Finite spaces, group actions, and complexity

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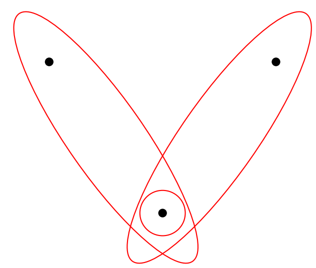
These are notes written to provide some background for participants in the 2021 REU in Topology at the University of Virginia. They rely quite heavily the references cited below, and should not be considered original.

Finite Spaces and posets.

Our aim will be to provide an overview of some constructions linking finite spaces, posets, and simplicial complexes; as well as equivariant versions of the same. As such, our basic object of study is the following:

Definition 0.1. A *finite topological space* X is a topological space such that the underlying set X is finite.[1] A topological space X is T_0 (or *Kolmogorov*) if for every pair of points $x, y \in X$, there is an open set $U \subset X$ which contains *precisely one* of x, y.

[1] It is quite easy to draw examples of finite topological spaces. We can simply circle all of the open sets other than $\emptyset$ and X itself. For instance, the space



Is the space with underlying set $X = \{a, b, c\}$ and topology

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}.$$

This space is, in fact, T_0.

We consider finite spaces together with continuous maps, in particular, if we say *a map of finite spaces*, we mean a continuous map

$$f : X \to Y$$

where X and Y are both finite. It turns out that finite T_0 spaces are closely related to posets — a connection which we can exploit to simplify our arguments.

Definition 0.2. A *poset* (*p*artially *o*rdered *set*) is a set X together with a binary relation $\leq$ on X which is

1. *Reflexive*: For all $x \in X$, $x \leq x$.
2. *Antisymmetric*: If $x \leq y$ and $y \leq x$, then $x = y$.
3. *Transitive*: If $x \leq y$ and $y \leq z$, then $x \leq z$.

A *monotone map* or *map of posets* is a map

$$f : X \to Y$$

between posets such that $x \leq y$ implies $f(x) \leq f(y)$.[2]

[2] We will often have cause to draw posets, too. If $x \leq y$ and $x \neq y$, we will draw an arrow from x to y. For instance

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet}$$

is the poset $0 < 1$.

It turns out that these two concepts are basically the same. From any finite T_0 space, we can construct a finite poset, and from any finite poset, we can construct a finite T_0 space.

Construction 0.3. Let X be a finite T_0 space. We define a poset $P(X)$, whose underlying set is X, and with the following order: Let $x, y \in X$. Then $x \geq y$ if and only if $x \in \overline{\{y\}}$, i.e., if and only if x is contained in the closure of the point y. In contexts where it is necessary to distinguish between multiple partial orders, we will denote the order on $P(X)$ by $\leq_X$.

Remark 0.4. It is worth here giving an equivalent characterization of the order relation on $P(X)$. For $x \in X$, denote by U_x the intersection of all open sets containing x. We could equivalently define $P(X)$ by declaring that $x \leq y$ if and only if $x \in U_y$. This is the definition given in [**?**] and [7].[3]

[3] To see that these two definitions are equivalent, suppose that $x \in \overline{\{y\}}$. now suppose that V is an open set containing x, but not y. Then V^c is a closed set containing y, but not x, which is a contradiction. Hence, $y \in U_x$.The other implication follows similarly.

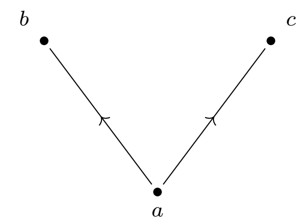
Lemma 0.5. *For any finite T_0 space X, $P(X)$ is a poset.*

Proof. It is definitional that, for any $x \in X$, $x \in \overline{\{x\}}$, and thus $x \leq x$.

Suppose that $x \neq y$. Then, since X is T_0, there is a closed set $U \subset X$ which contains only one of x and y. Without loss of generality, suppose $x \in U$, $y \notin U$. Then $\overline{\{x\}} \subset U$, and so $y \notin \overline{\{x\}}$. By contraposition, we thus see that if $y \leq x$ and $x \leq y$, we have $x = y$.

Finally, to see transitivity, we note that if $y \in \overline{\{z\}}$, then $\overline{\{y\}} \subset \overline{\{z\}}$. Thus, if $x \in \overline{\{y\}}$ and $y \in \overline{\{z\}}$, we have $x \in \overline{\{z\}}$. □

If we consider the three-point space X from the previous sidenote, we can draw the poset $P(X)$.



We can also go the other way, getting finite spaces out of posets.

Construction 0.6. Let Q be a finite poset. A subset $U \subset Q$ is called *upwards-closed* if, for every $y \in Q$, if there is a $x \in U$ such that $x \leq y$, then $y \in U$. A subset U is called *downwards-closed* if, for every $y \in Q$, if there is an $x \in U$ such that $y \leq x$, then $y \in U$.

We define a topological space $T(Q)$ to have the same underlying set as Q, with the topology whose open sets are the downwards-closed sets.[4]

[4] Notice that this means that the closed sets of this topology are the upwards-closed sets.

Exercise 0.7. Show that, for any finite poset Q, $T(Q)$ is a finite T_0 topological space.

Proposition 0.8. *Let X be a finite T_0 space. then $T(P(X)) = X$ (as topological spaces).*

Proof. Let X be a finite T_0 space, and denote the topology on X by τ, and the topology on $T(P(X))$ by τ_P. Suppose that $U \in \tau$. Let $y \in X$ such that there exists an $x \in U$ with $x \in \overline{\{y\}}$ (i.e, $y \leq x$). If $y \notin U$, then $\overline{\{y\}} \subset U^c$. This implies $x \in U^c$, which is a contradiction, so $y \in U$. Thus, U is downwards-closed, and so $U \in \tau_P$.

Now suppose $V \in \tau_P$. Then for every $x \in V$ and every $y \in V^c$, we have that $x \notin \overline{\{y\}}$. Consequently,

$$V^c = \bigcup_{y \in V^c} \overline{\{y\}}$$

which is a finite union of closed sets in τ, and thus is closed in τ. Hence, $V \in \tau$. □

For those of you initiated into the lore of categories, most of our work in this section can be subsumed into the statement that T and P are functors which define an equivalence of categories between the category of finite T_0 spaces, and the category of finite posets.

Proposition 0.9. *Let Q be a finite poset, then $P(T(Q)) = Q$ (as posets).*

Proof. Denote the original order on Q by $\leq_Q$, and the order on $P(T(Q))$ by $\leq_T$. Let $x \leq_Q y$. If V is an open set of $T(Q)$ containing y, then V^c is an upwards-closed set of Q which does not contain y. Hence, $x \notin V^c$, so $x \in V$. Since V was arbitrary, we see that $x \in U_y$, and therefore $x \leq_T y$.

Now suppose that $x \not\geq_Q y$. Then the set

$$U = \{z \in Q \mid z \geq_Q y\}$$

is closed in $T(Q)$, and does not contain x, so $x \not\geq_T y$. Hence, if $x \geq_T y$, then $x \geq_Q y$. □

Exercise 0.10. Let X and Y be finite T_0 spaces, and let $f : X \to Y$ be a map of underlying sets. Show that f is continuous if and only if f is a monotone map between the posets $P(X)$ and $P(Y)$.

Exercise 0.11. Let X and Y be T_0 spaces.

1. Denote by $X \times Y$ the product space. Show that $P(X \times Y) \cong P(X) \times P(Y)$ as posets.[5]
2. Denote by $X \amalg Y$ the disjoint union space, and denote by $P(X) \amalg P(Y)$ the poset whose underlying set is $X \amalg Y$, equipped with the order generated by those on $P(X)$ and $P(Y)$. Show $P(X \amalg Y) = P(X) \amalg P(Y)$.

[5] By a product of posets $(Q, \leq_Q)$ and $(R, \leq_R)$, we mean the poset whose underlying set is the cartesian product $Q \times R$, and such that

$$(a, b) \leq (p, q) \Leftrightarrow (a \leq_Q p) \wedge (b \leq_R q)$$

Exercise 0.12. Let Q be a finite poset. Show that Q has a *maximal point*, i.e., a $q \in Q$ such that, if $p \in Q$ with $p \geq q$, $p = q$. Formulate a definition of a *minimal point*, and show that Q has a minimal point.

We can also perform a quite important construction, which allows us to classify maps out of a product.

Construction 0.13. Let Q and R be finite posets. Write $\mathrm{Hom}(Q, R)$ for the set of all monotone maps $Q \to R$. We can define a partial order on $\mathrm{Hom}(Q, R)$ by setting

$$f \leq g \iff (\forall x \in Q, f(x) \leq g(x))$$

We denote the resulting finite poset by $\mathcal{M}\mathrm{ap}(Q, R)$.

Exercise 0.14. Denote by $[1]$ the poset $\{0 \leq 1\}$ and by $[2]$ the poset $\{0 \leq 1 \leq 2\}$. Draw the posets $\mathcal{M}\mathrm{ap}([1], [2])$ and $\mathcal{M}\mathrm{ap}([1], [1])$.

Proposition 0.15. *Let Q, R, and S be finite posets. Then there is a bijection*

$$\mathrm{Hom}(Q \times R, S) \cong \mathrm{Hom}(Q, \mathcal{M}\mathrm{ap}(R, S)).$$

Proof. Given a poset map

$$f : Q \times R \to S$$

define a map

$$\begin{aligned} \psi_f : Q &\longrightarrow \mathcal{M}\mathrm{ap}(R, S) \\ q &\longmapsto (r \mapsto f(q, r)) \end{aligned}$$

We leave it as an exercise to the reader to check that the map $f \mapsto \psi_f$ is, in fact, a bijection. □

Note that we can also view $\mathcal{M}\mathrm{ap}(R, S)$ as a finite T_0 space. Moreover, given X and Y finite T_0 spaces, we can find a finite T_0 space. $T(\mathcal{M}\mathrm{ap}(P(X), P(Y)))$ that has the same universal property, but now from T_0 spaces.

1.1 Homotopies

We will begin by recalling some definitions from topology, and then relating them to more esoteric combinatorial constructs via the relation with posets discussed above. Throughout, we denote by $I = [0,1]$ the unit interval with the standard topology.

Definition 1.1. Let X and Y be topological spaces, and let $f, g : X \to Y$ be continuous maps. A *homotopy from f to g* is a map

$$h : X \times I \to Y$$

such that $h|_{X\times\{0\}} = f$ and $h|_{X\times\{1\}} = g$. If there is a homotopy from f to g, we say that f and g are *homotopic*.

Definition 1.2. A *path* from x to y in a topological space X is a continuous map $\alpha : I \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Our aim now will be to shift some of the topological nature of these definitions into the world of posets. The benefit is that doing so will allow us to attack our problems combinatorially.

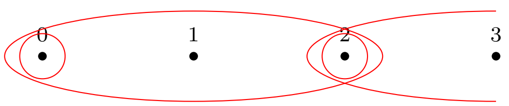
Definition 1.3. The *combinatorial interval J_m of length $m+1$* is the poset[6]

$$0 < 1 > 2 < \cdots m-1 > (<)m.$$

[6] While we keep the notation from [7], there, this poset is called a *finite fence*.

These posets should be thought of as finite analogues of the interval I.[7]

[7] As an example, let's draw the topological space associated to J_3.



Let's now consider a finite T_0 space X, and a path

$$\alpha : I \to X$$

from x to y. Our aim is to show that this exists if and only if there is a *combinatorial path*

$$\beta : J_m \to X$$

from x to y. We claim that this will follow from the next proposition.[8]

[8] This argument mostly follows [1].

Proposition 1.4. *Let X be a finite space. The following are equivalent:*

1. *X is path connected.*
2. *X is connected.*
3. *For any $x, y \in X$, there is a combinatorial path from x to y in X.*

Proof. By general topology, (1) implies (2). We will proceed by showing (2) implies (3) and (3) implies (1).

Suppose that X is connected. Fix $x \in X$, and consider the set

$$A_x := \{y \in X \mid \text{there is a combinatorial path from } x \text{ to } y\}.$$

If $y \in A_x$, there is a combinatorial path

$$x \leq x_1 \geq x_2 \leq \cdots x_{m-1} \leq (\geq) y$$

If $z \leq y$, we can form a concatenated combinatorial path[9] from x to z, and so $z \in A_x$. Similarly, we can show that if $z \geq y$ then $z \in A_x$. Thus, A_x must be both closed and open. Since X is connected, this implies that either $A_x = \emptyset$ or $A_x = X$. Since $x \in A_x$, we see that $A_x = X$. We thus see that fro any $x, y \in X$, there is a combinatorial path from x to y. Thus, (2) implies (3).

[9] Basically: stick one or two more inequalities on the end. If m is odd, then we get

$$x \leq x_1 \geq x_2 \leq \cdots x_{m-1} \leq y \geq z$$

and if m is even, we get

$$x \leq x_1 \geq x_2 \leq \cdots x_{m-1} \geq y \leq y \geq z$$

Finally, suppose that for any $x, y \in X$, there is a combinatorial path from x to y. We wish to show that there is also a topological path from x to y. Since we can reverse and concatenate topological paths, it will suffice to show that, if $x \leq y$, there is a topological path from x to y. We can assume, without loss of generality, that $x \neq y$. We define a path

$$\begin{aligned} f : I &\longrightarrow X \\ t &\longmapsto \begin{cases} x & t < 1 \\ y & t = 1 \end{cases} \end{aligned}$$

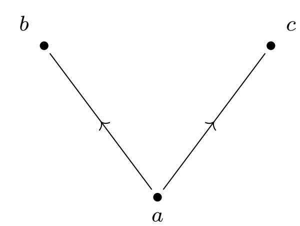
To see that this is continuous, we note that the non-empty open sets of the subspace $\{x, y\} \subset X$ are $\{x\}$ and $\{x, y\}$. Thus, we need only note that

$$f^{-1}(\{x\}) = [0, 1)$$

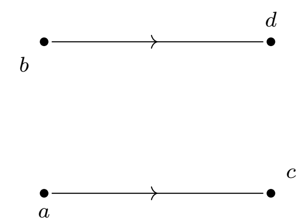
is an open subset of I. We therefore see that (3) implies (1), concluding the proof. $\square$

Definition 1.5. We say that X is *order-connected* if it satisfies condition (3) from 1.4.[10]

[10] The space



is order-connected. The space



is not.

Corollary 1.6. *Let X be a finite space, and let $x, y \in X$. Then there is a (topological) path from x to y if and only if there is a combinatorial path from x to y.*

Proof. Suppose that there is a topological path

$$f : I \to X$$

from x to y. Denote by $Y \subset X$ the image of f. Since I is connected, so is Y. By Proposition 1.4, this means there is a combinatorial path in Y from x to y.

Suppose there is a combinatorial path from x to y, and consider this path as a subspace $Y \subset X$. Then, by definition, Y is order connected, and thus path-connected by Proposition 1.4. Hence, there is a path from x to y. $\square$

Definition 1.7. Let Q and R be finite posets, and let $f, g : Q \to R$ be monotone maps. A *homotopy from f to g* is a map

$$h : Q \times J_m \to R$$

for some $m \geq 0$ such that $h|_{Q \times \{0\}} = f$ and $h|_{R \times \{m\}} = g$. If there is a homotopy from f to g, we say that f and g are *homotopic*.

Corollary 1.8. *Let X and Y be finite T_0 spaces and let $f, g : X \to Y$ be continuous maps. Then there is a homotopy from f to g if and only if there is a homotopy from f to g when considered as maps of posets.*

Exercise 1.9. Prove Corollary 1.8 using Proposition 1.4 and Proposition 0.15.[11]

[11] Hint: a homotopy from f to g is equivalently a path
$$h : I \to T(\mathcal{M}\mathrm{ap}(P(X), P(Y)))$$
from f to g.

We now notice that we can define homotopies of maps between posets by building them out of very simple maps.

Definition 1.10. Let Q and R be finite posets, and $f, g : Q \to R$ monotone maps. An *elementary homotopy* from f to g is a map

$$h : Q \times J_1 \to R$$

such that $h|_{Q\times\{0\}} = f$ and $h|_{Q\times\{1\}} = g$.

Exercise 1.11. Show that there is an elementary homotopy from f to g if and only if $f \leq g$ as elements of $\mathcal{M}\mathrm{ap}(Q, R)$.

Definition 1.12. Let X be a space, and $A \subset X$ a subspace. Denote by $i : A \to X$ the inclusion. If there is a map $r : X \to A$ such that

1. $r \circ i = \mathrm{id}_A$
2. There is a homotopy
$$H : X \times I \to X$$
from the identity on X to $i \circ r$ such that, for every $a \in A$ and every $t \in I$, $h(a, t) = a$.

we say that A is a *deformation retract* of X.

Deformation retraction is a particularly strong way of saying that a subspace A is homotopy equivalent to X. Heuristically, it means that we can 'squish X into A without moving the points in A'.

To make sure that this definition is sensible, we need a further lemma.

Lemma 1.13. *Let X be a topological space, and let $B \subset A \subset X$ be subspaces. If B is a deformation retract of A, and A is a deformation retract of X, then B is a deformation retract of X.*

Proof. We fix the data of the two deformation retracts. Let $i_B : B \to A$ and $i_A : A \to X$ be the inclusions; $r_B : A \to B$ and $r_A : X \to A$ the retractions; and $h_A : X \times I \to X$ and $h_B : A \times I \to A$ be the homotopies.

Define $i = i_A \circ i_B$, and $r = r_B \circ r_A$. We note that

$$r \circ i = r_B \circ r_A \circ i_A \circ i_B = r_B \circ i_B = \mathrm{id}_B.$$

We thus need only construct a homotopy from $i \circ r$ to id_X.

Consider first the composite

$$X \times I \xrightarrow{r_A \times \mathrm{id}_I} A \times I \xrightarrow{h_B} A \xrightarrow{i_A} X$$

This provides a homotopy H_1 from $i_A \circ \mathrm{id}_A \circ r_A$ to $i_A \circ i_B \circ r_B \circ r_A$. Note that, for $b \in B$ and $t \in T$, we have that $H_1(b,t) = b$.[12] Since h_A is a homotopy from id_X to $i_A \circ r_A$, we can then concatenate h_A and H_1 to get the desired homotopy from id_X to $i \circ r = i_A \circ i_B \circ r_B \circ r_A$.[13] $\square$

[12] Note that, from the definition of a deformation retraction, we have that r_A fixes every point in A.

[13] Recall that the concatenation of a homotopy $h : X \times I \to X$ from f to g and a homotopy $H : X \times I \to X$ from g to q is the homotopy

$$(H * h)(x,t) := \begin{cases} h(x, 2t) & 0 \le t \le \frac{1}{2} \\ H(x, 2(t - \frac{1}{2})) & \frac{1}{2} \le t \le 1 \end{cases}$$

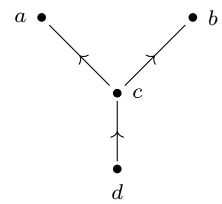
from f to q.

We now introduce some terminology and notation from [4] which is of some use in analyzing homotopies of finite spaces via the associated posets. Of particular use will be a kind of subspace of a finite space — called a *core* — which can be displayed as a deformation retract. The following material very closely parallels the final section of [4].

Definition 1.14. Let Q be a finite poset. We say that a point $q \in Q$ is

1. *upbeat*[14] if there is an $x \in Q$ with $x > q$ such that $y > q$ implies $y \ge x$.
2. *downbeat* if there is an $x \in Q$ with $x < q$ such that $y < q$ implies $y \le x$.

[14] Let's sketch an example of an upbeat point (the notion of a downbeat point is formally dual). Consider the poset



The point d is upbeat, because for every $y > d$, we have $y \ge c$.

The upbeat and downbeat points are in some sense, superfluous. Given an upbeat point q, for instance, we choose $x > q$ such that, for every $y > q$, $y \ge x$. We can then homotop q to x, and the poset will look much the same. We make this formal with the following definitions and lemmata.

Definition 1.15. A *minimal finite poset* is a finite T_0 space X such that $P(X)$ has neither upbeat nor downbeat points. A *core* of a finite T_0 space X is a minimal subspace $A \subset X$ which is a deformation retract of X.

Lemma 1.16. *Let X be a finite T_0 space. Then X has a core A.*

Proof. We will show that, for an upbeat (or downbeat) point $x \in X$, $X \setminus \{x\}$ is a deformation retract of X. The lemma then follows immediately from Lemma 1.13.

We work with the associated poset $P(X)$. Let x be an upbeat point of x, and choose $z > x$ such that, for every $y > x$ $y \ge z$. Define a map

$$h : P(X) \times J_1 \to P(X)$$

by setting

$$h(q,t) = \begin{cases} q & q \ne x \\ x & q = x, \quad t = 0 \\ z & q = x, \quad t = 1 \end{cases}$$

To see that this is a monotone map, we need only note that if $q > x$, then $h(q,1) = q \ge z = h(x,1)$ and, if $q < x$, then $h(q,1) = q < x < z = h(x,1)$.

This map clearly fixes $P(X) \setminus \{x\}$, and defines the desired deformation retract. $\square$

Exercise 1.17. Complete the proof of Lemma 1.16 by showing that, for any downbeat point $x \in X$, the subspace $X \setminus \{x\}$ is a deformation retract of X.

Cores are of substantial utility in the study of finite spaces, because they allow us to reduce the study of homotopy equivalences of finite spaces to the study of homeomorphisms of cores, as the following proposition shows.

Proposition 1.18. *Let X be a minimal finite T_0 space. If $f : X \to X$ is homotopic to the identity, then $f = \mathrm{id}_X$.*

Proof. We again work with the associated poset $P(X)$. Since any homotopy can be decomposed into elementary homotopies, it will suffice to prove the statement in two cases: $f \geq \mathrm{id}_X$, and $f \leq \mathrm{id}_X$.

Suppose $f \geq \mathrm{id}_X$. Let x be a maximal point of X. Then $f(x) = x$. Now let $x \in X$, and suppose, inductively, that $f(y) = y$ for any $y > x$. For $y > x$, this implies that

$$y = f(y) \geq f(x)$$

If $f(x) \neq x$, this means that x is upbeat, which contradicts the minimality of X. Thus $f(x) = x$. By induction, we see that $f = \mathrm{id}_X$

A completely analogous argument using downbeat points and minimal points shows that if $f \leq \mathrm{id}_X$, then $f = \mathrm{id}_X$. □

Corollary 1.19. *A map $f : X \to Y$ of minimal finite T_0 spaces is a homotopy equivalence if and only if it is a homeomorphism.*

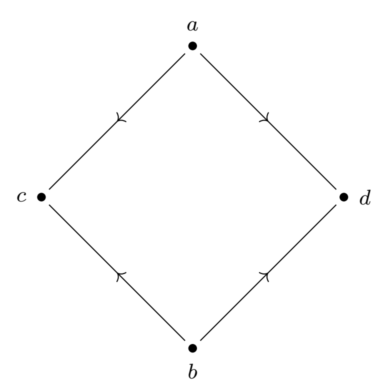
Corollary 1.20. *Finite T_0 spaces are homotopy equivalent if and only if they have homeomorphic cores.*

Exercise 1.21. Derive these two corollaries from Proposition 1.18

1.2 Finite G-spaces

We now want to add some algebraic information to our finite spaces. Throughout the rest of the document, we will fix the following conventions:

We will, throughout this section, consider a running example: the so-called *pseudo-circle*, which we will denote by $\mathfrak{S}^1$. This is the finite T_0 space whose associated poset is



The pseudo-circle functions, perhaps unsurprisingly, as a kind of discrete analogue of the circle.

- A group will mean a *finite* group.
- If a group G is considered as a topological space, we will consider it to be equipped with the *discrete* topology.

Definition 2.1. Let G be a group. A *finite G-space* is a space X equipped with a continuous G-action, i.e. such that the map

$$G \times X \to X$$

which defines the action is continuous.

If X and Y are G spaces, a continuous map

$$f : X \to Y$$

is called *equivariant* if, for every $g \in G$ and every $x \in X$,

$$f(g \cdot x) = g \cdot f(x).$$

Exercise 2.2. Let G be a group, and X a topological space. Show that an action of G on the underlying set of X makes X into a G-space if and only if, for every $g \in G$, the induced map

$$g \cdot (-) : X \to X$$

is continuous.[15]

[15] It is important to note that this is *only* true when we equip G with the discrete topology. If G is a more complicated topological group, additional conditions are necessary.

Since we have already seen that finite T_0 spaces and finite posets are, in effect, the same thing, it is probably not surprising that we can rewrite the definition of a G-space in terms of posets

Definition 2.3. Let G be a group. A G-poset is a poset Q equipped with a monotone action, i.e. an action such that the defining map

$$f : G \times Q \to Q$$

is monotone.[16]

[16] Here G is viewed as a discrete poset.

A map $f : Q \to R$ of G-posets is called *equivariant* if, for every $g \in G$ and every $q \in Q$, $f(g \cdot q) = g \cdot f(q)$.

Exercise 2.4. Convince yourself that a finite T_0 G-space is the same thing as a finite G-poset.

Our aim will be to study the homotopy theory of finite G-spaces.

Example. Lets consider the group $\mathbb{Z}/2$, and give some actions of $\mathbb{Z}/2$ on the pseudo-circle $\mathfrak{S}^1$. Note that a $\mathbb{Z}/2$-action is uniquely determined by an automorphism $\tau : \mathfrak{S}^1 \to \mathfrak{S}^1$ such that $\tau^2 = \mathrm{id}$. We list the possible $\mathbb{Z}/2$-actions on $\mathfrak{S}^1$.

- The trivial action, where τ is the identity.
- The 'vertical' action, where $\tau(a) = b$, $\tau(b) = a$, and τ fixes c and d.
- The 'horizontal' action, where $\tau(c) = d$, $\tau(d) = c$, and τ fixes a and b.
- The 'rotation' action, where $\tau(a) = b$, $\tau(b) = a$, $\tau(c) = d$, and $\tau(d) = c$

Definition 2.5. Let X and Y be G-spaces and let

$$f, g : X \to Y$$

be equivariant continuous maps. We can equip $X \times I$ with a G-action by letting G act trivially on I. A G-homotopy from f to g is an equivariant continuous map

$$h : X \times I \to Y$$

such that $h|_{X \times \{0\}} = f$ and $h|_{X \times \{1\}} = g$.

As ever, we can give a parallel definition for finite posets.

Definition 2.6. Let Q and R be finite G-posets and let

$$f, g : Q \to R$$

be equivariant monotone maps. We can equip $Q \times J_m$ with a G-action by letting G act trivially on I. A *G-homotopy* from f to g is an equivariant monotone map

$$h : Q \times J_m \to R$$

for some m such that $h|_{Q \times \{0\}} = f$ and $h|_{Q \times \{1\}} = g$. We call a G-homotopy *elementary* if $m = 1$.

Lemma 2.7. *Let G be a group and let Q and R be finite G-posets. Consider two G-equivariant monotone maps $f, g : Q \to R$. There is an elementary G-homotopy from f to g if and only if $f \leq g$ as elements of* $\mathcal{M}\mathrm{ap}(Q, R)$.[17]

[17] Note that this does *not* mean that two equivariant maps are G-homotopic if and only if they are homotopic. The intermediate maps may fail to be equivariant.

Proof. We already know that there is an elementary homotopy from f to g if and only if $f \leq g$. It thus will suffice for us to show that a homotopy

$$h : Q \times J_1 \to R$$

from f to g if and only if f and g are equivariant. This, however, follows immediately from the fact that, for $a \in G$

$$h(a \cdot (q, 0)) = h(a \cdot q, 0) = f(a \cdot q) = a \cdot f(q) = a \cdot h(q, 0)$$

and

$$h(a \cdot (q, 1)) = h(a \cdot q, 1) = g(a \cdot q) = a \cdot g(q) = a \cdot h(q, 1).$$

$\square$

Example 2.8. Consider the poset

$$\begin{tikzcd} b & & c \\ & a \arrow[ul] \arrow[ur] & \end{tikzcd}$$

equipped with the $\mathbb{Z}/2$-action which fixes a and swaps b and c. We define two $\mathbb{Z}/2$-equivariant maps, id_Q, and $g(q) = a$ for all $q \in Q$. Since $g \leq \mathrm{id}_Q$, these maps are G-homotopic.

Definition 2.9. Let X be a finite G-space. For $x \in X$, the *orbit of x under G* is the set

$$\mathcal{O}_x := \{y \in X \mid \exists g \in G \text{ s.t. } g \cdot x = y\}.$$

We denote the set of orbits of elements in X under G by X/G, and equip it with the topology whose open sets are those sets with open preimages in X. We call the resulting space the *quotient space*.

Definition 2.10. Let G be a group, and let Q be a finite G-poset. On underlying sets, we define the *quotient poset* Q/G exactly as we defined quotient spaces. Let $\mathcal{O}$ and $\mathcal{L}$ be orbits in Q/G. We define an order on Q/G by defining

$$\mathcal{O} \leq \mathcal{L} \Leftrightarrow \exists x \in \mathcal{O} \text{ and } \exists y \in \mathcal{L} \text{ s.t. } x \leq y.$$

A priori, it is not clear that this actually defines a poset. However, it will follow from the following lemma that it does.

Lemma 2.11. *Let G be a group, and let X be a finite T_0 G-space. Then the order $\leq_T$ on $P(X/G)$ is the same as the order $\leq_P$ on $P(X)/G$.*

Proof. Let $\mathcal{O}, \mathcal{L} \in P(X/G)$ such that $\mathcal{O} \leq_T \mathcal{L}$. We want to show that there are elements $x \in \mathcal{O}$ and $y \in \mathcal{L}$ such that $x \in U_y$. Suppose this is not the case. Choose $x \in \mathcal{O}$ and $y \in \mathcal{L}$ with an open set V such that $x \notin V$ and $y \in V$. Then, for every

$g \in G$, $g \cdot x \notin g \cdot V$ and $g \cdot y \in g \cdot V$. The set $[V]$ of orbits of elements in V is open (since its preimage is $\bigcup_{g \in G} g \cdot V$). By construction, $[V]$ contains $\mathcal{L}$, but does not contain $\mathcal{O}$, which is a contradiction. Thus $\mathcal{O} \leq_P \mathcal{L}$.

Now suppose that $\mathcal{O} \leq_P \mathcal{L}$. By definition, this means that there is $x \in \mathcal{O}$ and $y \in \mathcal{L}$ such that $x \in U_y$. Let $U \subset X/G$ be an open set containing $\mathcal{L}$. The preimage $\tilde{U} \subset X$ is an open set containing y, and thus contains x. However, this immediately implies that U contains $\mathcal{O}$, so $\mathcal{O} \leq_T \mathcal{L}$. □

Definition 2.12. Let G be a group. We say that a finite G-space X is *G-contractible* if there is a G-homotopy from id_X to a map $f : X \to X$ whose image lies in is a single G-orbit.[18]

This may seem like a strange definition. It implies that there are spaces which are not contractible, but *are* G-contractible. Since this seems a bit counterintuitive, let's try to see why we might want this definition.

Lemma 2.13. *Let X be a G-space, and consider $*$ to be the one-point G-space equipped with the trivial G-action. There is a bijection*

$$\left\{ \substack{\textit{equivariant maps} \\ x : * \to X} \right\} \cong \{G\textit{-fixed points of } X\}$$

Proof. A continuous map $f : * \to X$ is the same thing as a point $f(*) = x \in X$. This map is equivariant if and only if, for all $g \in G$,

$$x = f(*) = f(g \cdot *) = g \cdot f(*) = g \cdot x. \qquad \square$$

This tells us that only spaces with G-fixed points can be G-homotopy equivalent to the point. If we suppose that X has a G-fixed point, then we see that intuition matches with definition. Any equivariant map of G-spaces $f : X \to Y$ sends fixed points to fixed points. Consequently, we have

Lemma 2.14. *Suppose X is a G-space, and $x \in X$ is a G-fixed point. Then X is G-homotopy equivalent to the one-point G-space X if and only if X is G-contractible.*

Proof. It is clear that if $X \simeq_G *$, then X is G-contractible. We suppose instead that we have a G-homotopy

$$h : X \times I \to X$$

from id_X to a map f which takes values in a single orbit O. Since f is equivariant, it sends x to a fixed point. Since a fixed point is a G-orbit, $O = \{x\}$. Setting $\phi : * \to X$ to be the (equivariant) inclusion of X, and $\psi : X \to *$ to be the unique (equivariant) map, this implies that h is a homotopy from id_X to $\phi \circ \psi$. Thus, ϕ and ψ are G-homotopy inverse maps. □

What this says, in effect, is that, whenever the intuitive definition could possibly apply, it agrees with the formal definition. One further lemma makes clear that the formal definition is of significant use.

[18] This notion is *extremely* sensitive to the G-action on X. To illustrate this, consider the following two simple cases: Denote by $\underline{3} := \{1, 2, 3\}$ the set with three elements, treated as a discrete topological space. If we equip $\underline{3}$ with the trivial $\mathbb{Z}/3$ action, it is not $\mathbb{Z}/3$-contractible (since it is not contractible as a space). However, if we equip $\underline{3}$ with the $\mathbb{Z}/3$-action which cyclically permutes the elements, the identity map $\underline{3} \to \underline{3}$ already takes values in a single orbit, and so it is $\mathbb{Z}/3$-contractible.

Definition 2.15. Let X be a G-space. We denote by

$$X^G := \{x \in X \mid g \cdot x = x \ \ \forall g \in G\}$$

the set of G-fixed points. We equip this will the subspace topology induced by $X^G \subset X$.

Lemma 2.16. *Let G be a group and let X and Y be G-spaces. A G-equivariant map $f : X \to Y$ induces a continuous map $f^G : X^G \to Y^G$.*

Proof. Let $x \in X$ be a G-fixed point. Then $f(x)$ is a fixed point by Lemma 2.13. We thus see that f defines a map of sets $f^G : X^G \to Y^G$. Since this is simply the restriction of f to a subspace, it is continuous. □

Following [5], we now show that the notion of a core of a finite space X carries over nicely into the equivariant setting.

Definition 2.17. Let X be a G-space. We call a subspace $Y \subset X$ a *sub-G-space* if the G-action on X preserves Y. The subspace Y then naturally inherits a continuous G-action such that the inclusion

$$Y \hookrightarrow X$$

is G-equivariant.

For a finite T_0 space X, we will call a core $A \subset X$ an *equivariant core* of X if A is a sub-G-space and A is an equivariant deformation retract of X.[19]

[19] The eagle-eyed reader will note that we have not yet defined equivariant deformation retracts yet. It is a good check of your comprehension to work out what the correct definition should be.

Lemma 2.18. *Let X be a finite T_0 space. Then X has an equivariant core A.*

Proof. As in the proof of Lemma 1.16, we will show that we can remove upbeat points via an equivariant deformation retraction. Unlike in Lemma 1.16, however, we will have to remove more than one at a time.

Consider an upbeat point $x \in X$. Then, for any $g \in G$, the point $g(x)$ must be upbeat.[20] Consequently, the orbit $\mathcal{O}_x$ of x under G consists only of upbeat points. Since this is an orbit, the subspace $Y = X \setminus \mathcal{O}_x$ is a sub-G-space. We denote the inclusion by $i : Y \to X$.

[20]

Exercise. Explain why this must be the case.

Choose a point $z > x$ such that, for every $a > x$, $a \geq z$. We can then define a map

$$\begin{aligned} r : X &\longrightarrow Y \\ y &\longmapsto \begin{cases} y & y \in Y \\ gz & y = gx \end{cases} \end{aligned}$$

The map f is G-equivariant, and by construction $i \circ f \geq \mathrm{id}_X$. Since id_X is also equivariant, we see that this provides an equivariant homotopy fixing Y from $i \circ f$ to id_X. Thus, Y is an equivariant deformation retract of X. Applying this argument iteratively to all upbeat (or downbeat) points yields the lemma. □

Remark 2.19. Note that the equivariant core of a finite T_0 G-space X is the same as its non-equivariant core — both are obtained by remove the upbeat and downbeat points.

Corollary 2.20. *Let X be finite T_0 G-space, and suppose that X is (non-equivariantly) contractible. Then X is G-contractible.*

Proof. By Corollary 1.20, the core of X is a point. By Lemma 2.18, we see that X is G-homotopy equivalent to its core. □

Theorem 2.21. *Suppose X and Y are finite T_0 G-spaces, and let $f : X \to Y$ be an equivariant map. If f is a homotopy equivalence, then f is a G-homotopy equivalence.*

Proof. Let $A \subset X$ be an equivariant core, and let $B \subset Y$ be an equivariant core. Denote the maps defining the retractions by

$$A \xrightarrow{i_A} X \xrightarrow{r_A} A$$
$$B \xrightarrow{i_B} Y \xrightarrow{r_B} B$$

We then consider the composite map

$$\phi : A \xrightarrow{i_A} X \xrightarrow{f} Y \xrightarrow{r_B} B.$$

Since ϕ is a composite of G-equivariant maps, it is itself G-equivariant. However, by Corollary 1.20, ϕ is a homeomorphism (since it is a homotopy equivalence between minial finite T_0 spaces). Consequently, it has a G-equivariant inverse $\phi^{-1} : B \to A$.

We can then define a composite map

$$\psi : Y \xrightarrow{r_B} B \xrightarrow{\phi^{-1}} A \xrightarrow{i_A} X$$

which, as before, is equivariant. We claim that ψ is a G-homotopy inverse to f. We will write a sequence of G-equivariant homotopies

- First note that
$$\psi \circ f = i_A \circ \phi^{-1} \circ r_B \circ f \circ \mathrm{id}_X \simeq i_A \circ \phi^{-1} \circ r_B \circ f \circ i_A \circ r_A$$
where the last equivalence follows from the fact that A is a G-deformation retract of X.
- Then note
$$i_A \circ (\phi^{-1} \circ r_B \circ f \circ i_A) \circ r_A = i_A \circ \mathrm{id}_A \circ r_A$$
by the definition of ϕ.
- Finally, since A is a deformation retract of X, we have
$$i_A \circ \mathrm{id}_A \circ r_A = i_A \circ r_A \simeq \mathrm{id}_X.$$

Similar reasoning shows that there is a sequence of G-equivariant homotopies from $f \circ \psi$ to id_Y, completing the proof. □

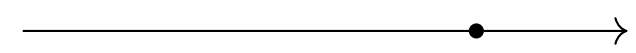
Exercise 2.22. Complete the proof of the theorem: construct a G-equivariant homotopy from $f \circ \psi$ to id_Y.

1.3 Order complexes

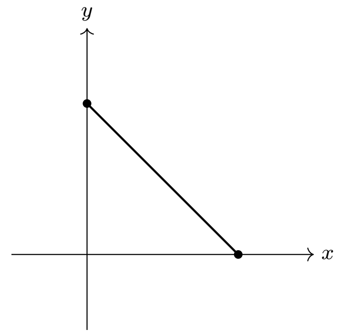
There is a very different topological interpretation of finite posets. We can view a finite poset Q as an *simplicial complex* — a space constructed by gluing together simplices.Simplices can be seen as higher-dimensional generalizations of triangles. We can use simplicies to build topological spaces out of conveniently well-understood components. This is analogous to sewing: to obtain a 2-dimensional surface which bends in 3-dimensions, we take flat pieces of fabric, and stitch them together to get a more complicated object.

Example. We can easily draw the first few standard n-simplices:

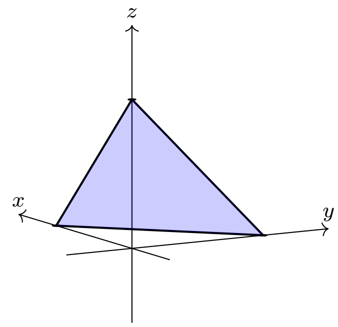
$n=0$



$n=1$



$n=2$



The standard 3-simplex is a tetrahedron. However, for obvious reasons, we cannot *draw* it as a subset of $\mathbb{R}^4$.

Definition 3.1. The *standard (geometric) n-simplex* $|\Delta^n|$ is the subset

$$|\Delta^n| := \left\{ x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1 \text{ and } x_i \geq 0 \ \ \forall 1 \leq i \leq n+1 \right\}$$

We will think of simplicial complexes as 'spaces built out of standard n-simplices'

Definition 3.2. A *simplicial complex* K consists of a set X together with a subset $\mathrm{Sim}(K) \subset \mathbb{P}(X) \setminus \emptyset$ of the power set such that:

1. Every singleton is in $\mathrm{Sim}(K)$.
2. Every set in $\mathrm{Sim}(K)$ is finite.
3. If $Y \in \mathrm{Sim}(K)$, and $Z \subset Y$, then $Z \in \mathrm{Sim}(K)$.

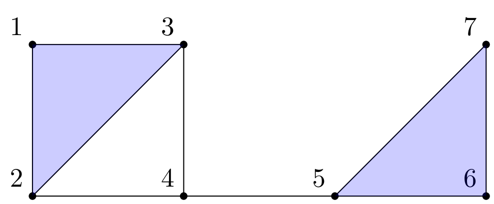
If $Z \subset Y \in \mathrm{Sim}(K)$, we call Z a *face of Y*.

While this definition may seem rather abstract, the intuition behind it is quite straightforward. The set X should be thought of as the vertices (or 0-simplicies) of the simplicial complex. More generally, we think of each $Y \in \mathrm{Sim}(K)$ with $|Y| = n+1$ as being an n-simplex with set of vertices Y.

Example 3.3. We define a simplicial complex K with $X = \{1,2,3,4,5,6,7\}$ and with

$$\begin{aligned} \mathrm{Sim}(K) = \{&\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\} \\ &\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{2,4\},\{4,5\},\{5,6\},\{6,7\},\{5,7\}, \\ &\{1,2,3\},\{5,6,7\}\}. \end{aligned}$$

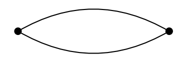
In a way that we will soon make rigorous, we can think of K as representing the following picture:



The drawing, like K, has seven 0-simplices, nine 1-simplices, and two 2-simplices.

Example 3.4. The standard (combinatorial) n-simplex Δ^n has vertex set $[n] := \{0, 1, \ldots, n\}$ and $\mathrm{Sim}(\Delta^n) := \mathbb{P}([n]) \setminus \emptyset$.

Non-example. The following drawings looks like it should correspond to simplicial complexes, but does not:



the reason for this is that, in a simplicial complex, there is at most one n-simplex with a given set of vertices. In the drawing above, however, both one simplices have the same set of vertices.

Construction 3.5. Let K be a simplicial complex with finite vertex set X (we call such a simplicial complex *finite*). We construct a metric space $|K|$ called *the geometric realization of* K as follows. The underlying set of $|K|$ is a set of formal linear combinations

$$\Lambda := \sum_{x \in X} \lambda_x x$$

with $\lambda_x \in \mathbb{R}$. To each such Λ, we associate a set $S_\Lambda := \{x \in X \mid \lambda_x \neq 0\}$. We can then define the underlying set

$$|K| := \left\{ \sum_{x \in X} \lambda_x x \;\middle|\; \lambda_x \geq 0, \quad \sum_{x \in X} \lambda_x = 1, \quad \text{and } S_\Lambda \in \mathrm{Sim}(K) \right\} \subset |\Delta^{|X|-1}|$$

Where we consider the coefficient vector $(\lambda_x)_{x \in X}$ as an element of $|\Delta^{|X|-1}|$. The taxicab metric[21] on $\mathbb{R}^{|X|}$ then induces a canonical metric on $|K|$, equipping it with the structure of a metric space, and thus, a topological space.

[21] Using the right notion of 'equivalence of metrics', where two metrics are equivalent if the identity is a continuous function from each to the other, this is equivalent to the Euclidean metric. We here take the taxicab metric for the sake of simplicity.

Remark 3.6. Note that the realization of the standard combinatorial n-simplex Δ^n is precisely the standard geometric n-simplex $|\Delta^n|$, justifying our notation and terminology.

Definition 3.7. We define a *map of simplicial complexes* $f : K \to K'$ to be a map $f : X \to X'$ of the underlying vertex sets such that, for each $S \in \mathrm{Sim}(K)$, $f(S) \in \mathrm{Sim}(K')$.

Lemma 3.8. *Let $f : K \to K'$ be a map of finite simplicial complexes. Then the assignment*

$$\begin{aligned} |f| : |K| &\to |K'| \\ \sum_{x \in X} \lambda_x x &\mapsto \sum_{x \in X} \lambda_x f(x) \end{aligned}$$

defines a continuous map.

Proof. We first check that $|f|$ is well defined. For $\Lambda \in |K|$ and $y \in X'$, the coefficient of y in $f(\Lambda)$ is

$$|f|(\Lambda)_y = \sum_{x \in f^{-1}(y)} \lambda_x$$

and is thus greater than 0. By definition

$$\sum_{y \in X'} |f|(\Lambda)_y = 1.$$

Moreover $S_{|F|(\Lambda)} = f(S_\Lambda)$, and therefore is in $\mathrm{Sim}(K')$, so we conclude that $|f|$ is a well-defined map of underlying sets.

To see that it is continuous, we note that

$$d(|f|(\Lambda), |f|(\Gamma)) = \sum_{y \in \mathrm{Im}(f)} \left| \sum_{x \in f^{-1}(y)} (\Gamma_x - \Lambda_x) \right| \leq \sum_{x \in X} |\Gamma_x - \Lambda_x| = d(\Lambda, \Gamma)$$

completing the proof. □

Exercise 3.9.

1. If $f : K \to K'$ induces a injection $\mathrm{Sim}(K) \to \mathrm{Sim}(K')$, $|f|$ is injective.

2. If $f : K \to K'$ induces a surjection $\mathrm{Sim}(K) \to \mathrm{Sim}(K')$, $|f|$ is surjective.

Corollary 3.10. *Let K be a finite simplicial complex with vertex set X, and let $S \in \mathrm{Sim}(K)$ with $|S| = n + 1$. Then there is a continuous injection*

$$|f_S| : |\Delta^n| \hookrightarrow |K|$$

Whose image is precisely those $\Lambda \in |K|$ such that S_Λ is a face of S.

Proof. Choosing a bijection between $\{0, 1, \dots n\}$ and S yields a map of simplicial complexes $f_S : \Delta^n \to K$. [22] □

[22] Note that this is something of an abuse of notation. While f_S depends on S, it also depends on our choice of bijection $[n] \cong S$.

Exercise 3.11. Check that the image of $|f_S|$ is precisely that described in the corollary, and thus is independent of our choice of bijection $[n] \cong S$.

Definition 3.12. We call the image of $|f_S|$ an *n-simplex of* $|K|$, and denote it by $|S|$.

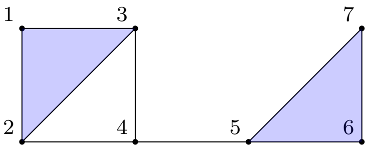
Lemma 3.13. *Let $f : K \to L$ and $g : L \to P$ be maps of finite simplicial complexes (with vertex sets X_K, X_L and X_P). Then*

1. *$g \circ f$ defines a map of simplicial complexes $K \to P$.*

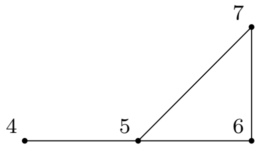
2. *On realizations $|g \circ f| = |g| \circ |f|$.*

Proof. To show 1., note that the underlying maps of vertex sets compose to give a map $g \circ f : X_K \to X_P$. Suppose $S \in \mathrm{Sim}(K)$. Then $f(S) \in \mathrm{Sim}(L)$ and thus $g(f(S)) \in \mathrm{Sim}(P)$. Part 2. follows immediately from direct computation. □

We return to our example simplicial complex K



An example of a subsimplicial complex is



This simplicial complex is *not* full, because it contains the vertices $5, 6$, and 7, but does not contain the simplex $\{5, 6, 7\}$, which is a simplex of K.

Definition 3.14. Let K be a simplicial complex with vertex set X_K. A *subcomplex of K* is a simplicial complex L such that $X_L \subset X_K$ and $\mathrm{Sim}(L) \subset \mathrm{Sim}(K)$.

Let $V \subset X$. We define the *full subcomplex of K on V* to be the simplicial complex L with

$$\mathrm{Sim}(L) := \{S \in \mathrm{Sim}(K) \mid S \subset V\}.$$

So how do we obtain a simplicial complex from a poset?

Definition 3.15. Let Q be a finite poset. We define the *order complex* $\mathcal{K}(Q)$ to be the simplicial complex whose vertex set is the underlying set of Q, and whose set of simplices is

$$\mathrm{Sim}(\mathcal{K}(Q)) := \{S \subset \mathbb{P}(Q) \mid S \text{ is totally ordered}\}.$$

By a *totally ordered set*, we mean a poset of the form

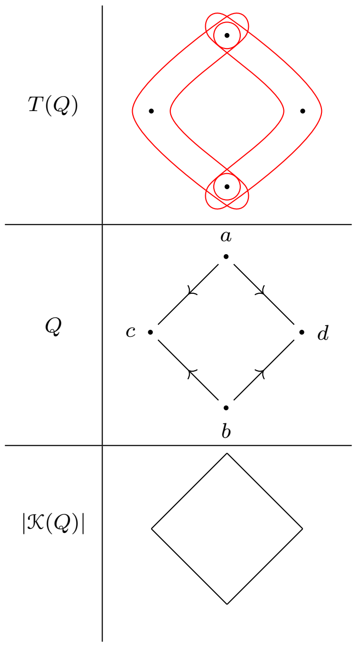
$$* \to * \to * \cdots \to *.$$

Thus, the condition that $S = \{q_0, q_1, \ldots, q_n\}$ is totally ordered amounts to saying that

$$q_0 \leq q_1 \leq q_2 \leq \cdots \leq q_n.$$

This gives us a somewhat strange result. There are two ways of associating a topological space to a finite poset Q. On the one hand, we can consider the finite T_0 space $T(Q)$, and on the other hand we can define the (infinite, Hausdorff) space $|\mathcal{K}(Q)|$. There are a number of subtle connections between these spaces, but we will, initially at least, only consider a loose heuristic — the spaces *look* similar.

Consider the pseudo-circle from the previous section. We draw the three ways we can look at the pseudo-circle: poset, finite space, and order complex.



There is a more explicit comparison which can be made between $|\mathcal{K}(Q)|$ and $T(Q)$, which was first constructed in [6]. To define it, we will need some additional terminology.

Definition 3.16. We define the *open n-simplex* to be the space

$$\Omega^n := \left\{ x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i 1 \text{ and } 0 \leq x_i < 1 \ \forall 1 \leq i \leq n+1 \right\} \subset |\Delta^n|$$

when $n \neq 0$, and to be $|\Delta^0|$ when $n = 0$.

Given a simplicial complex K and and n-simplex $S \in \mathrm{Sim}(K)$, we will denote by $S^\circ \subset |K|$ the image of the inclusion

$$\Omega^n \hookrightarrow |\Delta^n| \hookrightarrow |K|.$$

We will sometimes call S° and *open simplex of K*. Notice that, for $n \geq 1$, S° is an open subset of $|K|$.

Construction 3.17. Let Q be a finite poset. We define a map

$$\pi_Q : |\mathcal{K}(Q)| \to T(Q)$$

as follows. Given a point $x \in \mathcal{K}(Q)$, there is a unique simplex $S = \{q_0 < q_1 < \cdots < q_n\} \in \mathrm{Sim}(\mathcal{K}(Q))$ such that $x \in S^\circ$. We define

$$\pi_Q(x) := q_0.$$

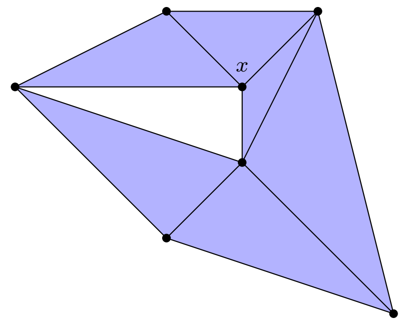
Exercise 3.18. Prove that, for any $x \in \mathcal{K}(Q)$, there is a unique simplex $S = \{q_0 < q_1 < \cdots < q_n\} \in \mathrm{Sim}(\mathcal{K}(Q))$ such that $x \in S^\circ$. Conclude the the map π_Q is well-defined.

Definition 3.19. Let K be a simplicial complex, and x a vertex of K. We define the *star of* x to be

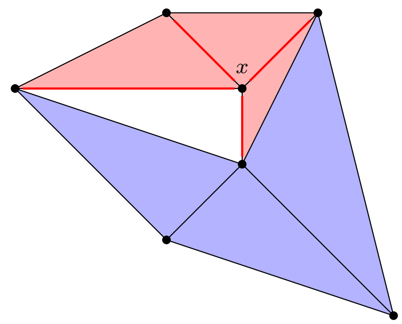
$$\mathrm{Star}(x) := \bigcup_{\substack{S\in \mathrm{Sim}(K)\\ x\in S}} S^\circ \subset |K|$$

Lemma 3.20. *For any simplicial complex K,* $\mathrm{Star}(x)$ *is an open subset of* $|K|$.

Let's try to get a sense of what the star of x looks like. We consider the following simplicial complex K:



The star of x is then the subset in red.



Proof. We first note that if we remove the point x from $\mathrm{Star}(x)$, we obtain

$$\mathrm{Star}(x)\setminus\{x\} = \bigcup_{\substack{S\in \mathrm{Sim}(K)\\ x\in S,\ S\neq\{x\}}} S^\circ \subset |K|$$

i.e., the union of the open n-simplices with $n \geq 1$ which have x as as a vertex. This is a union of open sets, and thus is open. It will thus suffice for us to show that there is an open neighborhood of x which is entirely contained in $\mathrm{Star}(X)$. However, this follows immediately from the fact that, for any vertex i of $|\Delta^n|$, there is an open subset of $|\Delta^n|$ around i which contains no vertex other than i. □

Lemma 3.21. *For any finite poset Q, the map π_Q is continuous.*

Proof. Let $U \subset T(Q)$ be an open subset. We claim that

$$\pi_Q^{-1}(U) = \bigcup_{x\in U} \mathrm{Star}(x)$$

Since this is a union of open sets, the lemma will then be proved.

Let $a \in \bigcup_{x\in U}\mathrm{Star}(x)$. Then there is a unique open simplex $q_0 < q_1 < \cdots q_n$ containing a, and $x = q_i$ for some i. Since U is open (i.e. downwards-closed) this means that $\pi_Q(a) = q_0 \leq x$ is in U.

Now suppose that $a \in \pi_Q^{-1}(U)$. Let $S = \{q_0 < q_1 < \cdots < q_n\}$ be the unique open simplex containing a. Then $\pi_Q(a) = q_0$, and so $q_0 \in U$. Consequently, $S^\circ \subset \mathrm{Star}(q_0) \subset \bigcup_{x\in U}\mathrm{Star}(x)$. □

The map π_Q turns out to be something like an equivalence (See [6] for details) — a *weak homotopy equivalence*. We won't go into detail here, but effectively this means that π_Q preserves a lot of topological information, and that, for some purposes, we can consider $T(Q)$ and $|\mathcal{K}(Q)|$ to be "almost the same".

1.4 G-simplicial complexes

In this final, brief section, we discuss G-actions on simplicial complexes. As we will see, there is a relation between G-posets and G-simplicial complexes. unfortunately, the relationship is rather more tangled than in the non-equivariant[23] setting.

[23] This is just a fancy way of saying "without a group action".

Definition 4.1. Let G be a group. A *G-simplicial complex* is a simplicial complex K with vertex set X together with an action

$$G \times X \longrightarrow X$$

$$(g, x) \longmapsto g\cdot x$$

such that, for each g, the induced map

$$g \cdot (-) : X \to X$$

is a map of simplicial complexes.

Lemma 4.2. *Let K be a G-simplicial complex with vertex set X. Then $|K|$ is a G-space.*

Proof. Since each $g \in G$ induces a map of simplicial complexes

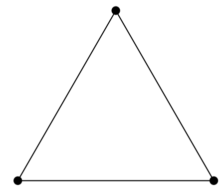
$$g \cdot (-) : K \to K,$$

Lemma 3.8 means that we get a continuous map

$$g \cdot (-) : |K| \to |K|.$$

It is immediate that the map induced by the identity element e_G is the identity on $|K|$, and compatibility with the group structure follows from Lemma 3.13. By Exercise 2.2, these maps then equip $|K|$ with the structure of a G-space. $\square$

Remark. It is important to note that not every continuous G-action on $|K|$ is induced by a simplical G-action. For instance, let K be the simplicial complex:



The realization $|K|$ is homeomorphic to the circle S^1, and thus there is a free, continuous $\mathbb{Z}/6$-action on $|K|$. However, every $\mathbb{Z}/6$-action on K is induced by an action on a three-point set, and thus, there must be a non-identity element with a fixed point by the orbit stabilizer theorem.

The import of this lemma, for us, is that it gives us a way to associate an *infinite* G-space to a finite G-space as follows.

Lemma 4.3. *Let G be a group, and let Q be a G-poset. Then $\mathcal{K}(Q)$ is a G-simplicial complex.*

Proof. The vertex set of $\mathcal{K}(Q)$ is simply Q, so we need only show that, for $S \in \mathrm{Sim}(\mathcal{K}(Q))$ and $g \in G$, we have $g \cdot S \in \mathrm{Sim}(\mathcal{K}(Q))$.

Let $S = \{q_0 < q_1 < \cdots < q_n\}$. Then

$$g \cdot S = \{g \cdot q_0, g \cdot q_1, \ldots, g \cdot q_n\}.$$

However, since the map $g \cdot (-)$ is monotone, we see that $g \cdot q_i \le g \cdot q_{i+1}$. Thus $g \cdot S \in \mathrm{Sim}(\mathcal{K}(Q))$.[24] $\square$

[24] We technically also need to check that $g \cdot S$ is still an n-simplex, but this follows from the fact that $g \cdot (-)$ is a bijection.

Ideally, we would be able to study quotients of G-simplicial complexes in terms of the corrsponding posets, but there's a problem: such quotients may not exist!

Example 4.4. Let K be the simplicial complex

and equip K with the simplicial $\mathbb{Z}/2$-action which swaps 0 and 1. The quotient of the underlying vertex set by this $\mathbb{Z}/2$ action yields a singleton set. However, it is clear that, for this quotient to make sense as a quotient of simplicial complexes, there should still be a 1-simplex. This is, of course, impossible.

This is a problem, but not an insurmountable one. Instead of taking quotients of simplicial complexes, we can first realize our G-simplicial complex to get a G-space. The quotient is then well defined. However, there is some more bad news: There are G-posets Q such that

$$|\mathcal{K}(Q/G)| \ncong |\mathcal{K}(Q)|/G.$$

So, while we have a relationship between G-posets and G-complexes, this relationship does not behave nicely under quotients.

The conditions under which $|\mathcal{K}(Q/G)|$ *is* homeomorphic to $|\mathcal{K}(Q)|/G$ are somewhat convoluted. In the (more general) context of simplicial sets, these are worked out in [2]. While we will not go further into the matter here, the interested student can either consult [2], or try to work out some such conditions themselves.

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