Exercise Sheet 1

Due: Monday, 5. Sept.

In the first two exercises, we consider the n-dimensional orthogonal group $\mathrm{O}(n)$ and the n-dimensional special orthogonal group $\mathrm{SO}(n)$. Throughout, we denote the standard basis for $\mathbb{R}^n$ by $(e_1, \ldots, e_n)$.

Exercise 1 (20 pts)**.** We will show that any matrix $B \in \mathrm{SO}(3)$ is a rotation about some axis.

1. Show that, for any $A \in \mathrm{SO}(2)$, A is a rotation about the origin.
2. Show that, for any $A \in \mathrm{SO}(3)$, there is some unit vector $v \in \mathbb{R}^3$ such that $Av = v$. (Hint: Consider $\det(A^T(A - I))$.)
3. Show that A preserves[1] the plane orthogonal to v. Show that there is a basis (v_1, v_2, v_3) such that, in this basis
$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$
where $B \in \mathrm{SO}(2)$.
4. Conclude that A is a rotation about the line $\{\lambda v \mid \lambda \in \mathbb{R}\}$.

Exercise 2 (10 pts)**.**

1. Write a matrix M which reflects e_1 and fixes the hyperplane spanned by $e_2, \ldots, e_n$. Show that M is orthogonal, and that M is *not* in $\mathrm{SO}(n)$.
2. Show that *any* orthogonal matrix B such that $B \notin \mathrm{SO}(n)$ can be written as a product of M and a matrix in $\mathrm{SO}(n)$.
3. Conclude that any isometry of $\mathbb{R}^3$ can be decomposed into reflections, rotations, and/or translations.

When we defined arc length, we required that our curves be C^1 on a closed interval. We now will see why this is necessary. In the following exercises, e_1 and e_2 will denote the Frenet frame of a curve.

Exercise 3 (10 pts)**.** A *piecewise linear curve* is a curve which consists of a sequence of straight line segments with endpoints $x^1, x^2, \ldots, x^k$.

For a C^1 curve $\gamma : [a, b] \to \mathbb{R}^n$, we call a PL curve defined by $x^1, \ldots x^k$ a *polygonal approximation to* γ if there is a sequence of parameter values

$$a = t_1 < t_2 < \ldots < t_{k-1} < t_k = b$$

such that $x^i = \gamma(t_i)$. Show that the length of γ is greater than the length of any polygonal approximation to γ. (Hint: The fundamental theorem of Calculus)

[1] In the sense that A sends vectors in the plane to vectors in the plane, *not* in the sense that A fixes every point in the plane.

Exercise 4 (10 pts.)**.** Consider the curve

$$\begin{aligned} \gamma : [0,1] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto \begin{cases} \left(t, t\sin\left(\frac{\pi}{t}\right)\right) & t > 0 \\ (0,0) & t = 0 \end{cases} \end{aligned}$$

1. Show that γ is continuous at 0, and note that for any $\epsilon > 0$, γ is C^1 on the interval $[\epsilon, 1]$.
2. Use the previous exercise to show that the length of γ on the interval $[\epsilon, 1]$ goes to ∞ as ϵ goes to zero.

Exercise 5. Find an explicit arc-length parameterized curve γ in $\mathbb{R}^2$ whose curvature is $\kappa(s) = \frac{1}{\sqrt{s}}$. (Hint: write the Frenet frame in terms of a single function $\theta(s)$, and reformulate the system of ODE's for $e_1(s)$ and $e_2(s)$ as a single ODE.)