is a colimit of F. (Hint: write down a cocone over F, and show that it is initial.)

2. Write down the dual statement to part 1. (Note: You do not have to prove the dual statement.)

3. Show that Grp and Set are cocomplete.

Exercise 3. Let C be a category. Notice that, for every $x \in \mathsf{C}$, there is a functor

$$\begin{aligned} \mathrm{ev}_x : \mathsf{Set}_{\mathsf{C}} &\longrightarrow \mathsf{Set} \\ A &\longmapsto A(x). \end{aligned}$$

1. Show that, given a functor $F : \mathsf{I} \to \mathsf{Set}_{\mathsf{C}}$, a cone over F is a limit cone if and only if, for every $x \in \mathsf{C}$, the induced cone over $\mathrm{ev}_x \circ F$ is a limit cone. Conclude that $\mathsf{Set}_{\mathsf{C}}$ is complete.

2. Show that given a functor $F : \mathsf{I} \to \mathsf{C}$ there is an isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(c, \lim_{\mathsf{I}} F) \cong \lim_{\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(c, F(-))$$

natural in c. (Where the latter limit is the limit of $よ \circ F : \mathsf{I} \to \mathsf{Cat}_{\mathsf{C}}$)

3. Write down the dual statement to part 2. (Note: you do not need to prove the dual statement.)