Exercise Sheet 5

Due: Monday, 3. Oct.

Exercise 1 (15 points)**.** Let $M \subset \mathbb{R}^n$ be a k-submanifold, and let $p \in M$. Show that, given k linearly independent tangent vectors $X_1, \dots, X_k$ in T_pM, there is a coordinate chart $\phi : U \to M$ with $p \in \phi(U)$ such that $p \in \phi(U)$ and $\partial_i \phi = V_i$ for $1 \leq i \leq k$. Conclude that we can always choose a coordinate chart such that $g_{i,j}(p)$ is diagonal.

Exercise 2 (15 points)**.** Let $\phi : U \to M$ be a chart on a k-submanifold of $\mathbb{R}^n$. Show that

$$g = (J\phi)^T(J\phi).$$

Argue that $\sqrt{\det(g)}$ is the (absolute value of the) k-dimensional volume of the parallelpiped spanned by $\partial_1\phi, \dots, \partial_k\phi$.

Exercise 3 (20 points)**.** Compute the matrix of the first fundamental form of each of the following manifolds in *one* coordinate chart of your choice.

1. The surface of revolution for a curve $\gamma : (a, b) \to \mathbb{R}^2$, where the first coordinate of γ is always positive.

2. The graph of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$.

3. The n-sphere $S^n \subset \mathbb{R}^{n+1}$.

Exercise 4 (15 points)**.** Let $\gamma : [a, b] \to M$ be a unit speed curve in M, and let $H : (-\epsilon, \epsilon) \times [a, b] \to M$ be a proper variation of γ, and write $\gamma_u(t) = H(u, t)$. Define the *energy* of γ to be

$$\frac{1}{2}\int_a^b \langle \gamma'(t)\gamma'(t)\rangle dt.$$

And write

$$E(u) =: \frac{1}{2}\int_a^b \langle\langle \gamma_u'(t), \gamma_u'(t)\rangle dt$$

Show that the *first variation of energy* $\partial E = \frac{dE}{du}|_{u=0}$ is given by

$$\left\langle \frac{\partial}{\partial u}\gamma_u(t) \mid_{u=0}, \gamma_0'(t) \right\rangle \Big|_{t=a}^b - \int_a^b \left\langle \frac{\partial}{\partial u}\gamma_u(t)\Big|_{u=0}, \frac{d^2}{dt^2}\gamma_u(t) \right\rangle dt$$

conclude that γ is a geodesic if and only if it is a critical point of energy.

Exercise 5 (15 points)**.** Suppose that $\phi : U \to M$ is a coordinate chart, where U is the open unit ball in $\mathbb{R}^3$. Consider the curve γ in U given by

$$\begin{aligned} \gamma : [-\tfrac{1}{2}, \tfrac{1}{2}] &\longrightarrow U \\ t &\longmapsto \tfrac{t}{\sqrt{3}}(1, 1, 1). \end{aligned}$$

Let g denote the matrix of the first fundamental form with respect to ϕ, and suppose that

$$g(\gamma(t)) = \begin{pmatrix} t^2 - 1 & 0 & 0 \\ 0 & 6t + 6 & 0 \\ 0 & 0 & 2t^2 - 2 \end{pmatrix}$$

compute the arc length of $\rho = \phi \circ \gamma$.