Math 4750, Fall 2022
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Exercise Sheet 1

Due: Monday, 12. Sept.

Definition. A *hyperplane* in $\mathbb{R}^n$ is an affine subspace of dimension $n-1$. More precisely, a hyperplane H is the zero set of an affine function

$$f(x) = \langle x - b, v \rangle$$

for some $b, v \in \mathbb{R}^n$.

Exercise 1 (15 points)**.** Let $\gamma : [a,b] \to \mathbb{R}^n$ be a Frenet curve with curvatures $\kappa_i(t)$ for $1 \leq t \leq n-1$. Show that if $\kappa_{n-1}(t) \equiv 0$, then γ is contained in a hyperplane.

Exercise 2 (20 points)**.** Let $\gamma : [a,b] \to \mathbb{R}^3$ be a Frenet curve in three dimensional space. Show that the formulae

$$\kappa_1(t) = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}$$
$$\kappa_2(t) = \frac{\det\left(\gamma'(t), \gamma''(t), \gamma'''(t)\right)}{|\gamma'(t) \times \gamma''(t)|^2}$$

hold.

Exercise 3 (10 points)**.** Explain, in your own words, why $\kappa_1(t)$ is strictly positive in dimension three, but can be negative in dimension two. Your explanation should *not* be a proof, but rather should clarify an underlying intuition.

Exercise 4 (15 points)**.** Compute the curvatures $\kappa_1(t)$ and $\kappa_2(t)$ for the curve

$$\gamma(t) := (a\cos(t), b\sin(t), ct)$$

under the assumptions $a \neq 0$ and $b \neq 0$. Compute the limit as $c \to \infty$ of $\kappa_1(t)$ when $a = b = 1$.

Exercise 5. We consider the bilinear form on $\mathbb{R}^2$ given by

$$\langle v, w \rangle_{1,1} := v_1 w_1 + v_2 w_2 = v^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w.$$

And write

$$G := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let A be a 2×2 real matrix. Show that the following are equivalent:

- The matrix A preserves the bilinear form $\langle -, - \rangle_{1,1}$ and $\det(A) = 1$.
- The matrix GA^TG is inverse to M and $\det(A) = 1$.
- A has the form
$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$
and $a^2 - b^2 = 1$.

Note that this imposes the strict requirement that $a \neq 0$.