Exercise Sheet 8

Due: Wednesday, 2. Nov.

Let $M \subset \mathbb{R}^3$ be a surface.

Exercise 1. Derive the Gauß equations in coordinates, i.e., show that the equation

$$\frac{\partial}{\partial x^k}\Gamma^n_{i,j} - \frac{\partial}{\partial x^j}\Gamma^n_{i,k} + \left(\Gamma^\ell_{i,j}\Gamma^n_{\ell k} - \Gamma^\ell_{i,k}\Gamma^n_{\ell,j}\right) = g^{\ell,n}\left(h_{i,j}h_{k,\ell} - h_{i,k}h_{j,\ell}\right)$$

holds.

Exercise 2. For tangent vector fields X, Y, and Z, compute the coefficient of $\partial_n\phi$ in the expression

$$\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Note that this coefficient is precisely the left-hand side of the Gauß equations.

Exercise 3. Show that the formula

$$\Gamma^\ell_{i,k}h_{j,\ell} - \Gamma^\ell_{j,k}h_{i\ell} + \frac{\partial}{\partial x^j}h_{i,k} - \frac{\partial}{\partial x^i}h_{j,k}$$

holds for any i, j, k. (Hint: Normal component.)