Model Categories

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These are notes written to provide some background for participants in the 2022 REU in Topology at the University of Virginia. They rely quite heavily the references cited below, and should not be considered original. My thanks to Tanner Carawan for catching a number of errors in the first draft of this document.

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Motivation

In many different branches of mathematics, we are interested in kinds of equivalence which are different from isomorphisms. These various kinds of ‘equivalence’ allow us to focus on the specific properties which are preserved by equivalence. At this point, some examples are in order:

1. In category theory, we often study *equivalences of categories*. These are functors which are not isomorphisms, but capture the notion that two categories are ‘basically the same’.

2. In algebraic topology, we often study *homotopy equivalences* between topological spaces. These are continuous maps which, while they may not have inverses, have ‘inverses up to shrinking or stretching’ (i.e. up to homotopy). A lot of algebraic invariants of topological spaces (Euler characteristic, fundamental group, etc.) are preserved by homotopy equivalences, and so the study of these invariants often involves the study of homotopy equivalences.

3. In homological algebra, we study “quasi-isomorphisms” — maps of chain complexes which induce isomorphisms on homology.

The aim of the theory of model categories is to take inspiration from these examples, and create a rigorous axiomatic framework in which we can study equivalences. To do this, we need to understand our first and second examples in more detail.

To start out with, we don’t only have one notion of ‘equivalence’ between topological spaces, we have two:

1. A *homotopy equivalence* is a continous map

   $$f : X \longrightarrow Y$$

   such that there is a continuous map

   $$g : Y \longrightarrow X$$

   and $g \circ f$ and $f \circ g$ can each be stretched or squished into the appropriate identity map.

2. A *weak homotopy equivalence* is a continuous map

$$f : X \longrightarrow Y$$

which induces isomorphisms on all *homotopy groups* — these are generalizations $\pi_n(X, x)$ of the fundamental group $\pi_1(X, x)$ which detect higher-dimensional information.

It turns out that, for a special class of topological spaces, these two notions coincide, and trying to define these classes leads to the notion of a model category.

1 Cell complexes and Whitehead's Theorem

A particularly well-behaved class of inclusions of topological spaces is the set of inclusions of disk boundaries

$$\partial D^n \longrightarrow D^n$$

where $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ and $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n \mid |x| = 1\}$.[1] We call the disk D^n the n*-cell*, and the subspace ∂D^n its *boundary*.

In the study of algebraic topology, a very important class of "good" spaces is the *cell complexes*. These are the spaces that can be "built out of disks" in the following way.

[1] There is an important point to note here: $\mathbb{R}^0 = \{0\}$ is a one-point space, equipped with the norm $|0| = 0$. As a result D^0 is a one-point space, and ∂D^0 is the empty space.

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Definition 1.1. We define a n*-dimensional cell complex* inductively as follows:

1. The unique -1-dimensional cell complex is the empty space.

2. An n-dimensional cell complex is a space X such that there are

   (a) An $(n-1)$-dimensional cell complex Y.

   (b) An indexing set I, and a collection of continuous maps (the *attaching maps*)

   $$\{f_i : \partial D^n \to Y\}_{i \in I}.$$

   Such that X fits into a pushout diagram of the form

$$\begin{array}{ccc} \coprod_{i\in I} \partial D^n & \longrightarrow & \coprod_{i \in I} D^n \\ {\scriptstyle \coprod_{i\in I} f_i}\downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

More colloquially: a n-dimensional cell complex is a space that can be build by gluing together cells of dimension $\leq n$.

Example 1.2.

1. There is a unique -1-dimensional cell complex: the empty space $\varnothing$.

2. Every 0-dimesional cell complex is either empty[2] or is a set equipped with the discrete topology.

[2] Every $n-1$-dimensional cell complex can be viewed as an n-dimensional cell complex, by assuming that the set I_n of n-dimensional attaching maps is empty.

3. A 1-dimensional cell complex is a *topological graph*: it consists of a set I_V of points, with a set of line segments I_E such that the endpoints of each line segment $e \in I_E$ are points in I_V.

4. As an example of a 2-dimensional cell complex, consider the 2-sphere S^2. We can define this by an iterative construction. We start with the empty space $\varnothing$, then obtain a 0-dimensional cell complex by adding a single point— so we get the one-point space. We don't add anything in the next step, so we simply consider the 1-point space as a 1-dimensional cell complex. In the final step, we can add a single 2-cell via the unique attaching map $f : \partial D^2 \to *$. The pushout of the diagram

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$$\begin{array}{ccc} \partial D^2 & \longrightarrow & D^2 \\ {\scriptstyle f}\big\downarrow & & \\ * & & \end{array}$$

gives us the quotient space $D^2/(\partial D^2)$, which we then identify with the 2-sphere S^2.

5. There are other ways to show that S^2 is a 2-dimensional cell complex. For example, we could define the first two steps to be $\varnothing$ and $*$ as above. The next stage — the 1-dimensional cell complex — will be the pushout

$$\begin{array}{ccc} \partial D^1 & \longrightarrow & D^1 \\ \big\downarrow & & \\ * & & \end{array}$$

Which quotients the interval $[0, 1] \cong D^1$ by the relation that $0 \sim 1$. Thus, the 1-dimensional stage is the circle S^1. Finally, we attach two 2-cells to S^1, both via the same map: the identity on S^1. We thus obtain the pushout of

$$\begin{array}{ccc} \partial D^2 \amalg \partial D^2 & \longrightarrow & D^2 \amalg D^2 \\ \big\downarrow & & \\ S^1 & & \end{array}$$

which amounts to gluing two disks together along their boundary. Again, this is the 2-sphere.

Notice that in the second-to-last example we built up S^2 as a sequence of spaces

$$\varnothing \hookrightarrow * \hookrightarrow * \hookrightarrow S^2$$

such that each step is the inclusion an $n-1$-dimensional cell complex into an n-dimensional cell complex, and each inclusion map is a pushout of some collection of disk boundary inclusions $\partial D^n \to D^n$.

However, our last example gives the sequence

$$\varnothing \hookrightarrow * \hookrightarrow S^1 \hookrightarrow S^2$$

A different way of seeing S^2 as a 2-dimensional cell complex. So that we can keep track of how we build our complexes, we introduce the following definition.

Definition 1.3. A *n-dimensional cell decomposition* of an n-dimensional cell complex X consists of a sequence of spaces

$$\varnothing \hookrightarrow X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \cdots \hookrightarrow X^n = X$$

sets of indices $I_0, \ldots, I_n$, and sets of continuous maps

$$\{f_i^k : \partial D^k \to X^{k-1}\}_{i \in I_k}$$

such that the morphism $X^{k-1} \to X^k$ fits into a pushout diagram

$$\begin{array}{ccc} \coprod_{i \in I^k} \partial D^k & \longrightarrow & \coprod_{i \in I^k} D^k \\ \downarrow & & \downarrow \\ X^{k-1} & \longrightarrow & X^k \end{array}$$

We will call the space X^k the *k-skeleton* of the cell decomposition.

We can push this definition even farther:

Definition 1.4. A *cell decomposition* of a topological space X is an infinite sequence of spaces

$$\varnothing \hookrightarrow X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \cdots$$

whose colimit (union) is X, and such that, for every $k \geq 0$, the sequence

$$\varnothing \hookrightarrow X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \cdots \hookrightarrow X^k$$

is a cell decomposition of X^k. We say that X is a (∞-dimensional) cell complex if there is a cell decomposition of X. Notice that every finite-dimensional cell complex is an infinite dimensional cell complex.

One final definition will give us (finally) the class of "good spaces" we want to consider.

Definition 1.5. A space X is called a *retract* of a space Y if there are continuous maps

$$X \xrightarrow{s} Y \xrightarrow{p} X$$

such that $p \circ s = \mathrm{id}_X$. Notice that this implies, among other things that s is the inclusion of a subspace.

Definition 1.6. We call a space X *cofibrant* when X is a retract of a cell complex.

The cofibrant spaces are a particularly good realm to study weak homotopy equivalences, as the following theorem shows.

Theorem 1.7 (Whitehead). *Let $f : X \to Y$ be a continuous map between cofibrant spaces. Then the following are equivalent:*

1. *The map f is a homotopy equivalence.*
2. *The map f is a weak homotopy equivalence.*

What this tells us is that we can show that a map between cofibrant spaces is a homotopy equivalence by checking algebraic data. If f induces isomorphisms on all homotopy groups, then f is guaranteed to have an inverse up to homotopy!

2 *Axiomatizing cofibrations*

When we defined cofibrant spaces above, we only made use of some basic categorical constructions: coproducts, pushouts, colimits of directed sequences, and retracts. We now axiomatize how we did this.

Definition 1.8. Let $\mathcal{C}$ be a category which admits all colimits (in particular, $\mathcal{C}$ has an initial object $\varnothing$). Let $\mathbf{C}$ be a class of morphisms in $\mathcal{C}$. We say that $\mathbf{C}$ is

1. *Closed under coproducts* if, for $f_i : X_i \to Y_i$ in $\mathbf{C}$, the coproduct

   $$\textstyle\coprod_{i\in\mathbf{C}} f_i : \coprod_{i\in\mathbf{C}} X_i \longrightarrow \coprod_{i\in\mathbf{C}} Y_i$$

   is also in $\mathbf{C}$.

2. *Closed under pushouts* if, given a pushout diagram

   $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow[g]{} & W \end{array}$$

   in $\mathcal{C}$ where $f \in \mathbf{C}$, then $g \in \mathbf{C}$.

3. *Closed under retracts* if, given a commutative diagram

   $$\begin{array}{ccccc} C & \xrightarrow{s_1} & X & \xrightarrow{p_1} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[s_2]{} & Y & \xrightarrow[p_2]{} & D \end{array}$$

   such that $p_1 \circ s_1$ and $p_2 \circ s_1$ are each identities[3], then if $f \in \mathbf{C}$ we also have $g \in \mathbf{C}$.

4. *Closed under transfinite composition* if, given an (infinite) diagram

   $$X^0 \xrightarrow{f_1} X^1 \xrightarrow{f_2} X^2 \xrightarrow{f_3} \cdots$$

   in $\mathcal{C}$ with colimit X such that each f_i is in $\mathbf{C}$, then the induced morphism $X_0 \to X$ is also in $\mathbf{C}$.

[3] Such a diagram is called a *retract diagram*, displaying g as a *retract of* f.

We say that $\mathbf{C}$ is *saturated* in $\mathcal{C}$ when $\mathbf{C}$ is closed under coproducts, pushouts, retracts, and transfinite composition, and, moreover, every isomorphism lies in $\mathbf{C}$.

Given a set I of morphisms in $\mathcal{C}$, we call the smallest saturated set of morphisms in $\mathcal{C}$ containing I then *saturated hull of* I. We denote the saturated hull of I by $\mathrm{Cof}(I)$.

Using this language, we can rephrase our definition of cofibrant spaces in the following form:

Definition 1.9. The class of *classical cofibrations* in Top is the saturated hull of the disk boundary inclusions, i.e.

$$\mathrm{Cof}_{\mathrm{cl}} = \mathrm{Cof}(\{\partial D^n \to D^n\}_{n\geq 0}).$$

We say a topological space X is *cofibrant* if the unique morphism

$$\varnothing \to X$$

from the initial (empty) topological space to X is a classical cofibration.

The proof of Whitehead's theorem makes use of two classes of morphisms, satisfying dual axioms. One is the class of classical cofibrations defined above, and the other is the class of *Serre fibrations*. In the next chapter, we will describe the structure that is used to prove this theorem.

3 *Fibrations and lifting problems*

The other key tool that we use to study the weak equivalences of topological spaces are the *Serre fibrations*. In the interest of time, we will not dwell overlong on the Serre fibrations, instead simply introducing a few relevant definitions, and giving two exercises. The basic idea here is that of *lifitng properties*:

Definition 1.10. A continuous map $\pi : X \to Y$ of topological spaces is called a *Serre fibration* if for any commutative diagram

$$\begin{array}{ccc} \{0\} \times D^n & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ I \times D^n & \longrightarrow & Y \end{array}$$

in Top, there is a continuous map $f : D^n \to X$ such that the diagram

$$\begin{array}{ccc} \{0\} \times D^n & \longrightarrow & X \\ \big\downarrow & \nearrow^{f} & \big\downarrow{\scriptstyle \pi} \\ I \times D^n & \longrightarrow & Y \end{array}$$

commutes.

The commutative square above is often called a *lifiting problem* in Top, and the morphism f is called a *lift* or *solution to the lifting problem.* More formally, we can define this in *any* category

Definition 1.11. Let $\mathcal{C}$ be a category, and let

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle \iota}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ B & \longrightarrow & Y \end{array}$$

be a commutative diagram in $\mathcal{C}$. We call such a diagram a *lifitng problem* If there is a morphism $f : B \to X$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle \iota}\big\downarrow & \nearrow^{f} & \big\downarrow{\scriptstyle \pi} \\ B & \longrightarrow & Y \end{array}$$

commutes, we say that f *solves this lifting problem* or that f *is a lift* of the diagram. If *every* lifting problem with vertical morphisms ι and π has a solution, we say ι *has the left lifting property (LLP) with respect to* π and π *has the right lifting property (RLP) with respect to* ι.

Given a set $\mathbf{U}$ of morphisms in $\mathcal{C}$, we denote by ${}_\perp\mathbf{U}$ the set of those morphisms in $\mathcal{C}$ which have the LLP with respect to $\mathbf{U}$, and we denote by $\mathbf{U}_\perp$ the set of those morphisms in $\mathcal{C}$ which have the RLP with respect to $\mathbf{U}$.

Exercise 1. Show that, for any topological space X, the unique map $X \to *$ to the 1-point space is a Serre fibration.

Exercise 2. Suppose $\mathcal{C}$ is a category with all colimits, and let $\mathbf{U}$ be a set of morphisms in $\mathcal{C}$.

1. Show that ${}_\perp\mathbf{U}$ is a saturated set.
2. Show that $\mathrm{Cof}(\mathbf{U}) \subset {}_\perp(\mathbf{U}_\perp)$.
3. Show that $\mathbf{U}_\perp$ is closed under retracts.

2

Model categories and examples

In the previous chapter, we tried to briefly motivate the definition of a saturated set of morphisms, claiming that it was an essential component of the proof of Whitehead's theorem. In this chapter, we will give the abstract, category-theoretic definition of a model structure — the precise structure necessary to prove Whitehead's theorem abstractly.[1]

[1] While we will do this at the end of this document, the statement is intended as motivation for the definition of a model category.

Definition 2.1. Let $\mathcal{C}$ be a category which admits all small limits and colimits. A *model structure* on $\mathcal{C}$ consists of three classes of morphisms in $\mathcal{C}$:

- A class $\mathrm{W}_{\mathcal{C}}$ called *weak equivalences.*
- A class $\mathrm{Cof}_{\mathcal{C}}$ called *cofibrations.*
- A class $\mathrm{Fib}_{\mathcal{C}}$ called *fibrations.*

We additionally call morphisms in $\mathrm{Fib}_{\mathcal{C}} \cap \mathrm{W}_{\mathcal{C}}$ (that is, morphisms which are both fibrations and weak equivalences) *trivial fibrations*, and morphisms in $\mathrm{Cof}_{\mathcal{C}} \cap \mathrm{W}_{\mathcal{C}}$ *trivial cofibrations.*

These three classes are required to satisfy the following conditions:

1. The class $\mathrm{W}_{\mathcal{C}}$ of weak equivalences contains every isomorphism in $\mathcal{C}$ and satisfies the 2-out-of-3 rule.[2]

[2] This means that, if f and g are morphisms in $\mathcal{C}$ such that we can compose to get $f \circ g$, then if any two of the morphisms $\{f, g, f \circ g\}$ are weak equivalences, so is the third. It can be instructive to check that the isomorphisms in any category satisfy the 2-out-of-3 rule.

2. Each of the classes $\mathrm{Cof}_{\mathcal{C}}$, $\mathrm{Fib}_{\mathcal{C}}$, and $\mathrm{W}_{\mathcal{C}}$ is closed under retracts.

3. Given a commutative diagram

$$\begin{array}{ccc} c & \longrightarrow & x \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ d & \longrightarrow & y \end{array}$$

in $\mathcal{C}$, there is a morphism $\ell : d \to x$ making the diagram

$$\begin{array}{ccc} c & \longrightarrow & x \\ {\scriptstyle i}\downarrow & \overset{\ell}{\nearrow} & \downarrow{\scriptstyle p} \\ d & \longrightarrow & y \end{array}$$

commute when either of the following conditions are satisfied:

- i is a cofibration and p is a trivial cofibration.
- i is a trivial cofibration and p is a fibration.

4. Any morphism in $\mathcal{C}$ can be factored as

$$x \xrightarrow{i} y \xrightarrow{p} z$$

where i is a cofibration and p is a trivial fibration.

5. Any morphism in $\mathcal{C}$ can be factored as

$$x \xrightarrow{i} y \xrightarrow{p} z$$

where i is a trivial cofibration and p is a fibration.

Example 2.2. Let $(\mathrm{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. Then $(\mathrm{C}^{\mathrm{op}}, \mathcal{F}\mathrm{ib}, \mathcal{C}\mathrm{of}, \mathcal{W})$ is a model category.

Example 2.3 (Motivating example)**.** One of the original motivations for this definition was the one we began with. In his original definition of a model category in [1], Quillen showed that there is a model structure on Top whose cofibrations are the classical cofibrations, whose fibrations are the Serre fibrations, and whose weak equivalences are the weak homotopy equivalences. We will not prove this here, as developing the tools to do so would be quite time-consuming.

Before giving an example in full, we will briefly turn to a discussion of lifting properties.

The model category axiom (3) is of interest, since it has implications for saturation. Together with the final exercise of the previous chapter, it implies that $\mathcal{C}\mathrm{of}_{\mathcal{C}} \subset {}_{\perp}(\mathcal{F}\mathrm{ib}_{\mathcal{C}} \cap \mathrm{W})$ and $\mathcal{C}\mathrm{of}_{C} \cap \mathrm{W}_{\mathcal{C}} \subset {}_{\perp}\mathcal{F}\mathrm{ib}_{\mathcal{C}}$. It turns out that these inclusions are actually equalities.

Proposition 2.4. *Let $(C, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. Then*

1. $\mathcal{C}\mathrm{of}_{\perp} = (\mathcal{F}\mathrm{ib} \cap \mathcal{W})$ *and* $\mathcal{C}\mathrm{of} = {}_{\perp}(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$

2. $(\mathcal{C}\mathrm{of} \cap \mathcal{W})_{\perp} = \mathcal{F}\mathrm{ib}$ *and* $(\mathcal{C}\mathrm{of} \cap \mathcal{W}) = {}_{\perp}\mathcal{F}\mathrm{ib}$.

Proof. It suffices for us to show $\mathcal{C}\mathrm{of} = {}_{\perp}(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$ and $(\mathcal{C}\mathrm{of} \cap \mathcal{W}) = {}_{\perp}\mathcal{F}\mathrm{ib}$. The other statements follow by duality. We prove one of these statements, and leave the other to the reader.

Note first that $\mathcal{C}\mathrm{of} \subset {}_{\perp}(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$ by (M3). Now suppose $i : A \to B$ is in ${}_{\perp}(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$. We can factor i as

$$A \overset{g}{\hookrightarrow} \widehat{B} \underset{\sim}{\overset{f}{\twoheadrightarrow}} B$$

by (M5). We can then form the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{g} & \widehat{B} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f}\ \sim \\ B & \xrightarrow[\mathrm{id}_B]{} & B \end{array}$$

which has a solution $r : B \to \widehat{B}$ by (M3). This then yields a retract diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[r]{} & \widehat{B} & \xrightarrow[f]{\sim} & B \end{array}$$

Thus, by (M2), $i \in \mathcal{C}$of. $\square$

Corollary 2.5. *In any model category, the classes* $\mathcal{C}$of *of cofibrations and* $\mathcal{C}$of $\cap$ W *of trivial cofibrations are saturated.*

1 The canonical model structure on Cat

We denote by Cat the 1-category of (small) categories, with functors as morphisms. We make the following definitions:

Definition 2.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories.

1. We call F a *categorical cofibration* if F is injective on objects.

2. We call F an *isofibration* if, for every $x \in \mathcal{C}$ and every isomorphism $f : F(x) \to y$ in $\mathcal{D}$, there is an isomorphism $\widetilde{f} : x \to \widetilde{y}$ such that $F(\widetilde{f}) = f$.

3. We call F a *weak equivalence* if F is essentially surjective and fully faithful.

Following [4], we will prove

Theorem 2.7. *There is a model structure on the category* Cat *whose cofibrations are the categorical cofibrations, whose fibrations are the isofibrations, and whose equivalences are the weak equivalences of categories.*

We will do this by verifying properties 1-5.

Property 1: Weak equivalences

It is immediate that every isomorphism of categories is a weak equivalence of categories. We are thus left to verify that the weak equivalences of categories satisfy the 2-out-of-three rule. We have already seen that a composite of equivalences is an equivalence.

To help us in our endeavors, we define a functor

$$\pi_0 : \mathsf{Cat} \longrightarrow \mathsf{Set}$$

which sends a category $\mathcal{C}$ to the set $\pi_0(\mathcal{C})$ of isomorphism classes of objects in $\mathcal{C}$. Given a functor $F : \mathcal{C} \to \mathcal{D}$, the map $\pi_0(F)$ sends and isomorphism class $[x]$ to the isomorphism class $[F(x)]$.

Lemma 2.8. *If* $F : \mathcal{C} \to \mathcal{D}$ *is a weak equivalence of categories, then* $\pi_0(F)$ *is an isomorphism of sets.*

Proof. Since F is essentially surjective, the induced map $\pi_0(F)$ is surjective. It thus only remains for us to show that $\pi_0(F)$ is injective.

Suppose that $x, y \in \mathrm{Ob}(\mathcal{C})$ such that there is an isomorphism $f : F(x) \to F(y)$ with inverse $f^{-1} : F(y) \to F(x)$. Since F is full, there are morphisms $\phi : x \to y$ and $\psi : y \to x$ such that $F(\phi) = f$ and $F(\psi) = g$. Moreover, we know that $F(\psi \circ \phi) = \mathrm{id}_{F(x)}$ and $F(\phi \circ \psi) = \mathrm{id}_{F(y)}$. Since f is faithful, this implies $\psi \circ \phi = \mathrm{id}_x$ and $\phi \circ \psi = \mathrm{id}_y$. Thus, x and y are isomorphic, and so $\pi_0(F)$ is injective. □

Suppose we have functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

between small categories such that $G \circ F$ is a weak equivalence.

Notice that if F is a weak equivalence, our lemma, and 2-out-of-3 applied to $\{\pi_0(F), \pi_0(G), \pi_0(G \circ F)\}$ shows that G is essentially surjective. Similarly, if G is a weak equivalence, then F is essentially surjective.

We now turn our attention to fully faithfulness.

1. Suppose that G is a weak equivalence. Then for any $x, y \in \mathcal{C}$, $G_{F(x),F(y)}$ is a bijection. Similarly, by hypothesis $G_{F(x),F(y)} \circ F_{x,y}$ is a bijection. Applying 2-out-of-3 for isomorphisms to

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \xrightarrow{F_{x,y}} \mathrm{Hom}_{\mathcal{D}}(F(x),F(y)) \xrightarrow{G_{F(x),F(y)}} \mathrm{Hom}_{\mathcal{E}}(G(F(x)),G(F(y)))$$

   we see that $F_{x,y}$ is a bijection, i.e., F is fully faithful. Since we have already seen that F is essentially surjective, this tells us that when G and $G \circ F$ are weak equivalences, so is F.

2. Now suppose that F is a weak equivalence, and let $c, d \in \mathcal{D}$. By essential surjectivity, choose isomorphisms $c \cong F(x)$ and $D \cong F(y)$. These, together with their images under G, induce bijections

$$\mathrm{Hom}_{\mathcal{D}}(c,d) \cong \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$$

   and

$$\mathrm{Hom}_{\mathcal{E}}(G(c),G(d)) \cong \mathrm{Hom}_{\mathcal{E}}(G(F(x)),G(F(y)))$$

   We then obtain a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Hom}_{\mathcal{D}}(c,d) & \xrightarrow{G_{c,d}} & \mathrm{Hom}_{\mathcal{E}}(G(c),G(d)) \\ & & \downarrow{\cong} & & \downarrow{\cong} \\ \mathrm{Hom}_{\mathcal{C}}(x,y) & \xrightarrow[F_{x,y}]{} & \mathrm{Hom}_{\mathcal{D}}(F(x),F(y)) & \xrightarrow[G_{F(x),F(y)}]{} & \mathrm{Hom}_{\mathcal{E}}(G(F(x)),G(F(y))) \end{array}$$

   of sets. Applying 2-out-of-3 for isomorphisms to the bottom row tells us $G_{F(x),F(y)}$ is a bijection. Applying 2-out-of-3 for bijections to the commutative square then tells us that $G_{c,d}$ is a bijection. Thus G is fully faithful, and so G is an equivalence.

We have thus shown that the weak equivalences of categories satisfy the first condition in our definition.

Property 2: Closure under retracts

We will first prove the categorical cofibrations are closed under retracts. We now define a *second* functor from Cat to Set:

$$\begin{aligned} \mathrm{Ob} : \mathrm{Cat} &\longrightarrow \mathrm{Set} \\ \mathcal{C} &\longmapsto \mathrm{Ob}(\mathcal{C}). \end{aligned}$$

Notice that functors — in this case, Ob in particular — send retract diagrams to retract diagrams. Moreover, a functor $F : \mathcal{C} \to \mathcal{D}$ is a categorical cofibration if and only if $\mathrm{Ob}(F)$ is injective.

It thus will suffice for us to show that injective maps of sets are closed under retracts in Set. Suppose that

$$\begin{array}{ccccc} X & \xrightarrow{s_X} & A & \xrightarrow{p_x} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[s_Y]{} & B & \xrightarrow{p_Y} & Y \end{array}$$

is a retract diagram of sets such that g is injective. We notice that since $p_X \circ s_X = \mathrm{id}_X$, s_X is injective, and thus $g \circ s_X$ is injective. This then immediately implies that f is injective as desired.

We then turn our attention to the weak equivalences. We will leave as an exercise that bijections in Set are closed under retracts, and use this freely in our argument. Given a retract diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{S_\mathcal{C}} & \mathcal{A} & \xrightarrow{P_\mathcal{C}} & \mathcal{C} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle F} \\ \mathcal{D} & \xrightarrow[S_\mathcal{D}]{} & \mathcal{B} & \xrightarrow{P_\mathcal{D}} & \mathcal{D} \end{array}$$

in Cat such that G is a weak equivalence, we obtain a retract diagram

$$\begin{array}{ccccc} \pi_0(\mathcal{C}) & \xrightarrow{\pi_0(S_\mathcal{C})} & \pi_0(\mathcal{A}) & \xrightarrow{\pi_0(P_\mathcal{C})} & \pi_0(\mathcal{C}) \\ {\scriptstyle \pi_0(F)}\downarrow & & \downarrow{\scriptstyle \pi_0(G)} & & \downarrow{\scriptstyle \pi_0(F)} \\ \pi_0(\mathcal{D}) & \xrightarrow[\pi_0(S_\mathcal{D})]{} & \pi_0(\mathcal{B}) & \xrightarrow{\pi_0(P_\mathcal{D})} & \pi_0(\mathcal{D}) \end{array}$$

by applying the functor π_0. Then $\pi_0(G)$ is a bijection, and so $\pi_0(F)$ is a bijection. In particular, F is essentially surjective.

To see fully faithfulness, fix objects $x, y \in \mathrm{Ob}(\mathcal{C})$. Then we obtain a retract diagram of sets

$$\begin{array}{ccccc} \mathrm{Hom}_\mathcal{C}(x,y) & \xrightarrow{(S_\mathcal{C})_{x,y}} & \mathrm{Hom}_\mathcal{A}(S_\mathcal{C}(x), S_\mathcal{C}(y)) & \xrightarrow{(P_\mathcal{C})_{S(x),S(y)}} & \mathrm{Hom}_\mathcal{C}(x,y) \\ {\scriptstyle F_{x,y}}\downarrow & & \downarrow{\scriptstyle G_{PS(x),S(y)}} & & \downarrow{\scriptstyle F_{x,y}} \\ \mathrm{Hom}_\mathcal{D}(Fx,Fy) & \xrightarrow[(S_\mathcal{D})_{Fx,Fy}]{} & \mathrm{Hom}_\mathcal{B}(S_\mathcal{D}(Fx), S_\mathcal{D}(Fy)) & \xrightarrow[(P_\mathcal{C})_{S(Fx),S(Fy)}]{} & \mathrm{Hom}_\mathcal{D}(Fx,Fy) \end{array}$$

of sets. Since G is fully faithful, $G_{S(x),S(y)}$ is a bijection. Since bijections are closed under retracts in Set, this implies that $F_{x,y}$ is a bijection. We thus see that F is fully faithful. Combining this with essential surjectivity, we see that F is a retract diagram.

Finally, we will prove a characterization of the fibrations which shows that they are closed under retracts

Definition 2.9. Define the *walking isomorphism* $\mathcal{I}$ to be the category with two objects 0 and 1, and two non-identity morphisms $f : 0 \to 1$ and $g : 1 \to 0$. This means that $f \circ g = \mathrm{id}_1$ and $g \circ f = \mathrm{id}_0$. There is a canonical inclusion of the discrete[3] category $\{0\}$.

[3] A *discrete* category on a set X is a category with the set X as objects, and no non-identity morphisms.

Exercise 3. Show that there is an *isomorphism* of categories $\mathcal{I}^{\mathrm{op}} \cong \mathcal{I}$.

Our characterization of the isofibrations is the following:

Lemma 2.10. *Write* Iso *for the class of isofibrations in* Cat*. Then*

$$\mathrm{Iso} \cong \{\{0\} \to \mathcal{I}\}_{\perp}.$$

Exercise 4. Prove the lemma by unwinding the definitions.

Applying this lemma, together with Exercise 2 from Chapter 1, we see that the class of isofibrations is closed under retracts.

Property 3: liftings

We now are studying a commutative diagram of categories

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathcal{B} & \xrightarrow[G]{} & \mathcal{D} \end{array}$$

in which ι is a categorical cofibration, and π is an isofibration. We want to show that, if either ι or π is an equivalence of categories, then there is a lift

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle\iota}\downarrow & \overset{L}{\nearrow} & \downarrow{\scriptstyle\pi} \\ \mathcal{B} & \xrightarrow[G]{} & \mathcal{D} \end{array}$$

making the diagram commute.

CASE 1: π IS A WEAK EQUIVALENCE

If π is a weak equivalence of categories, we construct L as follows.

First notice that since π is essentially surjective and an isofibration, π is surjective on objects.[4] We thus can choose, for every $b \in \mathrm{Ob}(\mathcal{B})$, an object $c_b \in \pi^{-1}(G(b))$. We specifically choose $c_b = F(a)$ whenever there is an $a \in \mathrm{Ob}(\mathcal{A})$ such that $F(a) = b$.[5]

[4] Essential sujectivity lets us choose an isomorphism $f(c) \cong d$ for every $d \in \mathrm{Ob}(\mathcal{D})$. Isofibrancy lets us lift this isomorphism; in particular, we obtain an object $\tilde{d}$ such that $\pi(\tilde{d}) = d$.

[5] We can make such a choice precisely because ι is injective on objects.

For any object $b \in \mathrm{Ob}(\mathcal{B})$, we define $L(b) = c_b$ to be the object we found above.

On homsets, defining L is a bit easier. For $b, e \in \mathrm{Ob}(\mathcal{B})$, we define the map

$$L_{b,e} : \mathrm{Hom}_{\mathcal{B}}(b, e) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(c_b, c_e)$$

to be the composite of

$$\mathrm{Hom}_{\mathcal{B}}(b, e) \xrightarrow{G_{b,e}} \mathrm{Hom}_{\mathcal{D}}(G(b), G(e)) \xrightarrow{\pi^{-1}_{c_b,c_e}} \mathrm{Hom}_{\mathcal{C}}(c_b, c_e).$$

The existence of $\pi^{-1}_{c_b,c_e}$ is guaranteed by the fully-faithfulness of π.

It is clear from the construction that this L will make the diagram commute, and thus provide a lift for our original square. However, we must first verify that L is, in fact, a functor.

Suppose that $f : b \to c$ and $g : c \to d$ are morphisms in $\mathcal{B}$. Then by definition

$$\pi(L(g) \circ L(f)) = G(g) \circ G(f) = G(g \circ f) = \pi(L(g \circ f)).$$

Since π is fully faithful, this means that $L(g \circ f) = L(g) \circ L(f)$. A virtually identical argument shows that L preserves identities. Thus L is a functor, and the argument is complete.

CASE 2: ι IS A WEAK EQUIVALENCE

When ι is a weak equivalence, our construction is somewhat more complicated. For each $b \in \mathrm{Ob}(\mathcal{B})$, we can use the essential surjectivity of ι to choose $a_b \in \mathrm{Ob}(\mathcal{A})$ and an isomorphism $\phi_b : \iota(a_b) \xrightarrow{\cong} b$ in $\mathcal{B}$. Moreover, since ι is injective on objects, we can choose $\phi_b = \mathrm{id}_b : \iota(a_b) \to b$ when b is in the image of ι.

Since π is an isofibration, we can choose isomorphisms $\psi_b : F(a_b) \to c_b$ in $\mathcal{C}$ such that $\pi(\psi_b) = F(\phi_b) : G(\iota(a_b)) \to G(b)$. As before, when b is in the image of ι, we can choose $c_b = F(a_b)$ and $\psi_b = \mathrm{id} c_b$.

We can then define L on objects by $L(b) = c_b$. On morphisms, given $f : b \to d$ in $\mathcal{B}$, we can use the fully faithfulness of ι to *uniquely* write f as

$$f = \phi_d \circ \iota(g_f) \circ \phi_b^{-1}$$

where $g_f : a_b \to a_d$ is a morphism in $\mathcal{A}$. We then define

$$L(f) = \psi_d \circ F(g_f) \circ \psi_b^{-1}.$$

We check functoriality. We first note that

$$L(\mathrm{id}_b) = \psi_b \circ F(\mathrm{id}_{a_b}) \circ \psi_b^{-1} = \psi_b \circ \psi_b^{-1} = \mathrm{id}_{L(b)}.$$

Given two composable morphisms in b, $f : b \to d$ and $h : d \to e$, we have

$$L(h \circ f) = \psi_e \circ F(g_{h\circ f}) \circ \psi_b^{-1} = \psi_e \circ F(g_h) \circ \psi_d^{-1} \circ \psi_d \circ F(g_f) \circ \psi_b^{-1} = L(h) \circ L(f)$$

as desired.

It is immediate from the construction that L solves the lifting problem above.

Properties 4 & 5: The factorizations

We now come to the factorizations. In what follows, we fix a functor $F : \mathcal{C} \to \mathcal{D}$. We will give a construction of two categories, and leave it as an exercise to show that these categories form the desired factorizations.

We define a category $\mathcal{L}$ whose objects are triples (c, d, ϕ) where $c \in \mathrm{Ob}(\mathcal{C})$, $d \in \mathrm{Ob}(\mathcal{D})$ and $\phi : F(c) \xrightarrow{\cong} d$ is an isomorphism in $\mathcal{D}$. We define

$$\mathrm{Hom}_{\mathcal{L}}((c, d, \phi), (a, b, \psi)) \cong \mathrm{Hom}_{\mathcal{C}}(c, a)$$

and let the composition and identities be those of $\mathcal{C}$.

Exercise 5. Show that F factors as

$$\mathcal{C} \xrightarrow{G} \mathcal{L} \xrightarrow{H} \mathcal{D}$$

where G is injective on objects, essentially surjective, and fully faithful, and where H is an isofibration.

We define a category $\mathcal{R}$ with

$$\mathrm{Ob}(\mathcal{R}) = \mathrm{Ob}(\mathcal{C}) \amalg \mathrm{Ob}(\mathcal{D}).$$

For $c_1, c_2 \in \mathcal{C}$ and $d_1, d_2 \in \mathcal{D}$, define

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{R}}(c_1, c_2) &:= \mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2)) \\
\mathrm{Hom}_{\mathcal{R}}(d_1, d_2) &:= \mathrm{Hom}_{\mathcal{D}}(d_1, d_2) \\
\mathrm{Hom}_{\mathcal{R}}(c_1, d_1) &:= \mathrm{Hom}_{\mathcal{D}}(F(c_1), d_1) \\
\mathrm{Hom}_{\mathcal{R}}(d_1, c_1) &:= \mathrm{Hom}_{\mathcal{D}}(d_1, F(c_1))
\end{aligned}$$

and let compositions and identities be those of $\mathcal{D}$.

Exercise 6. Show that F factors as

$$\mathcal{C} \xrightarrow{G} \mathcal{R} \xrightarrow{H} \mathcal{D}$$

where G is injective on objects, and where H is essentially surjective, fully faithful, and an isofibration.

3

Homotopies, and Whitehead's theorem

We now return to our initial motivation: Whitehead's theorem. We will provide enough background to state the general form of Whitehead's theorem, and then discuss some of its consequences. To do this properly, we will need four background notions in a model category: *fibrant objects*, *cofibrant objects*, *cylinder objects*, and *path objects*.

The trick here is that, with a model structure in place, we can define a notion of homotopy in the category in question, and then use this to greatly simplify the localization of the category at the weak equivalences.

Definition 3.1. Let $(\mathsf{C}, \mathcal{C}of, \mathcal{F}ib, \mathcal{W})$ be a model category, and let X be an object in C. A **cylinder object** $\mathrm{Cyl}(X) \in \mathsf{C}$ is an object equipped with a cofibration

$$X \amalg X \longrightarrow \mathrm{Cyl}(X)$$

and a weak equivalence

$$\mathrm{Cyl}(X) \longrightarrow X$$

such that the canonical map $X \amalg X \longrightarrow X$ factors as

$$X \amalg X \longrightarrow \mathrm{Cyl}(X) \xrightarrow{\simeq} X.$$

Dual Definition 3.1. Let $(\mathsf{C}, \mathcal{C}of, \mathcal{F}ib, \mathcal{W})$ be a model category, and let X be an object in C. A **path object** $\mathrm{Path}(X) \in \mathsf{C}$ is an object equipped with a fibration

$$\mathrm{Path}(X) \longrightarrow X \times X$$

and a weak equivalence

$$X \longrightarrow \mathrm{Path}(X)$$

Such that the canonical map $X \longrightarrow X \times X$ factors as

$$X \longrightarrow \mathrm{Path}(X) \longrightarrow X \times X.$$

Example 3.2.

1. In the model structure on Top described above, a cylinder object for X is the space $X \times I$, with structure maps

$$X \amalg X \cong X \times \{0,1\} \hookrightarrow X \times I \xrightarrow{\mathrm{proj}} X.$$

2. In the model structure on Cat, let I be the *walking isomorphism*, the category with two objects, and a single isomorphism between them. Then a cylinder object for any $\mathcal{C} \in \mathrm{Ob}(\mathsf{Cat})$ is

$$\mathcal{C} \amalg \mathcal{C} \cong \mathcal{C} \times \{0,1\} \hookrightarrow \mathcal{C} \times I \longrightarrow X$$

Dual Example 3.2.

1. In the model structure on Top described above, a path object for X is the space $\mathcal{M}\mathrm{ap}(I, X)$ of continuous maps with the compact-open topology, equipped with the structure maps

$$X \xrightarrow{\mathrm{const}} \mathcal{M}\mathrm{ap}(I, X) \xrightarrow{\mathrm{ev}_0 \times \mathrm{ev}_1} X \times X.$$

2. In the model structure on Cat, a path object for $\mathcal{C}$ is

$$\mathcal{C} \xrightarrow{\mathrm{const}} \mathrm{Fun}(I, \mathcal{C}) \xrightarrow{\mathrm{ev}_0 \times \mathrm{ev}_1} \mathcal{C} \times \mathcal{C}.$$

where I is the walking isomorphism.

Exercise 7. Show that, in any model category C, every object $X \in \mathsf{C}$ has both a path object and a cylinder object.

The main utility of cylinder and path objects is that they allow us to define a notion of homotopy in a model category.

Definition 3.3. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category, and let $f, g : X \to Y$ be a pair of morphisms in C. A **left homotopy** from f to g is a morphism

$$\mathrm{Cyl}(X) \xrightarrow{H} Y$$

from a cylinder object such that the composite map

$$X \amalg X \longrightarrow \mathrm{Cyl}(X) \xrightarrow{H} Y$$

is $f \amalg g : X \amalg X \to Y$.

Dual Definition 3.3. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category, and let $f, g : X \to Y$ be a pair of morphisms in C. A **right homotopy** from f to g is a morphism

$$X \xrightarrow{H} \mathrm{Path}(Y)$$

such that the composite

$$X \xrightarrow{H} \mathrm{Path}(Y) \longrightarrow Y \times Y$$

is $f \times g : X \to Y \times Y$.

Before we proceed, we briefly note that we can always obtain cylinder/path objects in a model category.

Lemma 3.4. *Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category, and let X be an object of C. Then there is a cylinder object for X.*

Proof. We factor the map $\mathrm{id} \amalg \mathrm{id} : X \amalg X \to X$ using (M5). This yields a factorization

$$X \amalg X \xrightarrow{i} C \xrightarrow{s} X$$

where i is a cofibration and s is a trivial fibration (in particular, a weak equivalence). □

Ideally, we'd like our two notions of homotopy to be the same, i.e. that two maps are left-homotopic if and only if they are right-homotopic. This is not always the case, but often, it will be so.

Example 3.5. Let X be a topological space, and let $a, b : * \to X$ be the inclusions of two points. A left homotopy from a to b is a factorization

$$* \amalg * \longrightarrow I \times * \xrightarrow{H} X$$

of the map $a \amalg b : * \amalg * \to X$. In other words, this is a path from a to b in X.

On the other hand, a right homotopy from a to b is a factorization

$$* \longrightarrow X^I \longrightarrow X \times X$$

of the map $a \times b : * \to X \times X$. This is *also* a path from a to b. So, in Top, our two notions agree.

In general, left and right homotopy do not always agree. More importantly, left/right homotopy do not always define an equivalence relation on the set of maps $\mathrm{Hom}_{\mathsf{C}}(X, Y)$.

To rectify these issues, we will need to introduce two special kinds of objects in a model category. We have already met both of these concepts in special cases.

Definition 3.6. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. Denote the initial and terminal objects of C by $\varnothing$ and $*$, respectively. An object X in C is called **fibrant** if the unique map

$$X \longrightarrow *$$

is a fibration. Dually, X is called **cofibrant** if the unique map

$$\varnothing \longrightarrow X$$

is a cofibration.

Example 3.7. In our standard model structure on Top, every object is fibrant, and the cofibrant spaces are retracts of cell complexes.

Lemma 3.8. *Let* $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ *be a model category and* $X \in \mathsf{C}$ *be a cofibrant object. Let* $X \amalg X \xrightarrow{i_1 \amalg j_1} \mathrm{Cyl}_1(X) \xrightarrow{s_1} X$ *and* $X \amalg X \xrightarrow{i_2 \amalg j_2} \mathrm{Cyl}_2(X) \xrightarrow{s_2} X$ *be two cylinder objects for* X*. Then*

1. *The morphisms* $i_1 : X \to \mathrm{Cyl}_1(X)$ *and* $j_1 : X \to \mathrm{Cyl}_1(X)$ *are trivial cofibrations.*
2. *The pushout* K *in the diagram*

$$\begin{array}{ccc} X & \xrightarrow{j_1} & \mathrm{Cyl}_1(X) \\ \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle p} \\ \mathrm{Cyl}_2(X) & \xrightarrow{q} & K \end{array}$$

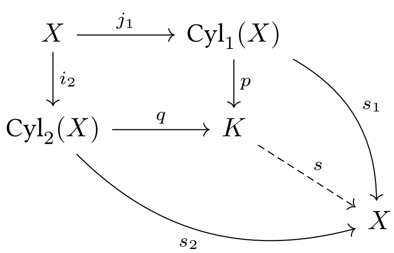
is a cylinder object for X *when equipped with the maps* $p \circ i_1$ *and* $q \circ j_2$ *from* $X \to K$ *and the map* $s : K \to X$ *obtained by universal property.*

Proof. To see (1), we note that by definition

$$X \xrightarrow{j_1} \mathrm{Cyl}_1(X) \xrightarrow{s_1} X$$

is a factorization of the identity, and $s_1 \in \mathcal{W}$. Since $\mathcal{W}$ satisfies 2-out-of-3, this means that $j_1 \in \mathcal{W}$. Moreover, since X is cofibrant, we can write the inclusion of the second factor $X \to X \amalg X$ as the coproduct of the identity on X with the cofibration $\varnothing \to X$. Since cofibrations are a saturated set, this implies this inclusion is a cofibration. Thus, j_1 is a composite of cofibrations, and so is itself a cofibration.

To see (2), we note that trivial cofibrations form a saturated class, and thus are closed under pushout. Since out argument for (1) shows that j_1 and i_2 are trivial cofibrations, we thus see that p and q must be trivial cofibrations as well. Universal property implies that the diagram



induces a unique dashed map s. Since s_1 and p are weak equivalences, 2-out-of-3 implies that s must be as well.

The final verification — that $i_1 \amalg j_2 : X \amalg X \to K$ is a cofibration — is left to the reader. ☐

Lemma 3.9. *Let* $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ *be a model category. If* X *is a cofibrant object and* Y *is any object, then left homotopy is an equivalence relation on* $\mathrm{Hom}_{\mathsf{C}}(X,Y)$.

Proof. First, let us show that f is left-homotopic to f. Let

$$X \amalg X \xrightarrow{\ i\ } \mathrm{Cyl}(X) \xrightarrow{\ s\ } X$$

be a cylinder object for X. It is then immediate that

$$X \amalg X \xrightarrow{\ i\ } \mathrm{Cyl}(X) \xrightarrow{f\circ s} Y$$

is a factorization of $f \amalg f : X \amalg X \to Y$. Thus $f \circ s$ is a homotopy from f to f.

Reflexivity of the relation follows immediately from the fact that if

$$X \amalg X \xrightarrow{i_1 \amalg i_2} \mathrm{Cyl}(X) \xrightarrow{\ s\ } X$$

is a cylinder object for X, then so is

$$X \amalg X \xrightarrow{i_2 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{\ s\ } X.$$

Transitivity follows immediately from Lemma 3.8, part (2). □

Dualizing, we immediately obtain

Dual Lemma 3.9. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. If Y is a fibrant object and X is any object, then right homotopy is an equivalence relation on $\mathrm{Hom}_{\mathsf{C}}(X,Y)$.

Finally, we aim to show that left and right homotopy describe the same equivalence relation on $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ when X is cofibrant and Y is fibrant.

Lemma 3.10. *Let* $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ *be a model category, and let* X *be a cofibrant object. Suppose that* $f, g : X \to Y$ *are left-homotopic. Then for any path object* $\mathrm{Path}(Y)$ *of* Y*, there is a right homotopy*

$$H : X \longrightarrow \mathrm{Path}(Y)$$

from f *to* g.

Proof. We begin with a cylinder object

$$X \amalg X \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(X) \xrightarrow{\ s\ } X$$

for X and a homotopy $G : \mathrm{Cyl}(X) \to Y$ from g to f. Additionally, we have fixed a path object

$$Y \xrightarrow{\ j\ } \mathrm{Path}(Y) \xrightarrow{\ p\ } Y \times Y.$$

We write a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{j \circ f} & \mathrm{Path}(Y) \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle p} \\ \mathrm{Cyl}(X) & \xrightarrow[(f\circ s)\times G]{} & Y \times Y \end{array}$$

By the dual of Lemma 3.8 (1), we note that the map $\mathrm{Path}(Y) \to Y \times Y$ is a fibration. Additionally, i_0 is a cofibration. This means we can solve the lifting problem represented by this commutative diagram, to get a map

$$\begin{array}{ccc} X & \xrightarrow{j \circ f} & \mathrm{Path}(Y) \\ {\scriptstyle i_0}\downarrow & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathrm{Cyl}(X) & \xrightarrow[(f \circ s) \times G]{} & Y \times Y \end{array}$$

making the diagram commute. We can then check that $\psi \circ i_0 : X \to Y$ defines a homotopy from f to g.

To see this, let $\pi_1, \pi_2 : Y \times Y \to Y$ be the two projections. We then compute

$$\begin{aligned} \pi_1 \circ p \circ \psi \circ i_0 &= \pi_1 \circ ((f \circ s) \times G) \circ i_0 \\ &= f \circ s \circ i_0 \\ &= f \circ \mathrm{id}_X = f \end{aligned}$$

and

$$\begin{aligned} \pi_2 \circ p \circ \psi \circ i_0 &= G \circ i_0 \\ &= g. \end{aligned}$$

completing the proof. □

Corollary 3.11. *Let* $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ *be a model category. Let* X *be a fibrant object, and* Y *a cofibrant object. Let* $f, g : X \to Y$ *be two morphisms. The following are equivalent.*

1. *For any cylinder object* $\mathrm{Cyl}(X)$ *of* X*, there is a left homotopy from* f *to* g *with cylinder object* $\mathrm{Cyl}(X)$.
2. f *and* g *are left-homotopic.*
3. *For any path object* $\mathrm{Path}(Y)$ *of* Y*, there is a right homotopy from* f *to* g *with path object* $\mathrm{Path}(Y)$.
4. f *and* g *are right-homotopic.*

Proof. Follows immediately from Lemma 3.10 and its dual. □

Definition 3.12. Let $(\mathsf{C}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib}, \mathcal{W})$ be a model category. We call an object $X \in \mathsf{C}$ **fibrant-cofibrant** if it is both fibrant and cofibrant.

Let $X, Y \in \mathsf{C}$ be fibrant-cofibrant objects. We call a morphism $f : X \to Y$ a **homotopy equivalence** in C if there is a morphism $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

Example 3.13.

1. In the model structure on Top, the homotopy equivalences in the model-categorical sense are precisely the classical homotopy equivalences.

2. In the model structure on Cat, the homotopy equivalences are precisely the homotopy equivalences of categories.

Proposition 3.14. *Let* $(\mathsf{C}, \mathcal{C}of, \mathcal{F}ib, \mathcal{W})$ *be a model category, and let* X *and* Y *be fibrant-cofibrant objects in* C*. If a morphism* $f : X \to Y$ *is in* $\mathcal{W}$*, then it is a homotopy equivalence.*

Proof. We first notice that, given a weak equivalence $f : X \to Y$, we can factor it as

$$X \xhookrightarrow[i]{\sim} Z \xrightarrow[s]{} Y$$

where i is a trivial cofibration. By 2-out-of-3, we see that s is also a weak equivalence, and thus a trivial fibration. It thus suffices to show the statement for trivial fibrations and trivial cofibrations. Since these two cases are dual, it will in fact suffice to prove one of them.

Suppose $f : X \to Y$ is a trivial cofibration. Fix a path object

$$Y \xrightarrow{i} \mathrm{Path}(Y) \xrightarrow{s} Y \times Y$$

for Y. Since f is a trivial cofibration and X is fibrant, we can solve the lifting problem

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle f}\downarrow & \overset{g}{\nearrow} & \downarrow \\ Y & \longrightarrow & * \end{array}$$

to obtain a dashed morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. We then consider the lifting problem

$$\begin{array}{ccc} X & \xrightarrow{i \circ f} & \mathrm{Path}(Y) \\ {\scriptstyle f}\downarrow & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[(f \circ g) \times \mathrm{id}_Y]{} & Y \times Y \end{array}$$

since f is a trivial cofibration and Y is a fibration, we can solve this lifting problem to obtain a morphism $\psi : Y \to \mathrm{Path}(Y)$. The map ψ is, by construction, a right-homotopy from $f \circ g$ to id_Y, completing the proof. □

As a corollary of this proposition, we obtain the most general form of Whitehead's theorem:

Theorem 3.15 (Whitehead)**.** *Let* $(\mathsf{C}, \mathcal{C}of, \mathcal{F}ib, \mathcal{W})$ *be a model category, and let* X *and* Y *be fibrant-cofibrant objects in* C*. A morphism* $f : X \to Y$ *is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. We need only show that the homotopy equivalences are weak equivalences. Consider a left homotopy

$$\begin{array}{ccc} \mathrm{Cyl}(X) & \xrightarrow{H} & X \\ {\scriptstyle (i_1, i_2)}\uparrow & \nearrow{\scriptstyle (f \circ g, \mathrm{id}_X)} & \\ X \amalg X & & \end{array}$$

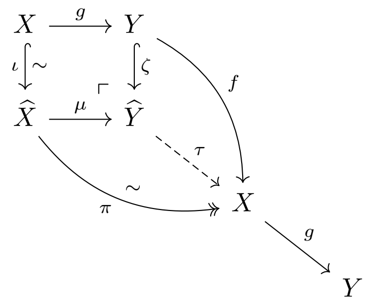
We notice that i_1 and i_2 are each weak equivalences. Thus, by 2-out-of-3, so is H. Applying 2-out-of-3 again, we see that $f \circ g$ is a weak equivalence.

Thus, if we have a homotopy equivalence

$$g : X \rightleftarrows Y : f$$

in $\mathcal{C}$, we see that $f \circ g$ and $g \circ f$ are weak equivalences.

To see that this implies that g is a weak equivalence (and thus, by 2-out-of-3, that f is as well), we need an additional argument. Our argument will chase the following diagram



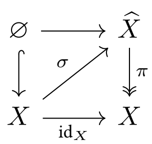
which we construct as follows. We factor the weak equivalence $f \circ g$ as a trivial cofibration $\iota : X \to \widehat{X}$ followed by a trivial fibration $\pi : \widehat{X} \to X$. We define $\widehat{Y}$, $\mu : \widehat{X} \to \widehat{Y}$, and $\zeta : Y \to \widehat{Y}$ via the pictured pushout square. The morphisms f and π then determine a unique morphism $\tau : \widehat{Y} \to X$ as pictured.

Since ζ is a pushout of a trivial cofibration, it is itself a trivial cofibration. Since

$$g \circ f = g \circ \tau \circ \zeta$$

is a weak equivalence, this implies by 2-out-of-3 that $g \circ \tau$ is a weak equivalence.

We then notice that there is a map $\sigma : X \to \widehat{X}$ such that $\pi \circ \sigma = \mathrm{id}_X$. This follows from solving the lifting problem

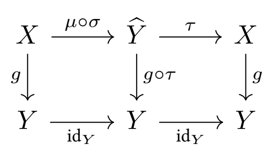


which admits a solution since X is cofibrant. Note that, by 2-out-of-3, σ is also a weak equivalence.

We then see that

$$\tau \circ \mu \circ \sigma = \pi \circ \sigma = \mathrm{id}_X.$$

This means that we can view $\widehat{Y}$ as a retract of X. Extending this, we obtain a retract diagram.



However, since $g \circ \tau$ is a weak equivalence, this implies that g is a weak equivalence, as desired. □

Remark 3.16. The argument above can be extended to prove a very general principle — the so-called 2-out-of-6 principle — in any model category. For a proof very similar to the above in a slightly more general setting, see Theorem 3.1 from this nLab page.

From this model categorical result, we can derive a number of better-known results. Firstly, the original theorem of Whitehead:

Corollary 3.17 (Whitehead)**.** *In* Top*, a morphism $f : X \to Y$ between cofibrant spaces is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Then the characterization of equivalences of categories:

Corollary 3.18. *A functor $F : \mathcal{C} \to \mathcal{D}$ of small categories is essentially surjective and fully faithful if and only if it is an equivalence of categories.*

There are a wide variety of other results which can be similarly extracted from Whitehead's Theorem. For those familiar with homological algebra, we offer one further concluding example:

Corollary 3.19. *A morphism $f : C_\bullet \to D_\bullet$ between chain complexes of projective R-modules is a quasi-isomorphism if and only if it is a chain homotopy equivalence.*

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