$\mathbb{R}$-enriched categories

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'Metric spaces' as enriched categories

There is a curious observation due to Lawvere in [2]: if you allow hom-sets to be something other than sets — letting them instead objects in a certain strange category — you can view the resulting notion of 'categories' as a generalization of metric spaces.

Definition 1.1. The category **R** is the category associated to the partially ordered set $[0, \infty]$, but in the opposite of the usual way: We say that there is a unique morphism $a \to b$ in **R** if and only if $a \geq b$.

There is a 'product' operation on this category: a functor[1]

$$\begin{aligned} (-) + (-) : \mathbf{R} \times \mathbf{R} &\longrightarrow \mathbf{R} \\ (a, b) &\longmapsto a + b \end{aligned}$$

[1] Here we use the conventions that $a + \infty = \infty$ and $a - \infty = 0$. In particular, we set $\infty - \infty = 0$.

This product has a neutral element: the object 0 in **R**. This is also the terminal object in **R**. Moreover, if we define

$$a^b := \max(a - b, 0)$$

there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{R}}(a + b, c) \cong \mathrm{Hom}_{\mathbf{R}}(a, c^b).$$

Remark 1.2. The category **R** also has all limits and colimits. Limits are, effectively, suprema, and colimits are, effectively, infima.

We can make use of this to define **R**-*enriched categories*. We will do this by comparison with the usual definition of categories. We fix a singleton $* \in \mathrm{Ob}(\mathrm{Set})$, and notice that, in the same way that $0 + a = a$ in **R**, we have $* \times A \cong A$ in Set.

Categories

A category $\mathcal{C}$ consists of

- A set $\mathrm{Ob}(\mathcal{C})$, whose elements are called the *objects* of $\mathcal{C}$
- For every pair of objects $x, y \in \mathrm{Ob}(\mathcal{C})$, a chosen object $\mathrm{Hom}_{\mathcal{C}}(x,y) \in \mathrm{Ob}(\mathrm{Set})$ called the *hom-set* from x to y in $\mathcal{C}$.
- For every object $x \in \mathrm{Ob}(\mathcal{C})$ a map $u_x : * \to \mathrm{Hom}_{\mathcal{C}}(x,x)$ in Set, called the *identity* on x.
- For every three objects $x, y, z \in \mathrm{Ob}(\mathcal{C})$, a map

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \times \mathrm{Hom}_{\mathcal{C}}(y,z) \to \mathrm{Hom}_{\mathcal{C}}(x,z)$$

in Set called *composition.*

These data must satisfy the following conditions.

- For any $x, y \in \mathrm{Ob}(\mathcal{C})$ the diagrams

$$\begin{array}{ccc} * \times \mathrm{Hom}_{\mathcal{C}}(x,y) & \xrightarrow{u_x \times \mathrm{id}_{\mathrm{Hom}}} & \mathrm{Hom}_{\mathcal{C}}(x,x) \times \mathrm{Hom}_{\mathcal{C}}(x,y) \\ & \searrow^{\cong} & \downarrow{\scriptstyle -\circ-} \\ & & \mathrm{Hom}_{\mathcal{C}}(x,y) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x,y) \times * & \xrightarrow{\mathrm{id}_{\mathrm{Hom}} \times u_y} & \mathrm{Hom}_{\mathcal{C}}(x,y) \times \mathrm{Hom}_{\mathcal{C}}(y,y) \\ & \searrow^{\cong} & \downarrow{\scriptstyle -\circ-} \\ & & \mathrm{Hom}_{\mathcal{C}}(x,y) \end{array}$$

in Set must commute.

- For any $x, y, z, w \in \mathrm{Ob}(\mathcal{C})$, the diagram

$$\begin{array}{ccc} \mathcal{C}(x,y) \times \mathcal{C}(y,z) \times \mathcal{C}(z,w) & \xrightarrow{\mathrm{id} \times \circ} & \mathcal{C}(x,y) \times \mathcal{C}(y,w) \\ {\scriptstyle \circ \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathcal{C}(x,z) \times \mathcal{C}(z,w) & \xrightarrow[\circ]{} & \mathcal{C}(x,w) \end{array}$$

must commute in Set

R-Categories

An **R**-category **C** consists of

- A set $\mathrm{Ob}(\mathbf{C})$, whose elements are called the *objects* of **C**
- For every pair of objects $x, y \in \mathrm{Ob}(\mathbf{C})$, a chosen object $\mathbf{C}(x,y) \in \mathrm{Ob}(\mathbf{R})$ called the *hom-object* from x to y in **C**.
- For every object $x \in \mathrm{Ob}(\mathbf{C})$ a map $u_x : 0 \to \mathbf{C}(x,x)$ in **R**, called the *identity* on x.
- For every three objects $x, y, z \in \mathrm{Ob}(\mathcal{C})$, a map

$$\mathbf{C}(x,y) + \mathbf{C}(y,z) \to \mathbf{C}(x,z)$$

in **R** called *composition.*

These data must satisfy the following conditions.

- For any $x, y \in \mathrm{Ob}(\mathbf{C})$ the diagrams

$$\begin{array}{ccc} 0 + \mathbf{C}(x,y) & \xrightarrow{u_x + \mathrm{id}_{\mathrm{Hom}}} & \mathrm{Hom}_{\mathcal{C}}(x,x) + \mathbf{C}(x,y) \\ & \searrow^{\cong} & \downarrow{\scriptstyle -\circ-} \\ & & \mathbf{C}(x,y) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{C}(x,y) + 0 & \xrightarrow{\mathrm{id}_{\mathrm{Hom}} + u_y} & \mathbf{C}(x,y) + \mathbf{C}(y,y) \\ & \searrow^{\cong} & \downarrow{\scriptstyle -\circ-} \\ & & \mathbf{C}(x,y) \end{array}$$

in **R** must commute.

- For any $x, y, z, w \in \mathrm{Ob}(\mathbf{C})$, the diagram

$$\begin{array}{ccc} \mathbf{C}(x,y) \times \mathbf{C}(y,z) \times \mathbf{C}(z,w) & \xrightarrow{\mathrm{id} \times \circ} & \mathcal{C}(x,y) \times \mathbf{C}(y,w) \\ {\scriptstyle \circ \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathbf{C}(x,z) \times \mathbf{C}(z,w) & \xrightarrow[\circ]{} & \mathbf{C}(x,w) \end{array}$$

must commute in **R**.

Notice that we really are just replacing Set with $\mathbf{R}$, and $\times$ with $+$. The result gives us a well-defined notion of a $\mathbf{R}$-*category*. Lawvere's key observation connected this with the notion of metric spaces:

Definition 1.3. A *Lawvere metric space*[2] (X, d) consists of a set X, together with a function

$$X \times X \longrightarrow [0, \infty]$$

satisfying the following conditions.

- For any $x \in X$, $d(x, x) = 0$.
- (Triangle inequality) For any $x, y, z \in X$,

$$d(x, y) + d(y, z) \geq d(x, z).$$

[2] Per the nLab, this concept could also answer to the name of *extended quasipseudometric space*, which is absurd.

These are weaker conditions than those of a metric space: we allow for infinite distances, do not require that $d(x, y) = 0$ if and only $x = y$, and do not require that $d(x, y) = d(y, x)$.[3]

[3] For ease of reference, let's recall two definitions:

Definition. A *metric space* (X, d) consists of a set X and a function

$$d : X \times X \longrightarrow [0, \infty)$$

such that

- (identity) For any $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
- (triangle inequality) For any $x, y, z \in X$,

$$d(x, z) \leq d(x, y) + d(y, z)$$

- (symmetry) For any $x, y \in X$,

$$d(x, y) = d(y, x).$$

An *extended metric space* (X, d) is a set X with a function

$$d : X \times X \longrightarrow [0, \infty]$$

satisfying the three axioms above.

Exercise 1. Convince yourself that an $\mathbf{R}$-category is the same thing as a Lawvere metric space.

Remark 1.4. Given a $\mathbf{R}$-category $\mathbf{C}$, we can form an $\mathbf{R}$-category $\mathbf{C}^{\mathrm{op}}$ with the same objects, and hom-objects given by

$$\mathbf{C}^{\mathrm{op}}(x, y) := \mathbf{C}(y, x).$$

If the Lawvere metric space associated to $\mathbf{C}$ additionally satisfies the symmetry axiom, then $\mathbf{C}^{\mathrm{op}} = \mathbf{C}$. The converse also holds.

We can also define functors between $\mathbf{R}$-categories.

Definition 1.5. Let $\mathbf{C}$ and $\mathbf{D}$ be two $\mathbf{R}$-categories. A $\mathbf{R}$-functor $F : \mathbf{C} \to \mathbf{D}$ consists of

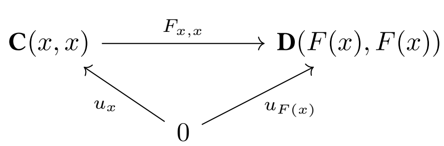
- A map $F : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$ of sets.
- For each $x, y \in \mathrm{Ob}(\mathbf{C})$ a morphism

$$F_{x,y} : \mathbf{C}(x, y) \longrightarrow \mathbf{D}(F(x), F(y))$$

  in $\mathbf{R}$.

These data must satisfy the conditions:

- For every $x \in \mathbf{C}$, the diagram



  in $\mathbf{R}$ commutes.

- For every $x, y, z \in \mathrm{Ob}(\mathbf{C})$, the diagram

$$\begin{array}{ccc} \mathbf{C}(x,y) + \mathbf{C}(y,z) & \xrightarrow{\circ} & \mathbf{C}(x,z) \\ {\scriptstyle F_{x,y}+F_{y,z}}\downarrow & & \downarrow{\scriptstyle F_{x,z}} \\ \mathbf{D}(x,y) + \mathbf{D}(y,z)' & \xrightarrow{\circ} & \mathbf{D}(x,z) \end{array}$$

Exercise 2. Show that an **R**-functor is equivalently a map $f : (X, d) \to (Y, \delta)$ of Lawvere metric spaces which is a *short map*, i.e., such that

$$\delta(f(x), f(y)) \leq d(x, y)$$

for any $x, y \in X$.

Definition 1.6. We denote by **R**Cat the category whose objects are small **R**-categories and whose morphisms are **R**-functors.

Example 1.7.

1. There is, as one might expect, an identity functor on any **R**-category, $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$, which acts as the identify on objects, and as the identity on hom-objects.

2. As we might expect, we can compose functors by composing the underlying maps of sets on objects, and composing the morphisms $F_{x,y}$ and $G_{F(x),F(y)}$ in **R**.

3. There is a *terminal* **R**-category **1**, which has a single object $*$, and has $\mathbf{1}(*, *) = 0$. For any **R**-category **C**, there is a unique **R**-functor $\mathbf{C} \to \mathbf{1}$.

4. There is a *initial* **R**-category $\varnothing$ which has no objects. For any **R**-category **C**, there is a unique **R**-functor $\varnothing \to \mathbf{C}$.

1 *Limits and colimits of* R*-categories*

The point of this section is to give a more-or-less explicit description of limits and colimits of **R**-enriched categories.

Construction 1.8. Let J be a small category, and let

$$\begin{aligned} \mathbf{F} : J &\longrightarrow \mathbf{R}\mathrm{Cat} \\ j &\longmapsto \mathbf{F}_j \end{aligned}$$

be a functor. If we compose F with the forgetful functor $\mathrm{Ob} : \mathbf{R}\mathrm{Cat} \to \mathrm{Set}$, we get a functor $\mathrm{Ob} \circ F$ from J to Set. Let (K, η) be a limit cone for this functor. We define an **R**-category **L** as follows:

1. Let $\mathrm{Ob}(\mathbf{L}) = K$.

2. For every $x, y \in K$ define

$$\mathbf{L}(x, y) := \sup_{j \in \mathrm{Ob}(J)} \mathbf{F}_j(\eta_j(x), \eta_j(y))$$

We then claim that

1. This defines an $\mathbf{R}$-category $\mathbf{L}$.
2. This definition makes each of the maps η_j into a functor $\mathbf{L} \to \mathbf{F}_j$, and thus makes $(\mathbf{L}, \eta)$ into a cone over F.
3. This cone is a limit cone.

To see (1), we must check that $\mathbf{L}(x,x) = 0$ for any $x \in \text{Ob}(\mathbf{L})$, and that the triangle inequality holds. For the first statement, we note that, for any $x \in \text{Ob}(\mathbf{L})$ and any $j \in J$ we have that $\mathbf{F}(\eta_j(x), \eta_j(x)) = 0$. Thus,

$$\mathbf{L}(x,x) = \sup_{j \in \text{Ob}(J)} (0) = 0.$$

For the triangle inequality, we note that for any $x, y, z \in \text{Ob}(\mathbf{L})$

$$\begin{aligned}
\mathbf{L}(x,y) &= \sup_{j \in \text{Ob}(j)} \mathbf{F}_j(\eta_j(x), \eta_j(y)) \\
&\leq \sup_{j \in \text{Ob}(j)} (\mathbf{F}_j(\eta_j(x), \eta_j(z)) + \mathbf{F}_j(\eta_j(z), \eta_j(y))) \\
&\leq \sup_{j \in \text{Ob}(j)} (\mathbf{F}_j(\eta_j(x), \eta_j(z))) + \sup_{j \in \text{Ob}(J)} (\mathbf{F}_j(\eta_j(z), \eta_j(y))) \\
&= \mathbf{L}(x,z) + \mathbf{L}(z,y)
\end{aligned}$$

as desired.

To see (2), we simply need to note that, for any $x, y \in \text{Ob}(\mathbf{L})$, we have

$$F_j(\eta_j(x), \eta_j(y)) \leq \sup_{j \in \text{Ob}(J)} \mathbf{F}_j(\eta_j(x), \eta_j(y)) = \mathbf{L}(x,y)$$

as desired.

To see (3), suppose we have another cone $(\mathbf{D}, \mu)$ over F. Then on sets of objects, there is a unique map

$$G : \text{Ob}(\mathbf{D}) \longrightarrow \text{Ob}(\mathbf{L})$$

which commutes with the cones η and μ.[4] Moreover, we see that for any $x, y \in \text{Ob}(\mathbf{D})$, we have

$$\begin{aligned}
\mathbf{L}(G(x), G(y)) = \sup_{j \in \text{Ob}(J)} \mathbf{F}_j(\eta_j(G(x)), \eta_j(G(y))) &= \sup_{j \in \text{Ob}(J)} \mathbf{F}_j(\mu_j(x), \mu_j(y)) \\
&\leq \sup_{j \in \text{Ob}(J)} \mathbf{D}(x,y) = \mathbf{D}(x,y)
\end{aligned}$$

so that this unique map G is, in fact, an $\mathbf{R}$-functor.

[4] This is because we defined the set of objects to be the limit in Set of F.

For the explict construction of colimits, we follow [**?**].

Construction 1.9. Let J be a small category, and let

$$\begin{aligned}
F : J &\longrightarrow \mathbf{R}\text{Cat} \\
j &\longmapsto \mathbf{F}_j
\end{aligned}$$

be a functor. If we compose F with the forgetful functor Ob : $\mathbf{R}$Cat $\to$ Set, we get a functor Ob $\circ F$ from J to Set. Let (K, η) be a colimit cone for this functor and define an $\mathbf{R}$-category $\mathbf{C}$ as follows.

1. Let $\mathrm{Ob}(\mathbf{C}) = K$.
2. Define the coproduct of underlying object sets to be

$$Z := \coprod_{j \in \mathrm{Ob}(J)} \mathrm{Ob}(\mathbf{F}_j)$$

   and define the structure of a $\mathbf{R}$-category $\mathbf{Z}$ on Z by setting

$$\mathbf{Z}(x,y) := \begin{cases} \mathbf{F}_j(x,y) & x,y \in \mathrm{Ob}(\mathbf{F}_j) \\ \infty & \text{else.} \end{cases}$$

   Denote by

$$\rho : Z \longrightarrow K$$

   the canonical quotient map.
3. For any x, y in K, define a *polygonal path* from x to y in K to be a finite ordered sequence of pairs $\{(x_i, y_i)\}_{i=0}^k$ with $x_i, y_i \in \mathrm{Ob}(\mathbf{F}_j)$ for some $j \in \mathrm{Ob}(J)$, and such that $\rho(x_0) = x$, $\rho(y_k) = y$, and $\rho(y_i) = \rho(x_{i+1})$. Denote by $\mathcal{P}_{x,y}$ the set of polygonal paths from x to y.
4. For every $x, y \in K$, define

$$\mathbf{C}(x,y) := \inf_{\{(x_i,y_i)\} \in \mathcal{P}_{x,y}} \left(\sum_{i=0}^{k} \mathbf{Z}(x_i, y_i) \right)$$

We then claim that

1. This definition makes $\mathbf{C}$ into a $\mathbf{R}$-category.
2. This definition makes each of the maps η_j into a functor $\mathbf{F}_j \to \mathbf{C}$, and thus makes $(\mathbf{C}, \eta)$ into a cone under F.
3. This cone is a colimit cone.

To see (1), we must again check that $\mathbf{C}(x,x) = 0$, and the triangle inequality. For the first of these properties, we simply need note that for any $x_0 \in \mathrm{Ob}(\mathbf{F}_j)$ such that $\rho(x_0) = x$, then $\{(x_0, x_0)\}$ is a polygonal path from x to x. Since

$$\sum_{i=0}^{0} \mathbf{Z}(x_0, x_0) = 0,$$

this immediately implies that $\mathbf{C}(x,x) = 0$.

For triangle inequality, notice that for $x, y, z \in \mathrm{Ob}(\mathbf{C})$ we can concatenate polygonal paths from x to y with polygonal paths from y to z. Thus, for every pair of polygonal paths $\vec{w} \in \mathcal{P}_{x,y}$ and $\vec{u} \in \mathcal{P}_{y,z}$ there is a polygonal path $\vec{w} \star \vec{u} \in \mathcal{P}_{x,z}$ such that the sum associated to $\vec{w} \star \vec{u}$ is the same as the sum of the sums associated to $\vec{w}$ and $\vec{u}$. To show triangle inequality, we then notice that for any $\epsilon > 0$, there are polygonal paths $\vec{u} \in \mathcal{P}_{x,y}$ and $\vec{v} \in \mathcal{P}_{y,z}$ such that

$$\left(\sum_{\vec{u}} \mathbf{Z}(x_i, y_i)\right) - \mathbf{C}(x, y) < \frac{\epsilon}{2}$$

and

$$\left(\sum_{\vec{v}} \mathbf{Z}(x_i, y_i)\right) - \mathbf{C}(y, z) < \frac{\epsilon}{2}$$

Thus, we see that

$$\left(\sum_{\vec{u}\star\vec{v}} \mathbf{Z}(x_i, y_i)\right) - (\mathbf{C}(x, y) + \mathbf{C}(y, z)) \leq \epsilon.$$

So for any $\epsilon \geq 0$, there is a $\vec{u} \star \vec{v} \in \mathcal{P}_{x,z}$ such that

$$\sum_{\vec{u}\star\vec{v}} \mathbf{Z}(x_i, y_i)$$

is ϵ-close to $\mathbf{C}(x, y) + \mathbf{C}(y, z)$. This implies that

$$\mathbf{C}(x, z) = \inf_{\vec{w}\in\mathcal{P}_{x,z}} \left(\sum_{\vec{w}} \mathbf{Z}(x_i, y_i)\right) \leq \mathbf{C}(x, y) + \mathbf{C}(y, z)$$

as desired.

The property (2) follows immediately, since, for any $a, b \in \mathrm{Ob}(\mathbf{F}_j)$, the pair (a, b) is a polygonal path from $\eta_j(a)$ to $\eta_j(b)$, so that

$$\mathbf{C}(\eta_j(a), \eta_j(b)) := \inf_{\vec{v}\in\mathcal{P}_{\eta_j(a),\eta_j(b)}} \left(\sum_{\vec{v}} \mathbf{Z}(x_i, y_i)\right) \leq \mathbf{F}_j(a, b)$$

as desired.

To see (3), suppose $(\mathbf{D}, \mu)$ is another cone under F. Notice that there is a unique underlying map of sets $G : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$. We need only show that this is an $\mathbf{R}$-functor. Given $x, y \in \mathbf{C}$, we notice that for any polygonal path $\vec{v} \in \mathcal{P}_{x,y}$, iterated triangle inequality for $\mathbf{D}$, together with the functoriality of the μ_j implies that

$$\mathbf{D}(G(x), G(y)) \leq \sum_{\vec{v}} \mathbf{Z}(x_i, y_i).$$

Thus,

$$\mathbf{D}(G(x), G(y)) \leq \mathbf{C}(x, y)$$

as desired.

Proposition 1.10. *The category* $\mathbf{R}$Cat *has all (small) limits and colimits.*

This also tells us that we can actually impose one more nice property on Lawvere metric spaces/$\mathbf{R}$-categories and still get a category of $\mathbf{R}$-categories which has all limits and colimits.

Definition 1.11. An $\mathbf{R}$-category $\mathbf{C}$ is called *symmetric* if

$$\mathbf{C}(x, y) = \mathbf{C}(y, x)$$

for any $x, y \in \mathrm{Ob}(\mathbf{C})$. We denote the full subcategory of $\mathbf{R}$Cat on the symmetric $\mathbf{R}$-categories by $\mathbf{R}\mathrm{Cat}^{\mathrm{sym}}$.

Corollary 1.12. *The subcategory* $\mathbf{R}\mathrm{Cat}^{\mathrm{sym}}$ *is closed under limits and colimits. Thus* $\mathbf{R}\mathrm{Cat}^{\mathrm{sym}}$ *has all small limits and colimits.*

Proof. Notice that if each $\mathbf{F}_j$ is a symmetric $\mathbf{R}$-category, we can switch the orders of $\eta_j(x)$ and $\eta_j(y)$ in the supremum defining the limit, thus, the resulting $\mathbf{R}$-category $\mathbf{L}$ is symmetric. For colimits we make the same argument — reversing the direction of each polygonal path, as well as the order of arguments in $\mathbf{Z}(-,-)$ in the infimum which defines the hom-objects of $\mathbf{C}$. ☐

2 R-enriched transformations

To define $\mathbf{R}$-enriched natural transformations, we once again simply rewrite definitions, using $\mathbf{R}$ instead of Set. Our main tool is the same observation we implicitly used above: an element in a set A is the same thing as a map of sets

$$* \longrightarrow A$$

from a singleton. We can thus replace the idea of 'a morphism in a hom-set $\mathrm{Hom}_{\mathcal{C}}(x, y)$' with 'a morphism $* \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x, y)$ in Set.'

In the context of a $\mathbf{R}$-category $\mathbf{C}$ — where we replace Set with $\mathbf{R}$ — we can think of 'a morphism in $\mathbf{C}(x, y)$' as 'a morphism

$$0 \longrightarrow \mathbf{C}(x, y)$$

in $\mathbf{R}$.'

Definition 1.13. Let $\mathbf{C}$ and $\mathbf{D}$ be $\mathbf{R}$-enriched categories, and let $F, G : \mathbf{C} \to \mathbf{D}$ be $\mathbf{R}$-enriched functors. An $\mathbf{R}$-*enriched natural transformation* from F to G is a collection of morphisms

$$\{\ \mu_c : 0 \longrightarrow \mathbf{D}(F(c), G(c))\ \}_{c \in \mathrm{Ob}(\mathbf{C})}$$

in $\mathbf{R}$ such that for any $c, d \in \mathrm{Ob}(\mathbf{C})$, the diagram

$$\begin{array}{ccccc}
\mathbf{C}(c,d) & \xrightarrow{=} & 0 + \mathbf{C}(c,d) & \xrightarrow{\mu_c + G} & \mathbf{D}(F(c), G(c)) + \mathbf{D}(G(c), G(d)) \\
\downarrow = & & & & \downarrow \circ \\
\mathbf{C}(c,d) + 0 & \xrightarrow[F+\mu_d]{} & \mathbf{D}(F(c), F(d)) + \mathbf{D}(F(d), G(d)) & \xrightarrow[\circ]{} & \mathbf{D}(F(c), G(d))
\end{array}$$

in $\mathbf{R}$ commutes.

Remark 1.14. The condition that there is a morphism $0 \to \mathbf{D}(F(c), G(d))$ in $\mathbf{R}$ means precisely that

$$\mathbf{D}(F(c), G(d)) \leq 0$$

so that $\mathbf{D}(F(c), G(d)) = 0$. Because of this, the commutative diagram condition becomes vacuous. Thus, there is a unique natural transformation from F to G if and only if, for every $c \in \text{Ob}(\mathbf{C})$, $\mathbf{D}(F(c), G(c)) = 0$.

We can compose natural transformations using the composition law in **D**, but since our **R**-enriched natural transformations have such simple descriptions, we don't really need to give a definition. Given two functors F and G, there are either no natural transformation for F to G, or precisely one. As such, composition is easy to define: Given **R**-enriched natural transformations $\mu : F \Rightarrow G$ and $\eta :\Rightarrow H$, then we note that, for any $x \in \text{Ob}(\mathbf{C})$, we have

$$0 \leq \mathbf{C}(F(x), H(x)) \leq \mathbf{C}(F(x), G(x)) + \mathbf{C}(G(x), H(x)) = 0 + 0 = 0$$

so that $\mathbf{C}(F(x), H(x)) = 0$. This means there is precisely one natural transformation from F to H, which we call the composite of μ and η. This also has an interesting consequence:

Lemma 1.15. *An* **R***-enriched natural transformation* $\mu : F \Rightarrow G$ *is an* **R***-enriched natural isomorphism if and only if, for every* $c \in \text{Ob}(\mathbf{C})$*,*

$$\mathbf{D}(F(c), G(c)) = 0 = \mathbf{D}(G(c), F(c))$$

Using this fact, let's define the notion of an **R**-equivalence.

Definition 1.16. A **R**-enriched functor $F : \mathbf{C} \to \mathbf{D}$ is said to be

1. A *homotopy equivalence* if there is a functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $F \circ G \cong \text{Id}_{\mathbf{D}}$ and $G \circ F \cong \text{Id}_{\mathbf{C}}$.
2. *Essentially surjective* if, for every object $d\,\text{Ob}(\mathbf{D})$, there is an object $c \in \text{Ob}(\mathbf{C})$ such that
$$\mathbf{D}(F(c), d) = 0 = \mathbf{D}(d, F(c))$$
3. *Fully faithful* if, for every $c, d \in \text{Ob}(\mathbf{C})$ the morphism
$$\mathbf{C}(c, d) \longrightarrow \mathbf{D}(F(c), F(d))$$
is an isomorphism (and thus equality) in **R**.

Exercise 3. Show that an **R**-enriched functor is an equivalence if and only if it is essentially surjective and fully faithful.

Exercise 4. Let **C** and **D** be *extended metric spaces.* Show that $F : \mathbf{C} \to \mathbf{D}$ is an equivalence if and only if it is a bijective isometry.

Conjecture 1

The first conjecture is that there is a model structure on **R**Cat (or **R**Cat$^{\text{sym}}$) which, in a sense, models the theory of extended metric spaces.

Conjecture 1. *There is a model structure on either* **R**Cat *or* **R**Cat$^{\text{sym}}$ *(or both) such that*

W) The weak equivalences are the essentially surjective and fully faithful **R***-functors.*

FCO) The fibrant-cofibrant objects are those (symmetric?) **R***-categories* **C** *such that* $\mathbf{C}(x, y) = 0$ *if and only if* $x = y$*.*

3 *Cauchy completeness*

Usually when dealing with metric spaces, we say that a metric space (X, d) is *complete* if every *Cauchy sequence* has a *limit*. We will define a similar notion in terms of **R**-categories, and show that this relates to the Cauchy completion of usual categories.

Definition 1.17. Let **C** be a symmetric **R**-category. We say that a sequence of objects $\{x_n\}_{n\in\mathbb{N}}$ in **C** is a *Cauchy sequence* if, for every $\epsilon > 0$, there is a number $N \in \mathbb{N}$ such that, for every $m, n > N$,

$$\mathbf{C}(x_n, x_m) < \epsilon.$$

Definition 1.18. Let **C** be a symmetric **R**-category. We say that a sequence of objects $\{x_n\}_{n\in\mathbb{N}}$ in **C** *converges* to an object $x \in \mathbf{C}$ if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that, for every $n > N$,

$$\mathbf{C}(x_n, x) < \epsilon.$$

We say that **C** is *(Cauchy) complete* if every Cauchy sequence has at least one limit.

Exercise 5.

1. Give an example of a symmetric **R**-category which contains a Cauchy sequence with two different limits.

2. Show that if a symmetric **R**-category **C** has the additional property that, if $\mathbf{C}(x, y) = 0$, then $x = y$, then any two limits of a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in **C** are equal.

   Let use briefly digress, and discuss the completion of usual metric spaces.

Construction 1.19. Let (X, d) be a metric space. We call two Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ *equivalent* if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

This defines an equivalence relation on the set of Cauchy sequences in X. We denote by $\overline{X}$ the set of equivalence classes of Cauchy sequences in X under this equivalence relation. We can then define a metric δ on $\overline{X}$ by declaring

$$\delta([\{x_n\}], [\{y_n\}]) := \lim_{n\to\infty} d(x_n, y_n).$$

We must then check that this is well-defined. Suppose that two Cauchy sequences $\{x_n\}$ and $\{z_n\}$ are equivalent. Then, for any Cauchy sequence $\{y_n\}$

$$\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n, z_n) + d(z_n, y_n) = \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(z_n, y_n)$$

An identical argument shows that

$$\lim_{n\to\infty} d(z_n, y_n) \leq \lim_{n\to\infty} d(x_n, y_n).$$

Thus we see that δ is well-defined on equivalence classes. Symmetry and triangle inequality follow from properties of limits, together with the symmetry and triangle inequality of d. By definition, $\delta(\{x_n\}, \{y_n\}) = 0$ if and only if $\{x_n\}$ and $\{y_n\}$ represent the same equivalence class. We thus obtain a metric space $(\overline{X}, \delta)$.

Viewing points in x as constant Cauchy sequences, we obtain an isometric injection

$$(X, d) \longrightarrow (\overline{X}, \delta).$$

Moreover, it is not too hard to show that every Cauchy sequence in $(\overline{X}, \delta)$ converges.

Returning to our regularly scheduled programming, we want to formulate the notion of an 'equivalence class of Cauchy sequences' categorically. We will first work this out in the case of symmetric **R**-categories, and then extend the definition to non-symmetric **R**-categories.

Simplifying assumption: We will temporarily assume that all of our **R**-categories are symmetric.

Given a symmetric **R**-category **C**, we can consider **R**-enriched functors

$$F : \mathbf{C} \longrightarrow \mathbf{R}.$$

We can unwind what these mean in more metric-space theoretic terms:

Definition 1.20. Let (X, d) be a metric space. A function $f : X \to [0, \infty]$ is called a *presheaf on* X if, for every $x, y \in X$, we have

$$f(y) - f(x) \leq d(x, y).$$

Construction 1.21. Given any point $z \in X$, we can define the *indicator presheaf* f_z on X by defining

$$f_z(y) = d(z, y).$$

To see this is a presheaf, we can notice that

$$f_z(y) - f_z(x) = d(z, y) - d(z, x) \leq d(z, x) + d(x, y) - d(z, x) = d(x, y).$$

The corresponding notion for symmetric **R**-categories is

Definition 1.22. A *presheaf* on an **R**-category **C** is an **R**-functor

$$f : \mathbf{C}^{\text{op}} \longrightarrow \mathbf{R}.$$

Given any $z \in \text{Ob}(\mathbf{C})$, we can define the *indicator presheaf on* z to be the functor

$$f_S : \mathbf{C}^{\text{op}} \longrightarrow \mathbf{R}.$$

defined by

$$f_z(x) = \mathbf{C}(x, z)$$

Lemma 1.23. *Let* **C** *be a symmetric* **R***-category, and let* $x, y \in \text{Ob}(\mathbf{C})$. *The following are equivalent.*

1. *The indicators presheaves are equal:* $f_x = f_y$.

2. *The hom-objects (distances) between the two points are zero:* $\mathbf{C}(x, y) = 0$.

Proof. First suppose that the indicator presheaves are equal. Then

$$0 = f_x(y) - f_y(y) = \mathbf{C}(y,x) - \mathbf{C}(y,y) = \mathbf{C}(y,x)$$

as desired.

On the other hand, suppose that $\mathbf{C}(x,y) = 0$, Then we notice that for any $z \in \mathrm{Ob}(\mathbf{C})$,

$$f_x(z) = \mathbf{C}(z,x) \leq \mathbf{C}(z,y) + \mathbf{C}(y,x) = \mathbf{C}(z,y) = f_y(z)$$

and

$$f_y(z) = \mathbf{C}(z,y) \leq \mathbf{C}(z,x) + \mathbf{C}(x,y) = \mathbf{C}(z,x) = f_x(z)$$

so that $f_x(z) = f_y(z)$. □

This should be interpreted as the beginnings of an $\mathbf{R}$-categorical version of the Yoneda lemma — associating a presheaf to an object only collapses points which are isomorphic. To make this more precise, we should define an $\mathbf{R}$-category of $\mathbf{R}$-enriched functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{R}$.

Definition 1.24. Let $\mathbf{C}$ be an $\mathbf{R}$-category. We define the $\mathbf{R}$*-category of* $\mathbf{R}$*-enriched functors*

$$\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})$$

as follows. The objects are the $\mathbf{R}$-enriched functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{R}$. The hom-objects are given by

$$\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(f,g) := \sup_{z \in \mathrm{Ob}(\mathbf{C})} \mathbf{C}(f(z), g(z))$$

To show that this is an $\mathbf{R}$-category, we must check two properties.

1. We show that the hom-object from f to itself is $0 \in \mathbf{R}$.

$$\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(f,f) := \sup_{z \in \mathrm{Ob}(\mathbf{C})} \mathbf{R}(f(z), f(z)) = \sup_{z \in \mathrm{Ob}(\mathbf{C})} 0 = 0$$

2. We must show that triangle inequality holds. Let f, g, h be presheaves

$$\begin{aligned}
\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(f,h) &= \sup_{z \in \mathrm{Ob}(\mathbf{C})} \mathbf{R}(f(z), h(z)) \\
&\leq \sup_{z \in \mathrm{Ob}(\mathbf{C})} (\mathbf{R}(f(z), g(z)) + \mathbf{R}(g(z), h(z))) \\
&\leq \sup_{z \in \mathrm{Ob}(\mathbf{R})} (\mathbf{R}(f(z), g(z))) + \sup_{z \in \mathrm{Ob}(\mathbf{C})} (\mathbf{R}(g(z), h(z))) \\
&= \mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(f,g) + \mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(g,h)
\end{aligned}$$

Proposition 1.25 (Yoneda embedding)**.** *Let* $\mathbf{C}$ *be a* $\mathbf{R}$*-category. Then the assignment on objects*

$$\begin{aligned}
\mathcal{Y} : \mathbf{C} &\longrightarrow \mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R}) \\
x &\longmapsto f_x
\end{aligned}$$

defines a fully-faithful $\mathbf{R}$*-functor.*

Proof. It will suffice to show that

$$\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(f_x, f_y) = \mathbf{C}(x, y).$$

To this end, let $x, y \in \mathbf{C}$. We note that, for any $z \in \mathbf{C}$, we have

$$\begin{aligned}
\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(f_x, f_y) &= \sup_{z \in \mathrm{Ob}(\mathbf{C})} \mathbf{R}(f_x(z), f_y(z)) \\
&= \sup_{z \in \mathrm{Ob}(\mathbf{C})} \max\left(f_y(z) - f_x(z), 0\right) \\
&= \sup_{z \in \mathrm{Ob}(\mathbf{C})} \max\left(\mathbf{C}(z, y) - \mathbf{C}(z, x), 0\right) \\
&\geq \max\left(\mathbf{C}(x, y) - \mathbf{C}(x, x), 0\right) \\
&= \max\left(\mathbf{C}(x, y), 0\right) \\
&= \mathbf{C}(x, y).
\end{aligned}$$

On the other hand, we can apply triangle inequality

$$\begin{aligned}
\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(f_x, f_y) &= \sup_{z \in \mathrm{Ob}(\mathbf{C})} \mathbf{R}(f_x(z), f_y(z)) \\
&= \sup_{z \in \mathrm{Ob}(\mathbf{C})} \max\left(f_y(z) - f_x(z), 0\right) \\
&= \sup_{z \in \mathrm{Ob}(\mathbf{C})} \max\left(\mathbf{C}(z, y) - \mathbf{C}(z, x), 0\right) \\
&\leq \sup_{z \in \mathrm{Ob}(\mathbf{C})} \max\left(\mathbf{C}(z, x) + \mathbf{C}(x, y) - \mathbf{C}(z, x), 0\right) \\
&= \sup_{z \in \mathrm{Ob}(\mathbf{C})} \max\left(\mathbf{C}(x, y), 0\right) \\
&= \max\left(\mathbf{C}(x, y), 0\right) = \mathbf{C}(x, y).
\end{aligned}$$

Thus $\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})(f_x, f_y) = \mathbf{C}(x, y)$ as desired. □

Now that we have established our Yoneda embedding, we want to try and view Cauchy sequences in $\mathbf{C}$ as contained in $\mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})$. We will first define a presheaf associated to any Cauchy sequence.

Definition 1.26. Let $\mathbf{C}$ be a symmetric $\mathbf{R}$-category, and let $\overline{x} = \{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathbf{C}$. We define a presheaf

$$f_{\overline{x}} : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{R}$$

by[5]

$$f_{\overline{x}}(z) = \lim_{n \to \infty} \mathbf{C}(x_n, z).$$

Notice that the convergence of this limit is conditioned on $\{x_n\}$ being Cauchy.

Lemma 1.27. *Let $\mathbf{C}$ be a symmetric $\mathbf{R}$-category, and let $\overline{x} = \{x_n\}_{n \in \mathbb{N}}$ and $\overline{y} = \{y_n\}_{n \in \mathbb{N}}$ be two Cauchy sequences in $\mathbf{C}$. Then $f_{\overline{x}} = f_{\overline{y}}$ if and only if $\overline{x}$ and $\overline{y}$ are equivalent.*

[5] To see that this is a presheaf, we can use the triangle inequality to show

$$\begin{aligned}
f_{\overline{x}}(z) - f_{\overline{x}}(y) &= \left(\lim_{n \to \infty} \mathbf{C}(x_n, z)\right) - \left(\lim_{n \to \infty} \mathbf{C}(x_n, y)\right) \\
&= \lim_{n \to \infty} \left(\mathbf{C}(x_n, z) - \mathbf{C}(x_n, y)\right) \\
&\leq \lim_{n \to \infty} \left(\mathbf{C}(x_n, y) + \mathbf{C}(y, z) - \mathbf{C}(x_n, y)\right) \\
&= \mathbf{C}(y, z)
\end{aligned}$$

so that we do, indeed, obtain a presheaf.

Proof. First suppose the sequences are equivalent. Then for any $\epsilon > 0$, choose $N_\epsilon \in \mathbb{N}$ such that, for all $n > N_\epsilon$,

$$\mathbf{C}(x_n, y_n) < \epsilon.$$

Then on the one hand, for any $n > N_\epsilon$ and any $z \in \mathrm{Ob}(\mathbf{C})$, we have

$$\begin{aligned}\mathbf{C}(x_n, z) - \mathbf{C}(y_n, z) &\leq \mathbf{C}(x_n, y_n) + \mathbf{C}(y_n, z) - \mathbf{C}(y_n, z)\\ &= \mathbf{C}(x_n, y_n)\end{aligned}$$

and similarly

$$\mathbf{C}(y_n, z) - \mathbf{C}(x_n, z) \leq \mathbf{C}(x_n, y_n)$$

so that

$$|\mathbf{C}(y_n, z) - \mathbf{C}(x_n, z)| \leq \mathbf{C}(x_n, y_n) < \epsilon$$

Thus, $f_{\overline{x}} = f_{\overline{y}}$.

On the other hand, suppose that $f_{\overline{x}} = f_{\overline{y}}$. Let $\epsilon > 0$, and choose $N_\epsilon \in \mathbb{N}$ such that, for any $n, m > N_\epsilon$

$$\mathbf{C}(y_n, y_m) < \frac{\epsilon}{2}.$$

Since $f_{\overline{x}} = f_{\overline{y}}$, for any $n > N_\epsilon$, we have

$$\lim_{m\to\infty} \mathbf{C}(x_m, y_n) = \lim_{k\to\infty} \mathbf{C}(y_k, y_n) < \frac{\epsilon}{2}$$

We can then choose $M_\epsilon \in \mathbb{N}$ such that, for $n > N_\epsilon$ and $m > M_\epsilon$ we have

$$|\mathbf{C}(x_m, y_n) - \frac{\epsilon}{2}| < \frac{\epsilon}{2}$$

Consequently, for any $m, n > \max(N_\epsilon, M_\epsilon)$, we have

$$\mathbf{C}(x_m, y_n) < \epsilon.$$

Thus,

$$\lim_{n\to\infty} \mathbf{C}(x_n, y_n) = 0$$

and the Cauchy sequences are equivalent. □

We thus are left to characterize those presheaves $f : \mathbf{C}^{\mathrm{op}} \to \mathbf{R}$ such that f comes from a Cauchy sequence.

Definition 1.28. Let $\mathbf{C}$ be a symmetric $\mathbf{R}$-category. We say that a presheaf f on $\mathbf{C}$ is *dual* to a presheaf g on $\mathbf{C}$ if

$$0 = \inf_{z\in\mathrm{Ob}(\mathbf{C})} (f(z) + g(z))$$

and

$$f(y) + g(z) = \mathbf{C}(y, z)$$

for any $y, z \in \mathrm{Ob}(\mathbf{C})$.

Proposition 1.29. *Let* $\mathbf{C}$ *be a symmetric* $\mathbf{R}$*-category and let* $f : \mathbf{C}^{\mathrm{op}} \to \mathbf{R}$ *be a presheaf. The following are equivalent.*

1. *The presheaf f has a dual.*

2. *There is a Cauchy sequence $\overline{x} = \{x_n\}_{n\in\mathbb{N}}$ such that $f = f_{\overline{x}}$.*

Proof. First, suppose that f has a dual g, so that

$$0 = \inf_{z\in\mathrm{Ob}(\mathbf{C})} (f(z) + g(z))$$

and

$$f(y) + g(z) = \mathbf{C}(y, z)$$

for any $y, z \in \mathrm{Ob}(\mathbf{C})$. By the first of these conditions, for any $n \in \mathbb{N}$, we can choose an element $z_n \in \mathrm{Ob}(\mathbf{C})$ such that

$$f(z_n) + g(z_n) \leq \frac{1}{n}.$$

By the second equation, we then have

$$\mathbf{C}(z_n, z_m) \leq f(z_n) + g(z_m) \leq \frac{1}{n} + \frac{1}{m}$$

So that the sequence $\{z_n\}$ is Cauchy.

Since f is a presheaf, we have that

$$f(y) - f(x) \leq \mathbf{C}(x, y)$$

and

$$f(x) - f(y) \leq \mathbf{C}(y, x).$$

Since $\mathbf{C}$ is symmetric, this means that

$$|f(x) - f(y)| \leq \mathbf{C}(x, y) \leq f(x) + g(y)$$

In particular, we have

$$\lim_{n\to\infty} |f(x) - f(z_n)| \leq \lim_{n\to\infty} \mathbf{C}(x, z_n) \leq \lim_{n\to\infty} (f(x) + g(z_n))$$

so that

$$f(x) = |f(x)| \leq \mathbf{C}(x, z_n) \leq f(x)$$

so, indeed, setting $\overline{z} = \{z_n\}$, we have $f_{\overline{z}} = f$.

On the other hand, if $f = f_{\overline{z}}$ for some Cauchy sequence $\{z\}$, we can note that f is dual to f. We leave it to the reader to check this fact. □

Definition 1.30. Let $\mathbf{C}$ be a symmetric $\mathbf{R}$-category. The *Cauchy completion* of $\mathbf{C}$ is full subcategory

$$\overline{\mathbf{C}} \subset \mathbf{R}\,\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{R})$$

on those presheaves which have duals.

Remark 1.31. This notion of Cauchy completeness is, in fact a variant of the notion of Cauchy completeness for usual categories. For more details (and a corresponding increae in technicality), see [2] or [1].

Conjecture 2

The second conjecture concerns an analogue of the model structure for Cauchy-complete categories.

Conjecture 2. *There is a model structure on* $\mathbf{R}\mathrm{Cat}^{\mathrm{sym}}$ *such that*

FCO) The fibrant-cofibrant objects are the Cauchy-complete symmetric $\mathbf{R}$*-categories.*

W) The weak equivalences are something like $\mathbf{R}$*-pastoral equivalences — morphisms which induce equivalences on completions.*

Bibliography

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