Exercise Sheet 4

Due: Monday, 26. Sept.

Exercise 1. Let $n > 1$, and consider a smooth function $f : \mathbb{R}^n \to \mathbb{R}$.

1. Let $U \subset \mathbb{R}^n$ be an open subset. Show that the graph of f on U, i.e. the set

$$\mathrm{Graph}_U(f) := \{(x, y) \in U \times \mathbb{R} \mid y = f(x)\} \subset \mathbb{R}^{n+1}$$

is a smooth submanifold of dimension n.

2. Suppose that df_x has rank 1 for every x such that $f(x) = 0$. Show that the set

$$M = \{x \in \mathbb{R}^n \mid f(x) = 0\} \subset \mathbb{R}^n$$

is a smooth submanifold of dimension $n-1$. (Hint: Implicit function theorem.)

3. Let

$$M = \{x \in \mathbb{R}^n \mid f(x) = 0\} \subset \mathbb{R}^n$$

as in part (2). Show that $v \in T_pM$ if and only if v is orthogonal to the vector

$$\mathrm{grad}(f) = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right).$$

at p. Note that this is the same thing as requiring that v is in the kernel of the linear map

$$df_p : T_pM \longrightarrow T_{f(p)}\mathbb{R}$$

Exercise 2. The n-sphere S^n is the subset of $\mathbb{R}^{n+1}$ consisting of points which are unit distance from the origin.

1. Show that S^n is a smooth submanifold of $\mathbb{R}^{n+1}$ for $n \geq 1$.

2. Consider the odd-dimensional sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Let $(x^1, \dots, x^n, y^1, \dots, y^n)$ be coordinates on $\mathbb{R}^{2n}$. Write e_{x^i} and e_{y^j} for constant vector fields on $\mathbb{R}^{2n}$ corresponding to the standard basis. Show that the vector field

$$X(x^1, \dots, x^n, y^1, \dots, y^n) := \sum_{i=1}^{n} y^i e_{x^i} - x^i e_{y^i}$$

on $\mathbb{R}^{2n}$ restricts to a smooth tangent vector field on S^{2n-1}. Show that X is non-zero for every point in S^{2n-1}.

Remark 1. An fascinating fact about vector fields on spheres is the *Hedgehog Theorem*: There is a non-zero tangent vector field on the n-sphere $S^n \subset \mathbb{R}^{n+1}$ if and only if n is odd. (You can't comb a hedgehog, or at least not well.) For an analytic proof see this document.

Exercise 3. Find two smooth tangent vector fields X and Y on the torus $T^2 \subset \mathbb{R}^3$, which has a parameterization

$$\phi(u^1, u^2) = \big(\sin(u^1)(\cos(u^2) + 2), \cos(u^1)(\cos(u^2) + 2), \sin(u^2)\big) .$$

Your vector fields satisfy the following condition.

- At each $p \in T^2$, $X(p)$ and $Y(p)$ form a basis of T_pT^2.

Verify that your vector fields are indeed smooth, and show that they satisfy this condition. (Hint: draw a picture, then try to formal definition.)

Exercise 4. Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be two submanifolds, and let $f : M \to N$ be a smooth function. Let $\phi : U \to M$ be a chart around p and $\psi : V \to N$ a chart around $f(p)$. Define the *differential of f* at a point $p \in M$

$$df_p : T_pM \longrightarrow T_{f(p)}N$$

to be the unique linear map such that the diagram

$$\begin{array}{ccc} T_pM & \xrightarrow{\;df_p\;} & T_{f(p)}N \\ {\scriptstyle d(\phi)_p}\Big\uparrow & & \Big\uparrow{\scriptstyle d(\psi)_{f(p)}} \\ \mathbb{R}^k & \xrightarrow[d(\psi^{-1}\circ f\circ\phi)_{\phi(p)}]{} & \mathbb{R}^\ell \end{array}$$

commutes. Define

$$df : TM \longrightarrow TN$$

to send (p, v) to $(f(p), df_p(v))$.

1. Show that df is independent of the choice of charts in the definition.

2. Show that df is a smooth map between manifolds.

3. Show that, given a smooth curve $\gamma : (-a, a) \to M$ with $\gamma(0) = p$, the tangent vectors of γ and $f \circ \gamma$ at 0 are related by

$$df_p(\gamma'(0)) = \frac{d}{dt}(f \circ \gamma)|_{t=0}.$$