# Exercise Sheet 3

**Due: Monday, 19. Sept.**

**Exercise 1** (5 points)**.** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be non-empty open subsets. Show that if $\phi : U \to V$ is a diffeomorphism, then $n = m$.

In the following exercises, we make use of the following three maps.

$$\begin{aligned} \chi : B_1(0) &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto (x_1, y_2, \sqrt{1 - x_1^2 - x_2^2}), \end{aligned}$$

$$\begin{aligned} \psi : (0,\pi) \times (0, 2\pi) &\longrightarrow \mathbb{R}^3 \\ (\phi, \theta) &\longmapsto (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)), \end{aligned}$$

and

$$\begin{aligned} \sigma : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x_1, x_2) &\longrightarrow \left( \tfrac{2x_1}{1+x_1^2+x_2^2}, \tfrac{2x_2}{1+x_1^2+x_2^2}, \tfrac{x_1^2+x_2^2-1}{1+x_1^2+x_2^2} \right). \end{aligned}$$

**Exercise 2** (20 points)**.** For each of the maps $\chi$, $\psi$, and $\sigma$:

1. Argue that each map is smooth using Fact 2.12 from the notes.
2. Show that each map is regular on its given domain, i.e., that the Jacobian has maximal rank.

**Exercise 3** (20 points)**.**

1. Find the intersections of the images of the following pairs of functions
   - The functions $\psi$ and $\chi$
   - The functions $\psi$ and $\sigma$.

   and determine the corresponding subsets of these functions' domains.
2. Show that, where they are defined, $\sigma^{-1} \circ \psi$ and $\chi^{-1} \circ \psi$ are diffeomorphisms.

**Exercise 4** (15 points)**.** Let $\gamma : (a, b) \to \mathbb{R}^2$ be a smooth regular curve which is injective.

1. Show that the image of $\gamma$ is a smooth 1-dimensional submanifold of $\mathbb{R}^2$.

2. Suppose that the first coordinate of $\gamma$ is always positive. Define the corresponding *surface of revolution* $R_\gamma$ as follows. Say that a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ lies in $R_\gamma$ precisely when the point

$$\left( \frac{\langle (x_1, x_2, x_3), (x_1, x_2, 0) \rangle}{|(x_1, x_2, 0)|}, x_3 \right)$$

lies in the image of $\gamma$. Give a geometric interpretation (in words) of this condition, and draw a picture to accompany your explanation.

3. Show that $R_\gamma$ can be equipped with the structure of a 2-dimensional submanifold of $\mathbb{R}^3$.

**Exercise 5** (10 points)**.** Consider the map

$$\begin{aligned} \psi : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x_1, x_2) &\longmapsto (x_1^3 - x_1, x_1^2 - 1, x_2) \end{aligned}$$

1. Show that $\psi$ is smooth and regular.

2. Is the image of $\psi$ a submanifold of $\mathbb{R}^3$? Explain why or why not. Draw a picture.