Math 8850, Fall 2021
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Homework 0

This is a homework assignment designed to help me get a sense of your categorical background before the semester starts. *Don't feel discouraged* if you don't know how to complete these exercises. Depending on the responses, the first few weeks of class will go over categorical background.

Please complete as many of these exercises as you can, and send solutions to me at ws7jx@virginia.edu before the first day of class.

Exercise 1. Let C be a category, and denote by $\mathsf{Set}_{\mathsf{C}}$ the functor category $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$.

1. Show that the assignment on objects

$$\begin{aligned} よ : \mathsf{C} &\longrightarrow \mathsf{Set}_{\mathsf{C}} \\ c &\longmapsto \mathrm{Hom}_{\mathsf{C}}(-, c) \end{aligned}$$

   defines a functor.

2. Show that よ is fully faithful.

Exercise 2. Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ and $G : \mathsf{D} \longrightarrow \mathsf{C}$ are adjoint functors with unit $\epsilon : \mathrm{id}_{\mathsf{C}} \Rightarrow G \circ F$ and counit $\eta : F \circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$.

1. Show that ϵ is a natural isomorphism if and only if F is fully faithful.
2. Show that η is a natural isomorphism if and only if G is fully faithful.

Exercise 3. Let $\phi : I \to J$ be a fully faithful functor, and let C be a cocomplete category. Denote by

$$\begin{aligned} \phi^* : \mathrm{Fun}(J, \mathsf{C}) &\longrightarrow \mathrm{Fun}(I, \mathsf{C}) \\ F &\longmapsto F \circ \phi \end{aligned}$$

the restriction functor. Denote by

$$\phi_! : \mathrm{Fun}(I, \mathsf{C}) \longrightarrow \mathrm{Fun}(J, \mathsf{C})$$

the left adjoint of ϕ^*, i.e. the functor which sends each $F : I \to \mathsf{C}$ to the left Kan extension of F along ϕ. Prove that the unit

$$\eta : \mathrm{id}_{\mathrm{Fun}(I,\mathsf{C})} \Longrightarrow \phi^* \circ \phi_!$$

is an isomorphism.

Exercise 4. Let C, $\mathcal{Y}$, and $\mathsf{Set}_{\mathsf{C}}$ be as in Exercise 1. Suppose given a cocomplete category D and a functor $F : \mathsf{C} \to \mathsf{D}$. Denote by $\mathcal{Y}_! F : \mathsf{Set}_{\mathsf{C}} \to \mathsf{D}$ the left Kan extension of F along $\mathcal{Y}$.

1. Show that $\mathcal{Y}_! F$ has a right adjoint, given on objects by

$$\begin{aligned} R : \mathsf{D} &\longrightarrow \mathsf{Set}_{\mathsf{C}} \\ d &\longmapsto \mathrm{Hom}_{\mathsf{D}}(F(-), d) \end{aligned}$$

2. For a functor $G : \mathsf{Set}_{\mathsf{C}} \to \mathsf{D}$, show that the counit $\mathcal{Y}_!(\mathcal{Y}^*(G)) \Rightarrow G$ is an isomorphism if and only if G sends colimit cones in $\mathsf{Set}_{\mathsf{C}}$ to colimit cones in D.

3. Denote by $\mathrm{Fun}^{\mathrm{colim}}(\mathsf{Set}_{\mathsf{C}}, \mathsf{D})$ the full subcategory of $\mathrm{Fun}(\mathsf{Set}_{\mathsf{C}}, \mathsf{D})$ on the functors which preserve colimit cones. Conclude that

$$\mathcal{Y}_! : \mathrm{Fun}(\mathsf{C}, \mathsf{D}) \longrightarrow \mathrm{Fun}^{\mathrm{colim}}(\mathsf{Set}_{\mathsf{C}}, \mathsf{D})$$

is an equivalence of categories.