Homework 1

DUE: Wednesday, 8. Sept. 2021

Exercise 1. By construction, categories that arise from posets have the following property: between any given pair of objects, there is at most one morphism. Does every category with this property arise from a poset? What additional properties are needed to characterize those categories that arise from posets?

Exercise 2. A 2-category $\mathbb{C}$ consists of

- a set $\mathrm{ob}(\mathbb{C})$ of objects,
- for every pair (x, y) of objects, a category $\mathbb{C}(x, y)$ of morphisms from x to y,
- for every object x an object $\mathrm{id}_x \in \mathbb{C}(x, x)$ called the identity morphism,
- for every triple x, y, z, a functor

  $$\mu : \mathbb{C}(x, y) \times \mathbb{C}(y, z) \to \mathbb{C}(x, z)$$

  called a composition law,

subject to the conditions

1. for every objects $x, y \in \mathbb{C}$, the functors

   $$\mu(-, \mathrm{id}_y) : \mathbb{C}(x, y) \to \mathbb{C}(x, y)$$

   and

   $$\mu(\mathrm{id}_x, -) : \mathbb{C}(x, y) \to \mathbb{C}(x, y)$$

   are the identity functors on $\mathbb{C}(x, y)$,

2. for every 4-tuple (x, y, z, w) of objects of $\mathbb{C}$, the diagram

   $$\begin{array}{ccc} \mathbb{C}(x,y) \times \mathbb{C}(y,z) \times \mathbb{C}(z,w) & \xrightarrow{\mu \times \mathrm{id}} & \mathbb{C}(x,z) \times \mathbb{C}(z,w) \\ \downarrow{\scriptstyle \mathrm{id} \times \mu} & & \downarrow{\scriptstyle \mu} \\ \mathbb{C}(x,y) \times \mathbb{C}(y,w) & \xrightarrow{\mu} & \mathbb{C}(x,w) \end{array}$$

   commutes.

Show that the set of (small) categories forms the objects of a 2-category $\mathbb{C}\mathsf{at}$ where, for small categories C,D, we define $\mathbb{C}\mathsf{at}(\mathsf{C}, \mathsf{D}) = \mathrm{Fun}(\mathsf{C}, \mathsf{D})$.

Exercise 3. Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ and $G : \mathsf{D} \longrightarrow \mathsf{C}$ are adjoint functors with unit $\epsilon : \mathrm{id}_{\mathsf{C}} \Rightarrow G \circ F$ and counit $\eta : F \circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$.

1. Show that ϵ is a natural isomorphism if and only if F is fully faithful.
2. Show that η is a natural isomorphism if and only if G is fully faithful.

Exercise 4. For a category C, we will use the notation $\mathsf{Set}_{\mathsf{C}} := \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$.

1. Show that the Yoneda embedding

$$よ : \mathsf{C} \longrightarrow \mathsf{Set}_{\mathsf{C}}$$

   defined in class is a functor.
2. Use the Yoneda lemma to show that よ is fully faithful.
3. Is よ essentially surjective?