Exercise Sheet 6

Due: Wednesday, 19. Oct.

Exercise 1 (20 points)**.** Let $M \subset \mathbb{R}^n$ be a submanifold, and $\phi : U \to M$ a chart. Show that, for tangent vector fields X, Y on $\phi(U) \subset M$ and smooth functions $f, g, h : M \to \mathbb{R}$,

1. $X(fg) = X(f)g + fX(g)$
2. $[fX, gY](h) = fg[X,Y](h) + fX(g)Y(h) - gY(f)X(h)$.
3. $\nabla_X V - \nabla_V X = [X, V]$

Exercise 2 (20)**.** Let $\gamma : (a,b) \to \mathbb{R}^2$ be a smooth regular curve with $\gamma^1(t) > 0$ for all $t \in (a,b)$, and let R_γ be the corresponding surface of revolution. Consider the parameterization

$$\phi(t,\theta) = (\gamma_1(t)\cos(\theta), \gamma_1(t)\sin(\theta), \gamma_2(t))$$

1. Compute the matrix of the first fundamental form with respect to the chart ϕ
2. Compute the Christoffel symbols with respect to the chart ϕ.

Remark 1. Fix a k-submanifold $M \subset \mathbb{R}^n$. The remainder of this problem set is devoted to understanding the way in which geodesics are locally (on small enough open sets of M) length-minimizing. We will need to use without proof several facts from the theory of ordinary differential equations.

Firstly, we will need local existence and uniqueness of geodesics.

Let $M \subset \mathbb{R}^n$ be a k-submanifold, $p \in M$ a point, and $v \in T_pM$ a non-zero tangent vector. Then there is an $\epsilon > 0$ and a unique geodesic

$$\gamma_v : (-\epsilon, \epsilon) \to M$$

such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$.

This result follows from solving the (second-order, non-linear) initial value problem $\nabla_{\gamma'}\gamma' = 0$, $\gamma(0) = p$, and $\gamma'(0) = v$. Notably, for any constant $\lambda \in \mathbb{R}_{>0}$, this implies that $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$, meaning that if we choose small enough tangent vectors v, γ_v is always defined on (at least) $[-1, 1]$.

A very similar result is the existence of the *exponential map*

Let $p \in M$ be a point. Denote by $B_\epsilon(0) \subset T_pM$ the ball of radius ϵ around 0 in $T_pM \cong \mathbb{R}^k$. Then there is an $\epsilon > 0$ and a smooth map

$$\exp_p : B_\epsilon(0) \longrightarrow M$$

satisfying the following conditions:

- $\exp_p(0) = p$.
- $\exp_p(v) = \gamma_v(1)$.

Letting $v \in B_\epsilon(0) \subset T_pM$ be a tangent vector, we notice that this implies, for any $t \in \mathbb{R}$ such that $tv \in B_\epsilon(0)$, we have

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$$

Exercise 3 (10 points)**.** Argue that there is an $\epsilon > 0$ such that the restriction $\exp_p : B_\epsilon(0) \to M$ is a diffeomorphism onto its image. Give an explicit formula for the exponential map $\exp_{(0,0,1)}$ for the sphere $S^2 \subset \mathbb{R}^3$.

Exercise 4 (24 points)**.** Using exercise 3, we view the exponential map $\exp_p : B_\epsilon(0) \to M$ as a coordinate chart by identifying $T_pM \cong \mathbb{R}^k$. Notice that under the identification $T_pM \cong \mathbb{R}^k$, for any chart ρ around p, the vectors $\partial_1\rho, \ldots \partial_k\rho$ give coordinates on $B_0(r)$. We will write y^i for these coordinates on $B_0(r)$. This coordinate chart is sometimes referred to as *local normal coordinates*. For ease of notation, we write $\phi = \exp_p$.

1. Argue that, for any coordinate chart $\mu : V \to S^{k-1}$, the function

   $$\begin{aligned} \overline{\mu} : V \times (0,\epsilon) &\longrightarrow B_\epsilon(0) \setminus \{0\} \\ (x, r) &\longmapsto r\mu(x). \end{aligned}$$

   defines a coordinate chart. Conclude that $\phi \circ \overline{\mu}$ is a coordinate chart on $U \setminus \{p\}$.

2. Let $\psi = \phi \circ \overline{\mu}$, and define a vector field $S = \partial_r\psi$ Show that, for any point in the image of ψ with coordinates $y = r\mu(x)$,

   $$r = \sqrt{\sum_{i=1}^{k} (y^i)^2}.$$

   Further show that, with respect to the chart ϕ,

   $$S = \frac{y^i}{r}\partial_i\phi.$$

   Conclude that S is a smooth vector field on all of $U \setminus \{p\}$.

3. Given a unit vector $v = (v^1, \ldots, v^k) \in \mathbb{R}^k \cong T_pM$, consider the geodesic through p given by

   $$\exp_p(tv) = \gamma_v(t).$$

   Show that $\gamma_v'(t) = S(\gamma_v(t))$. Conclude that $\nabla_S S = 0$.

4. Show that, at every point in $U \setminus \{p\}$, $\langle S, S\rangle = 1$. (Hint: Use the curve $\gamma_v(t)$.)

Exercise 5 (25 points)**.** Let $M \subset \mathbb{R}^n$ be a k-submanifold. We will show that geodesics in M are locally of minimal length in the following way:

Given $p \in M$, there is an open neighborhood $U \subset M$ containing p such that, for every other point $q \in U$, a geodesic γ from p to q is the unique minimal length curve in U from p to q.

We fix $p \in M$, and let $B_\epsilon(0)$, U, ϕ, and S be as defined in the previous exercise.

1. Consider r as a function $U \setminus \{p\} \to \mathbb{R}$, given by

   $$r(y) = \sqrt{\sum_{i=1}^{k} (y^i)^2}.$$

   Show that, for any tangent vector field X on $U \setminus \{p\}$

   $$dr \circ X = \langle S, X \rangle.$$

2. Let $q \in U \setminus \{p\}$, and note that there is a unit vector v such that γ_v defines a geodesic from p to q in U. Suppose that β is another curve in U. Show that

   $$L(\beta) \geq L(\gamma_v)$$

   and that equality holds if and only if β is a reparameterization of γ. (Hint: bound the speed below by $\langle \beta'(t), S \rangle$.)

3. Show that if β is any curve in M from p to q,

   $$L(\beta) \geq L(\gamma_v)$$

   and that if β is not a curve in U, then $L(\beta) > L(\gamma)$.