Exercise Sheet 7

Due: Wednesday, 26. Oct.

Exercise 1. Let $M \subset \mathbb{R}^n$ be a k-submanifold, and $f : M \to \mathbb{R}$ a smooth function.

1. Show that, for $p \in M$, there is a unique vector $\operatorname{grad}(f)_p \in T_pM$ such that for every $v \in T_pM$,
$$\langle \operatorname{grad}(f)_p, v\rangle = df_p(v).$$

2. Let $\phi : U \to M$ be a chart containing p. Compute an expression for $\operatorname{grad}(f)_p$ with respect to the coordinate chart in terms of the first fundamental form, the derivatives of f, and the basis $\partial_i\phi$. Show that when $M = \mathbb{R}^n$, $\operatorname{grad}(f)$ is the usual gradient of a function.

3. Show that $\operatorname{grad}(f)$ defines a smooth tangent vector field on M.

Exercise 2. Consider a smooth function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) > 0$ for all x, and $1 + f'(x)^2 > 0$ for all x. Take the parameterization

$$\phi(u, v) = (f(u)\cos(v), f(u)\sin(v), u)$$

of the surface of revolution R_f of the graph of f.

1. Show that, with respect to the coordinate basis defined by ϕ,
$$L = \begin{pmatrix} \frac{f''(u)}{(f'(u)^2+1)^{3/2}} & 0 \\ 0 & \frac{-1}{f(u)\sqrt{1+f'(u)^2}} \end{pmatrix}.$$
Compute the Gaußian and mean curvatures.

2. A *minimal surface* is a hypersurface in $\mathbb{R}^3$ such that the mean curvature H is identically zero. Give a second order non-linear ordinary differential equation for f, whose solutions are the profile curves of minimal surfaces.

3. Show that the class of functions
$$f(u) = c\cosh\left(\frac{u}{c} + k\right)$$
satisfies your equation from part 1.

Exercise 3. Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Let γ be a smooth regular curve in M.

1. Show that
$$\frac{d^2}{dt^2} f(\gamma(t)) = \left\langle \frac{d}{dt}(\operatorname{grad} f)_{\gamma(t)}, \gamma'(t)\right\rangle + \left\langle \operatorname{grad}(f)_{\gamma(t)}, \gamma''(t)\right\rangle.$$

2. Suppose f has a local maximum at a point $p = \gamma(0)$, and that df_p has maximum rank. Show that grad_p is normal to T_pM. (Hint: test against coordinate vector fields)

Exercise 4. Suppose that $M \subset \mathbb{R}^{n+1}$ is a hypersurface, and let

$$\begin{aligned} f : \mathbb{R}^{n+1} &\longrightarrow \mathbb{R} \\ x &\longmapsto \textstyle\sum_{i=1}^{n+1} (x^i)^2 \end{aligned}$$

be the square of the radius. Suppose that $p \in M$ is a local maximum (on M) of f. Show that each principal curvature κ of M at p satisfies

$$\kappa > \frac{1}{|p|} = \frac{1}{\sqrt{f(p)}}.$$

(Hint: Let γ be a curve with $\gamma(0) = p$ whose tangent vector is a unit principal direction.)