Exercise Sheet 9

Due: Monday, 14. Nov.

Definition. Let $M \subset \mathbb{R}^3$ be a surface. We call a coordinate chart $\phi : U \to M$ defines *geodesic coordinates* (u^1, u^2) on M if the following conditions are satisfied:

1. The coordinate curves in which u^2 is constant are geodesics.
2. The metric has the form
$$g = \begin{pmatrix} 1 & 0 \\ 0 & g_{22} \end{pmatrix}$$

Fact 1. For any point p in a surface $M \subset \mathbb{R}^3$, there exists a geodesic coordinate chart containing p.

Exercise 1. Show that, in geodesic coordinates (u^1, u^2),
$$\Gamma^1_{1,1} = \Gamma^1_{1,2} = \Gamma^2_{1,1} = 0$$
and
$$\Gamma^2_{1,2} = \frac{\partial}{\partial u^1} \ln(\sqrt{g_{2,2}})$$

Exercise 2. Show that, in geodesic coordinates (u^1, u^2), the Gaußian curvature is given by
$$K = -\frac{1}{\sqrt{g_{2,2}}} \frac{\partial^2}{\partial (u^1)^2} \left(\sqrt{g_{2,2}}\right)$$

Exercise 3. Define
$$\begin{aligned} f : \mathbb{R} &\longrightarrow S^1 \subset \mathbb{R}^2 \\ t &\longmapsto (\cos(t), \sin(t)) \end{aligned}$$
and let $\mu : [a, b] \to S^1$ be a smooth function.

1. Show that, given $\tilde{p} \in \mathbb{R}$ and a semi-circle $U \subset S$ containing $p := f(\tilde{p})$, there is a *unique* open interval $(c, c + \pi) \subset \mathbb{R}$ containing p such that
$$f|_{(c,c+\pi)} : (c, c + \pi) \longrightarrow U$$
is a diffeomorphism.
2. Show that there is a smooth function $\theta : [a, b] \to \mathbb{R}$ such that
$$\mu(t) = (\cos(\theta(t)), \sin(\theta(t))).$$
Further show that any two such functions differ by a constant multiple of 2π.

Exercise 4. Let $x_1, \ldots, x_k \in \mathbb{R}^n$ be a set of points. A *convex combination* of $x_1, \ldots, x_k$ is a sum

$$\sum_{i=1}^{k} \lambda_i x_i$$

where $0 \leq \lambda_i \leq 1$ and

$$\sum_{i=1}^{k} \lambda_i = 1.$$

The *convex hull* of the set $S = \{x_1, \ldots, x_k\}$ is the set $\mathrm{Conv}(S) \subset \mathbb{R}^n$ of all convex combinations of points in S.

1. Show that, for any $y, z \in \mathrm{Conv}(S)$, the line segment $\overline{yz}$ from y to z lies in $\mathrm{Conv}(S)$.

2. Show that $\mathrm{Conv}(S)$ is compact.

3. We say that S is *convex independent* if, for every $x_i \in S$, $x_i \notin \mathrm{Conv}(S \setminus \{x_i\})$. If $S \subset \mathbb{R}^2$ is convex independent, show that, for every three distinct indices i, j, k, the vectors $x_j - x_i$ and $x_k - x_i$ are linearly independent.

Exercise 5. Let $S \subset \mathbb{R}^2$, and let $x \in \mathrm{Conv}(S)$. We say that a line

$$L = \{v \in \mathbb{R}^2 \mid \langle v - x, n \rangle = 0\}$$

through x in $\mathbb{R}^2$ *supports* S when $y \in \mathrm{Conv}(S)$ implies that $\langle y - x, n \rangle \geq 0$.

1. Let $S \subset \mathbb{R}^2$. Show that for any $y \notin \mathrm{Conv}(S)$, there is a line $L = \{\langle v - x, n \rangle = 0\}$ which supports S and such that $\langle y - x, n \rangle < 0$. (Hint: let x be the point of $\mathrm{Conv}(S)$ whose distance to y is minimal.)

2. We say that S is *convex independent* if, for every $x_i \in S$, $x_i \notin \mathrm{Conv}(S \setminus \{x_i\})$. If $S \subset \mathbb{R}^2$ is convex independent, show that, for every three distinct indices i, j, k, the vectors $x_j - x_i$ and $x_k - x_i$ are linearly independent.

3. Let $S \subset \mathbb{R}^2$ be a convex independent subset. Show that if

$$x = \sum_{i=1}^{\ell} \lambda_i x_i$$

is a convex combination in which at least three of the λ_i are non-zero, then there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset \mathrm{Conv}(S)$.