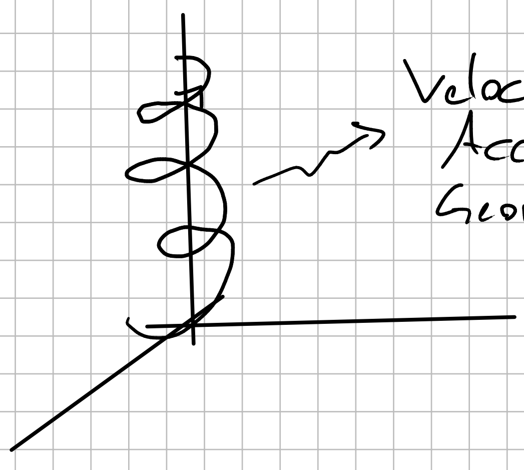
Introduction

Problem// Physics & Astronomy are usuall 3-dim.

This course// Transport Calculus to 3+ dim.

We consider//

- Paths in 3d-space



Velocities, Accelerations? Geometric quantities

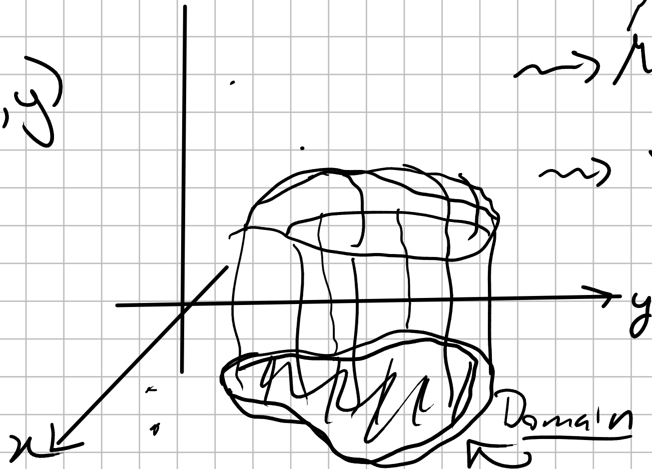
— Functions of 2-var.

Derivatives

Econ

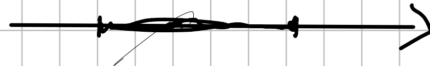
⇝ Maxima/minima

$f(x,y)$



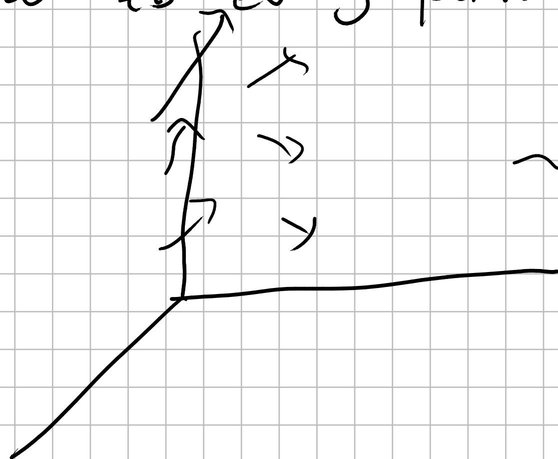
⇝ Volume

E&M



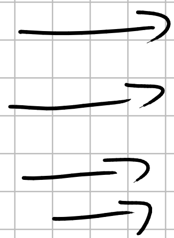
— Vector Fields

"arrow to every point in 3d space"

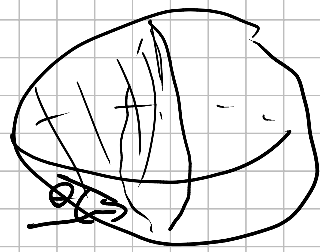


⇝ Electric/
mag.
fields

- Integrating Vectorfields



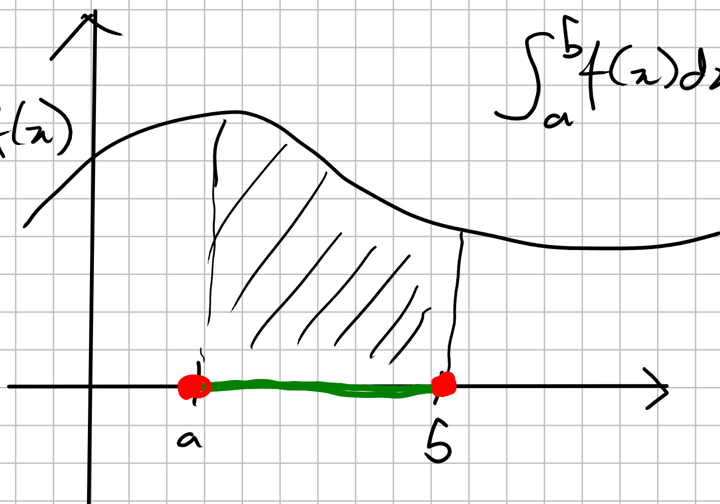
Earth



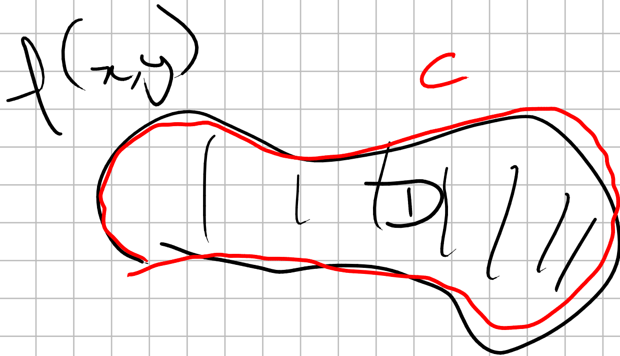
- Other: Magnetic flux

- Generalizations of Fund Thm of Calc.

$$\int_a^b f(x)\,dx = \underbrace{F(b) - F(a)}_{\text{0-dim integral}}$$



Integrate over 2-d.



One gen. will relate
2d integrals over D
to 1d integrals over C.

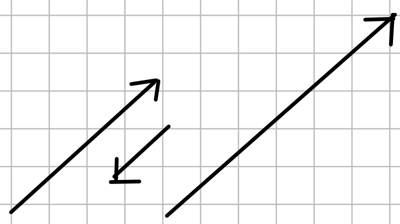
Class: 27th Aug

- HW#1 is on Webassign - Due next Thurs.
- Error in Lect1a

$$d(\bar{a},\bar{b}) = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$$

- Parametric curves in $\mathbb{R}^2$

Terminology: Call two non-zero vectors $\vec{v}, \vec{w}$ parallel if they have either the same direction, or opposite dirs



This is true if & only if $\vec{v} = \lambda \vec{w}$
$\lambda \in \mathbb{R}$ $\lambda \neq 0$.

Defn A vector $\vec{v}$ is called a **unit vector** $|\vec{v}| = 1$

Given $\vec{v} \in \mathbb{R}^3 \rightsquigarrow$ get unit vector in the same direction

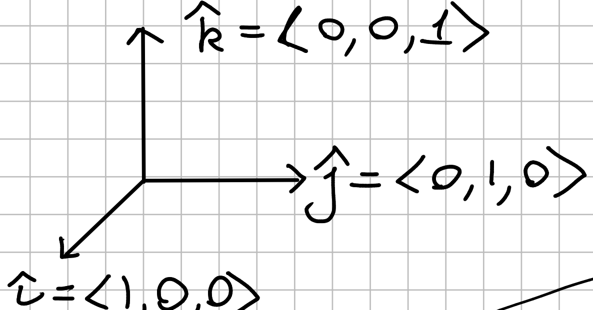
$$\hat{v} := \frac{1}{|\vec{v}|} \cdot \vec{v} = \frac{\vec{v}}{|\vec{v}|} \qquad |\hat{v}| = \left|\frac{1}{|\vec{v}|}\vec{v}\right|$$

$$= \left|\frac{1}{|\vec{v}|}\right| |\vec{v}|$$

$$= 1$$

$$\vec{v} = |\vec{v}|\, \hat{v}$$

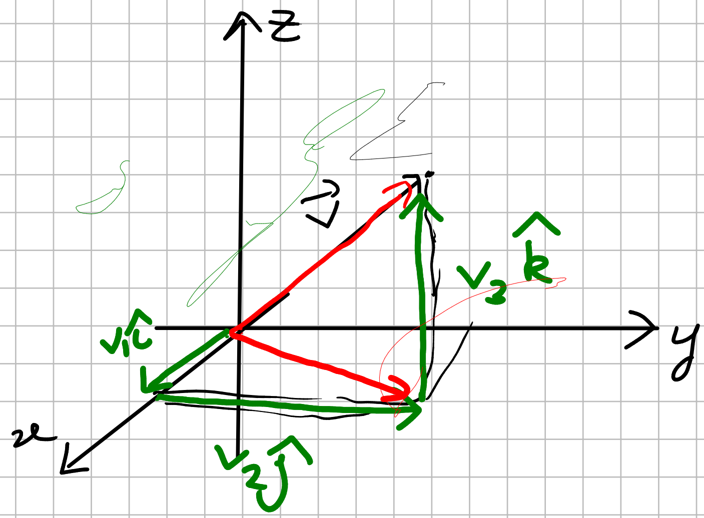
Standard unit vectors



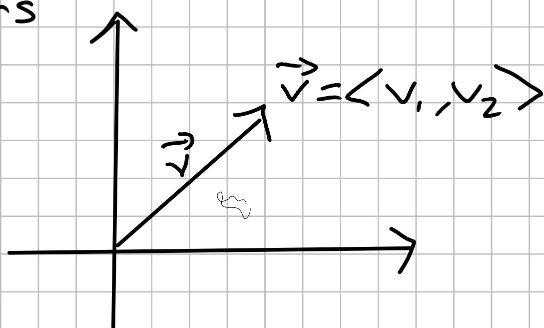
$$\vec{v} = \langle v_1, v_2, v_3\rangle$$

$$\vec{v} = v_1\hat{\imath} + v_2\hat{j} + v_3\hat{k}$$

$$= \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0, 0, v_3\rangle$$



2d Vectors



$\vec{v} \in \mathbb{R}^2$

- Same operations

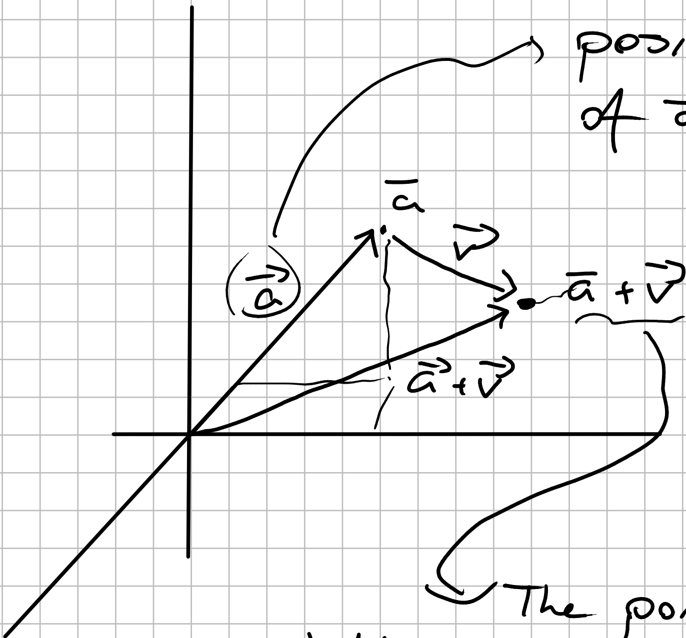
$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

$$\lambda \vec{v} = \langle \lambda v_1, \lambda v_2 \rangle \qquad \lambda \in \mathbb{R}$$

- Also have vectors in higher dim

Use of vectors: Translating points

position vector of $\bar{a}$

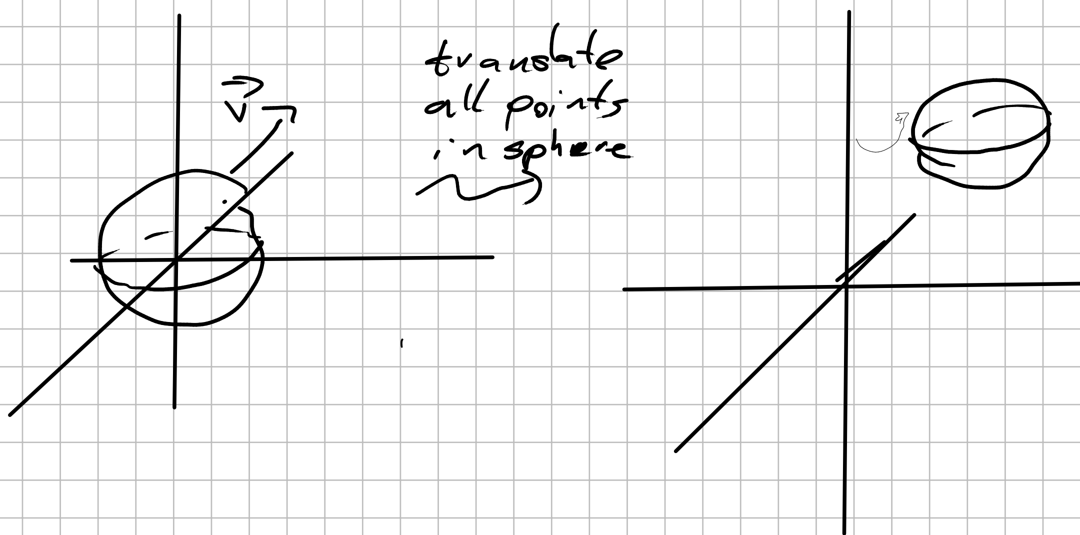


The point $\bar{a}$ shifted in $\vec{v}$-dir by a length $|\vec{v}|$

We call this

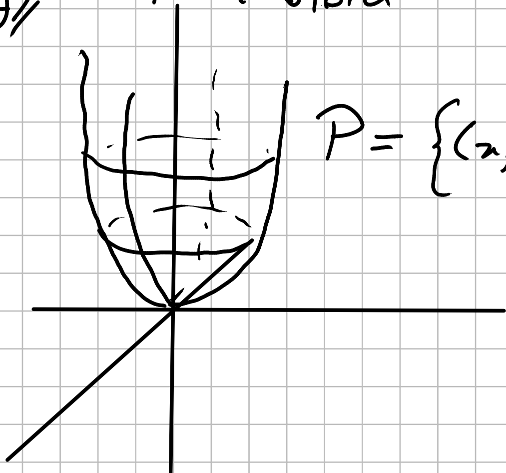
The point $\bar{a}$ __translated__ by $\vec{v}$

Can translate geometric figures



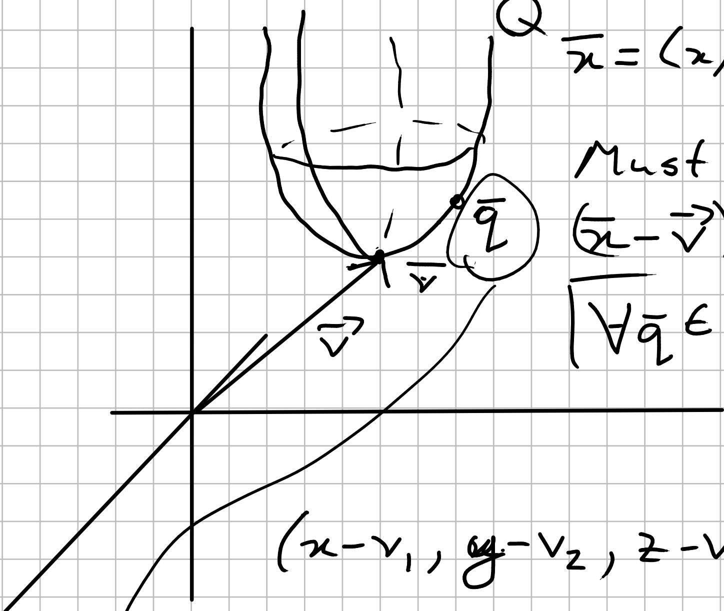
Can use this to find eqs

Eg// Paraboloid



$$P = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$$

Find eq of



$\bar{x} = (x, y, z) \in Q$

Must

$(\bar{x} - \vec{v}) \in P$

$\forall \bar{q} \in Q \; \exists \bar{p} \in P$ such that $\bar{p} + \vec{v} = \bar{q}$

$$(x - v_1, y - v_2, z - v_3) \in P$$

$$(z - v_3) = (x - v_1)^2 + (y - v_2)^2$$

$$Q = \{ (x, y, z) \in \mathbb{R}^3 \mid (z - v_3) = (x - v_1)^2 + (y - v_2)^2 \}$$

$\exists \bar{p} \in P$ such that

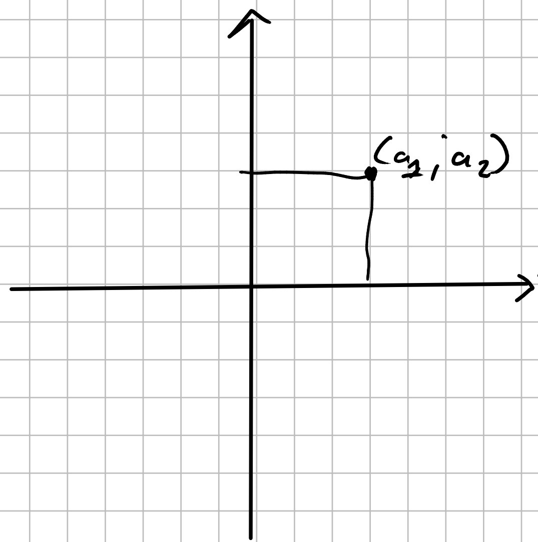
$\bar{p} + \vec{v} = \bar{q} \rightsquigarrow \bar{q} - \vec{v} = \bar{p} \in P$

$\exists$ = "there exists" $\forall$ = "for all"

$\mathbb{R}^2 = \{$ ordered pairs of real #'s $\}$

$(1, \pi) \in \mathbb{R}^2$

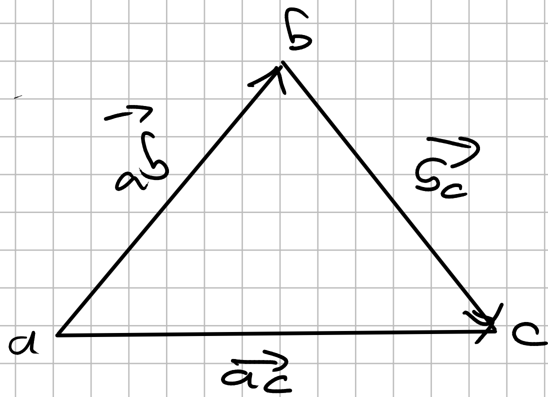
$(-3.2, 0) \in$



$\mathbb{R}^3 = \{$ordered triples of real #'s$\}$

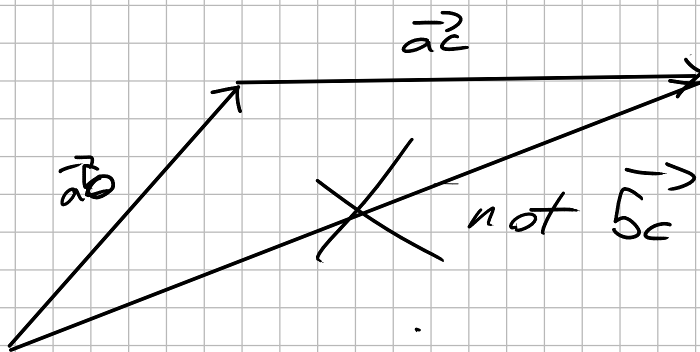
$(0, 1, 0)$

we think of $\mathbb{R}^3$ as 3d space



$$\vec{ab} + \vec{bc} = \vec{ac}$$

$$\vec{bc} = \vec{ac} - \vec{ab}$$



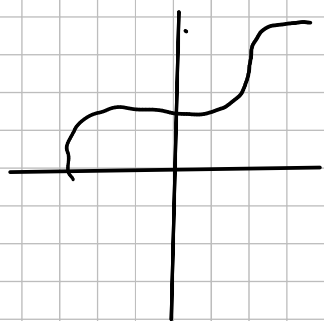
Class 3

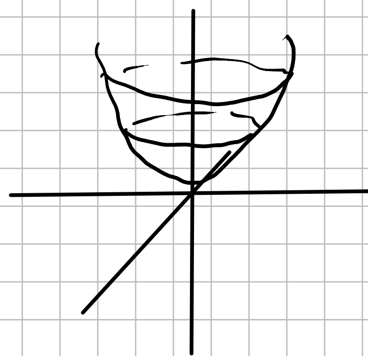
We have seen

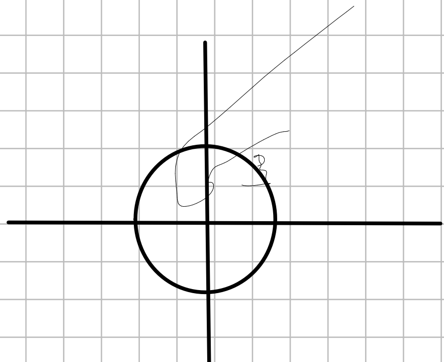
- Vectors, distances/lengths
- Vector sum, scalar mult.
- dot prod, components
- Cross product.

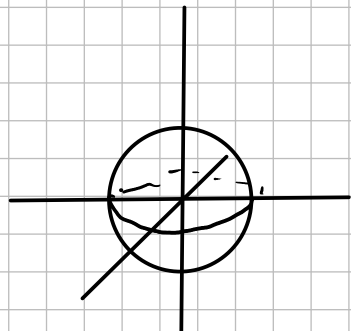
Q: How do we specify geometric shapes in $\mathbb{R}^2$ & $\mathbb{R}^3$

egs









Two main ways

(1) As solutions to equations
(2) Parametricly

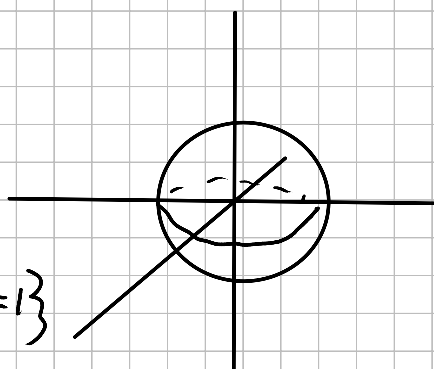
Today (1)

Idea// write an arbitrary point in coords
$\bar{x} = (x, y, z) \in \mathbb{R}^3$

& Specify some eqns that $\bar{x}$ must satisfy

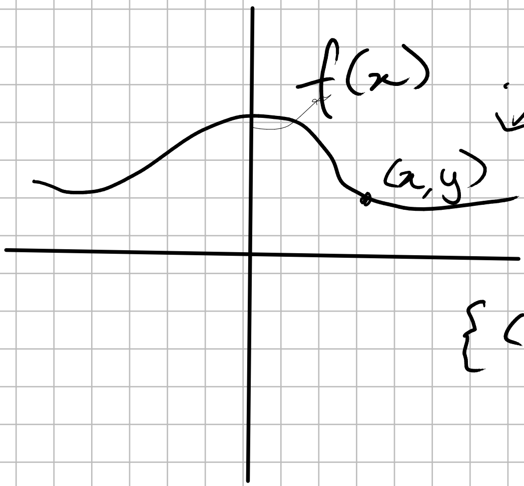
$$x^2 + y^2 + z^2 = 1$$

the figure is the solution set to eqn:



$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

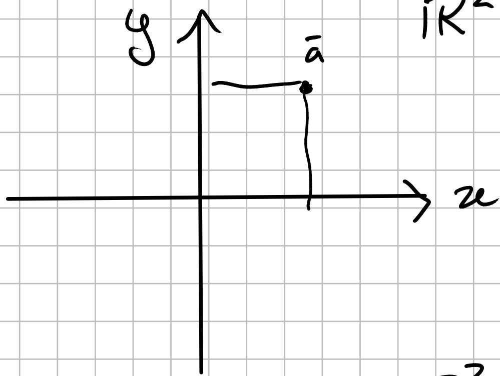
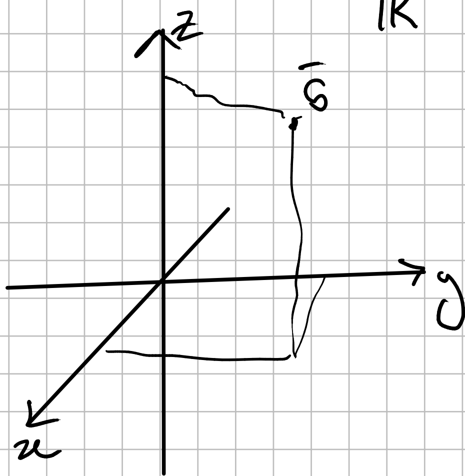
Eg// $f : \mathbb{R} \longrightarrow \mathbb{R}$

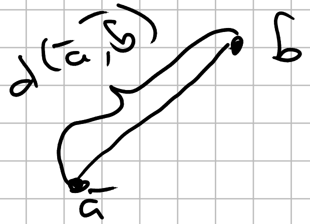


graph of f.

$$\{(x,y) \in \mathbb{R}^2 \mid y = f(x)\}$$

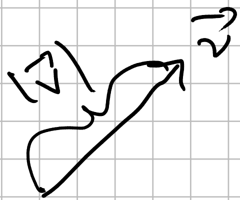
Class 4: Recap & Eqns

Geometry	Algebra
$\mathbb{R}^2$ 	$\bar{a} = (a_1, a_2) \in \mathbb{R}^2$
$\mathbb{R}^3$ 	$\bar{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$

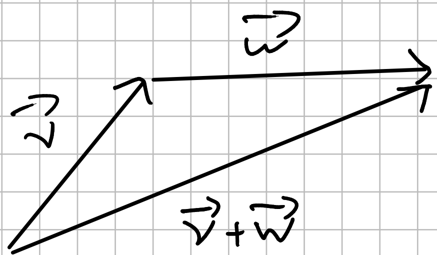


$$d(\bar a,\bar b)=\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$$

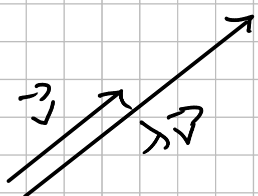
Vectors



$$\vec v=\langle v_1, v_2, v_3\rangle\in\mathbb{R}^3$$

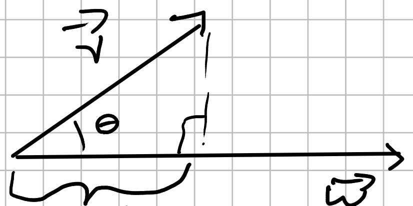


$$\vec v+\vec w=\langle v_1+w_1,\ v_2+w_2,\ v_3+w_3\rangle$$

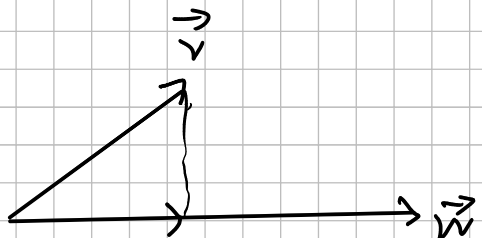


$$\lambda\vec v=\langle\lambda v_1, \lambda v_2, \lambda v_3\rangle$$

Dot Product



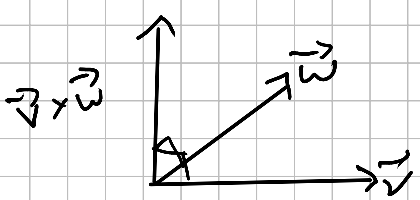
$$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} = \text{comp}_{\vec{w}}(\vec{v})$$

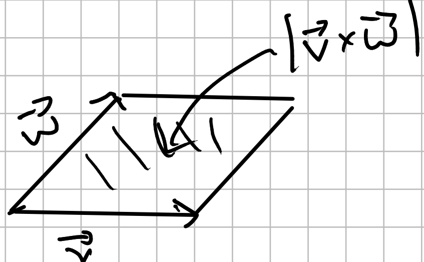


$\text{proj}_{\vec{w}}(\vec{v})$

$$\vec{v}\cdot\vec{w} := v_1w_1 + v_2w_2 + v_3w_3$$
$$= |\vec{v}||\vec{w}|\cos(\theta)$$

Cross product



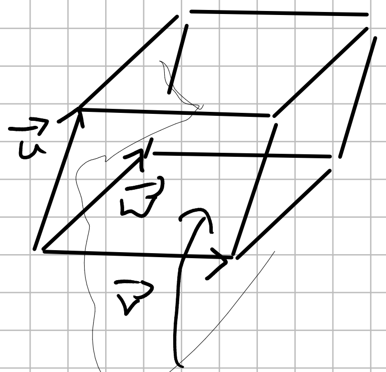


$$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\hat{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\hat{j}$$
$$+ \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\hat{k}$$

$$|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin(\theta)$$

Triple product

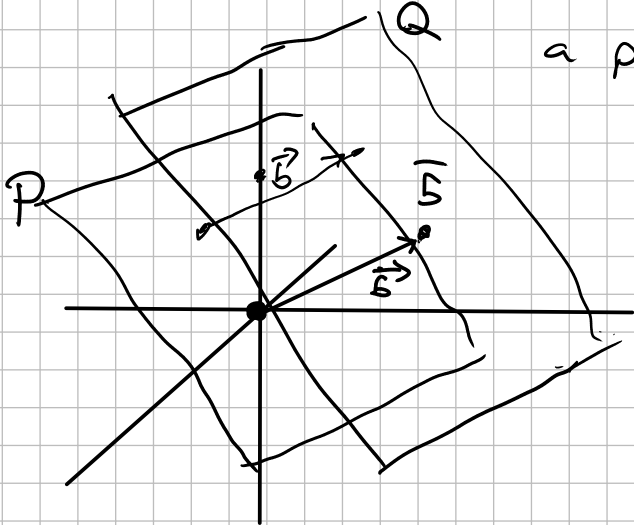


$$\text{Volume} = |\vec{u}\cdot(\vec{v}\times\vec{w})|$$

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

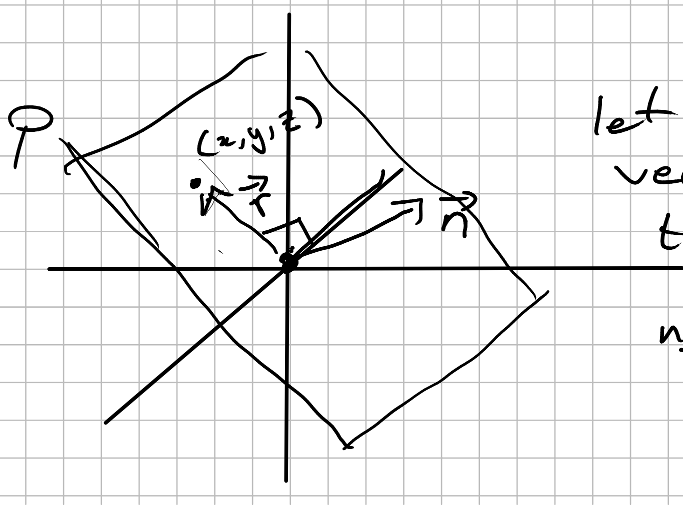
Today: Planes & Lines in $\mathbb{R}^3$

Given a plane Q in $\mathbb{R}^3$, and a point $\bar{b} \in Q$



Can translate Q by $-\vec{b}$ to get a parallel plane P through origin

$$(x,y,z) \in Q \iff (x-b_1, y-b_2, z-b_3) \in P$$



let $\vec{n}$ be a vector orthogonal to P, called <u>normal vector</u>

$\vec{r}$ is orthogonal to $\vec{n}$

$\vec{r} \cdot \vec{n} = 0$ vector eqn of P

Expand

$xn_1 + yn_2 + zn_3 = 0$ scalar eqn of P

$$P = \{ (x,y,z) \in \mathbb{R}^3 \mid xn_1 + yn_2 + zn_3 = 0 \}$$

Suppose $\vec{r} \in Q$, then

$\underbrace{(\vec{r} - \vec{b})} \in P$

$$\boxed{(\vec{r} - \vec{b}) \cdot \vec{n} = 0 \quad \text{vector eqn } Q}$$

$$\boxed{(x - b_1) n_1 + (y - b_2) n_2 + (z - b_3) n_3 = 0}$$

scalar eqn Q

$$(x-b_1)n_1 + (y-b_2)n_2 + (z-b_3)n_3 = 0$$

Expand & collect terms

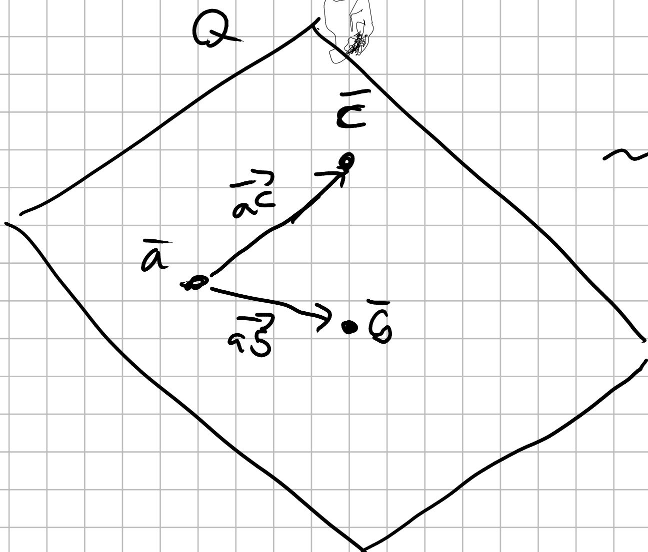
$$\boxed{ax + by + cz + d = 0}$$

linear eqn of Q

$a, b, c, d \in \mathbb{R}$

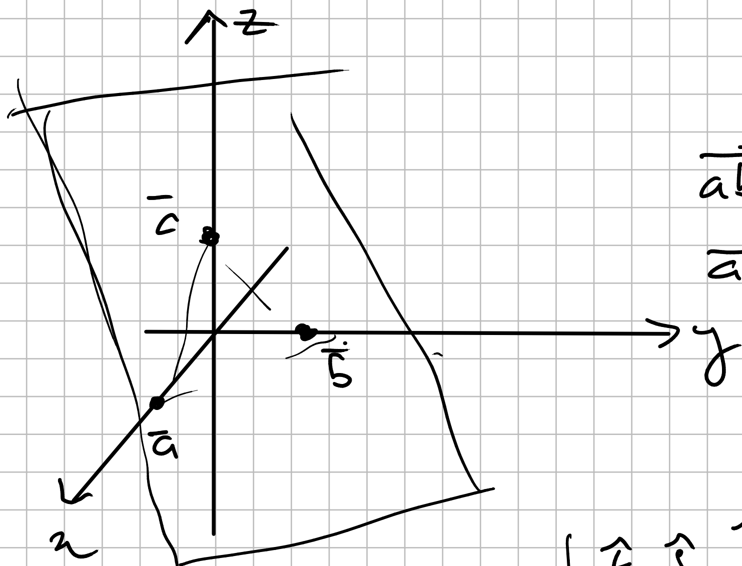
Euclidean geometry ⇝ Plane

- is defined by 3 non-colinear pts



⇝ $\vec{n} := \vec{ab} \times \vec{ac}$ is normal to Q

Eg// Let Q be the plane containing the points $\bar{a} = (1,0,0)$, $\bar{b} = (0,1,0)$, $\bar{c} = (0,0,1)$



$\vec{ab} = \langle -1, 1, 0 \rangle$

$\vec{ac} = \langle -1, 0, 1 \rangle$

$$\vec{n} := \vec{ab} \times \vec{ac} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} \hat{j} + \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} \hat{k}$$

$$= \hat{i} - (-1)\hat{j} + \hat{k} = \langle 1, 1, 1 \rangle$$

$\vec{n} = \langle 1, 1, 1 \rangle$

$\vec{a} = \langle 1, 0, 0 \rangle \in Q$

$\vec{r} = \langle x, y, z \rangle$

Vector eqn//

$$(\vec{r} - \langle 1, 0, 0 \rangle) \cdot \langle 1, 1, 1 \rangle = 0$$

$\Updownarrow$

Scalar eqn

$$(x-1) + y + z = 0$$

$\Updownarrow$

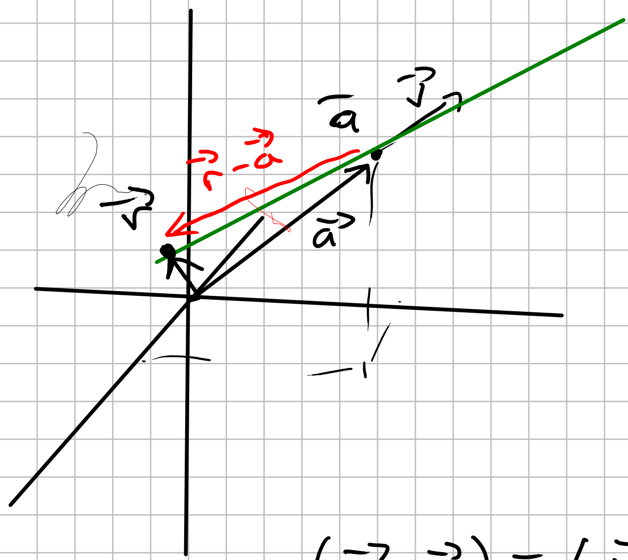
Linear eqn

$$x + y + z - 1 = 0$$

Rmk// "Angle between two planes defined to be angle between their normals" $\rightsquigarrow$ Parallel $\Leftrightarrow$ normals parallel

Planes perp. $\Leftrightarrow$ normals perpendicular

Lines in $\mathbb{R}^3$



Want / describe line in the $\vec{v}$-direction through $\vec{a}$

$$(\vec{r}-\vec{a}) = t\vec{v} \quad \text{for some } t \in \mathbb{R}$$

solve for $\vec{r}$

$$\vec{r}(t) = \vec{a} + t\vec{v}$$

parametric vector eqn of the line

$$\vec{r}(t) = \vec{a} + t\vec{v}$$

} expand to components

$$\left.\begin{aligned} x(t) &= a_1 + t v_1 \\ y(t) &= a_2 + t v_2 \\ z(t) &= a_3 + t v_3 \end{aligned}\right\}$$ (parametric) scalar equs of line

$$\frac{x-a_1}{v_1} = t, \quad \frac{y-a_2}{v_2} = t, \quad \frac{z-a_3}{v_3} = t$$

$$\frac{x-a_1}{v_1} = \frac{y-a_2}{v_2} = \frac{z-a_3}{v_3}$$

$\rightsquigarrow$ symmetric equs of line

If eg $v_3 = 0$, scalar eqns become

$x(t) = a_1 + t v_1$

$y(t) = a_2 + t v_2$

$z(t) = a_3$

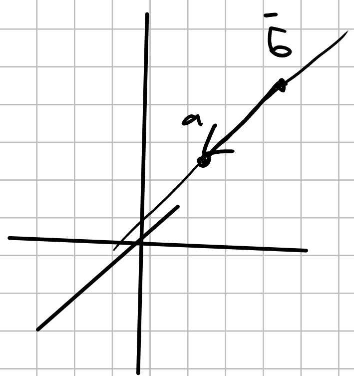
then sym. eqns

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2}, \quad z = a_3$$

Eg// line goes through

$\bar{a} = \langle 3, 2, 1 \rangle$

$\bar{b} = (1, 1, 0)$ $\rightsquigarrow$ $\overrightarrow{ba} = \langle 2, 1, 1 \rangle$



$\bar{b}$ is a point on line

$\rightsquigarrow$ $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \bar{b} + t\overrightarrow{ba}$

$= \langle 1, 1, 0 \rangle + t\langle 2, 1, 1 \rangle$.

$\rightsquigarrow$
$x(t) = 1 + 2t$
$y(t) = 1 + t$
$z(t) = t$

solve for t

$$\frac{x-1}{2} = y - 1 = z$$

Class 5// Reparameterization

From Lecture, $\vec{r} : [a, b] \longrightarrow \mathbb{R}^3$

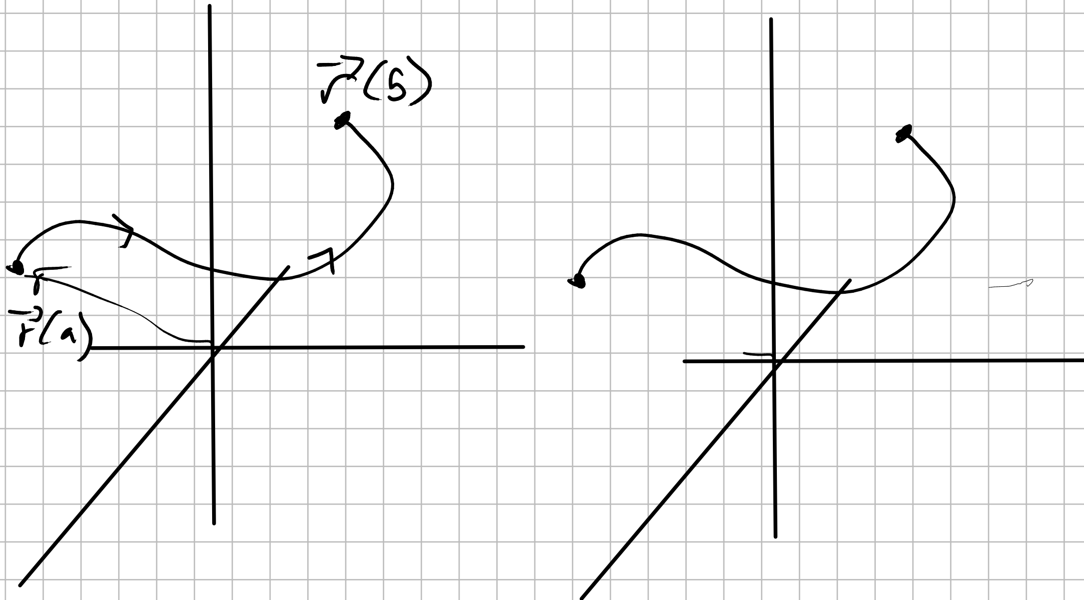
$t \longmapsto \langle r_1(t), r_2(t), r_3(t) \rangle$

Two ways of thinking about these
- Trajectories of particles in 3d
  - $\frac{d\vec{r}}{dt} = \vec{r}\,'(t) \rightsquigarrow$ velocity vector
  - $\frac{d^2\vec{r}}{dt^2} = \vec{r}\,''(t) \rightsquigarrow$ acceleration vector
- As a parameterization of a geometric object: a curve
  - Set of points

"there exists"

$$C = \{ (x, y, z) \in \mathbb{R}^3 \mid \exists\, t \in [a, b] \text{ with } \vec{r}(t) = \langle x, y, z \rangle \}$$

- Can be parametrized in many ways



How do we get another parameterization of the curve param. by $\vec{r}(t)$?

$f:[c,d] \longrightarrow [a,b]$ one-to-one, onto

$\vec{s}(u) = \vec{r}(f(u)) \quad c \le u \le d$

This is called a <u>reparameterization</u> of $\vec{r}(t)$ (or of C)

Eg//

(1) $\vec{r}(t) = \langle t^2, t, t^3 \rangle \qquad 1 \le t \le 5$

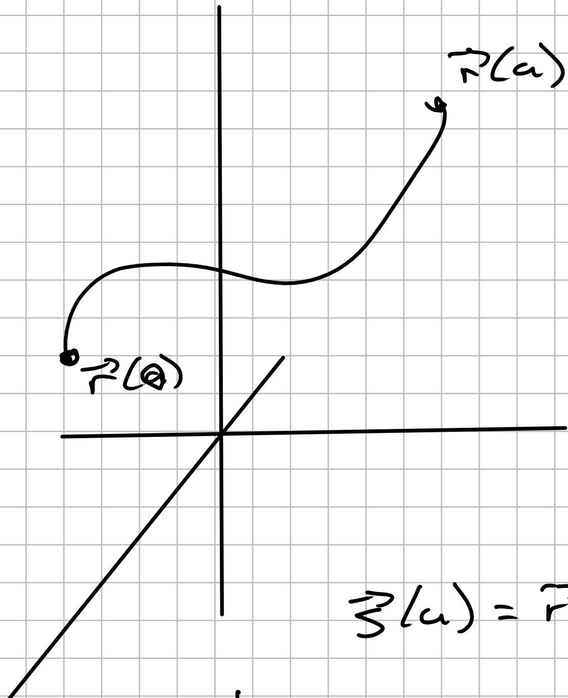
$$f : [0, \ln(5)] \longrightarrow [1, 5]$$
$$u \longmapsto e^u$$

so reparameterizing, we get

$$\vec{s}(u) = \vec{r}(f(u)) = \langle (e^u)^2, e^u, (e^u)^3 \rangle$$
$$= \langle e^{2u}, e^u, e^{3u} \rangle \qquad 0 \le u \le \ln(5)$$

(2)

$$\vec{r} : [0, a] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \cdots$$



$$f : [0, \tfrac{a}{2}] \to [0, a]$$
$$u \longmapsto 2u$$

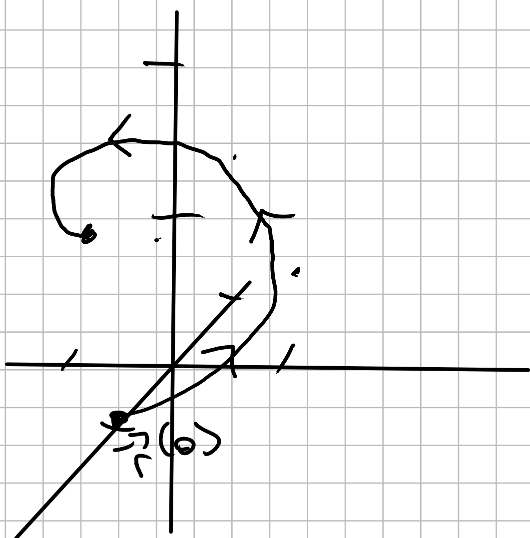
$$\vec{s}(u) = \vec{r}(f(u)) = \vec{r}(2u)$$

$$\frac{d}{du}\vec{s}(u) = \vec{r}\,'(f(u))\,2$$
$$= 2\,\vec{r}\,'(t)$$

(2) If $f'(u) < 0$

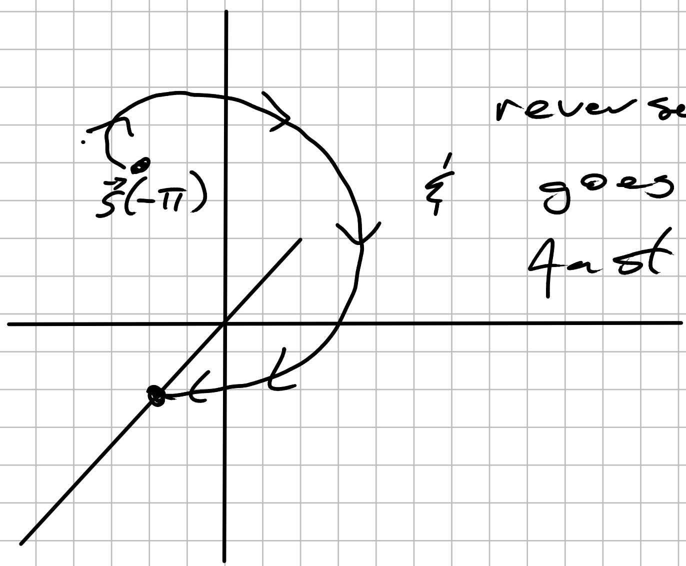
$\Rightarrow \vec{r}(f(u))$ traces through the curve "backwards" ie in the opposite direction to that in which $\vec{r}(t)$ traces it.

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle \qquad 0 \le t \le 2\pi$



$f(u) = -2u \qquad -\pi \le u \le 0$

$\vec{s}(u) = \vec{r}(f(u))$



reverses direction & goes twice as fast.

Class 6

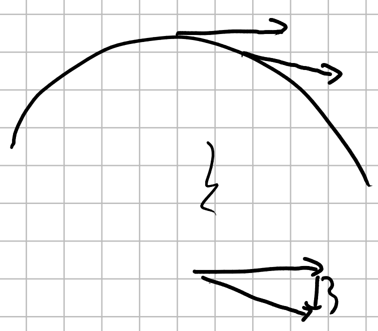
Previously ($\vec{r}(t)$ param of curve C in $\mathbb{R}^3$)

We defined curvature as the rate of change of the unit tangent vector $\vec{T}$ w.r.t. arc length

with respect to

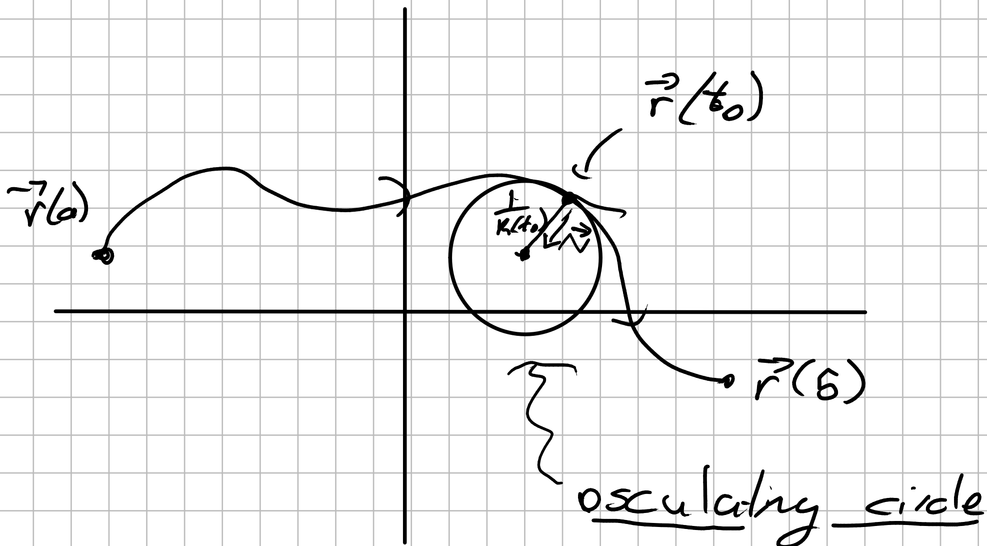
$$K(s) = \left|\frac{d\vec{T}}{ds}\right|$$

Nice geometric interpretation



Visualization

(1) in the x-y plane

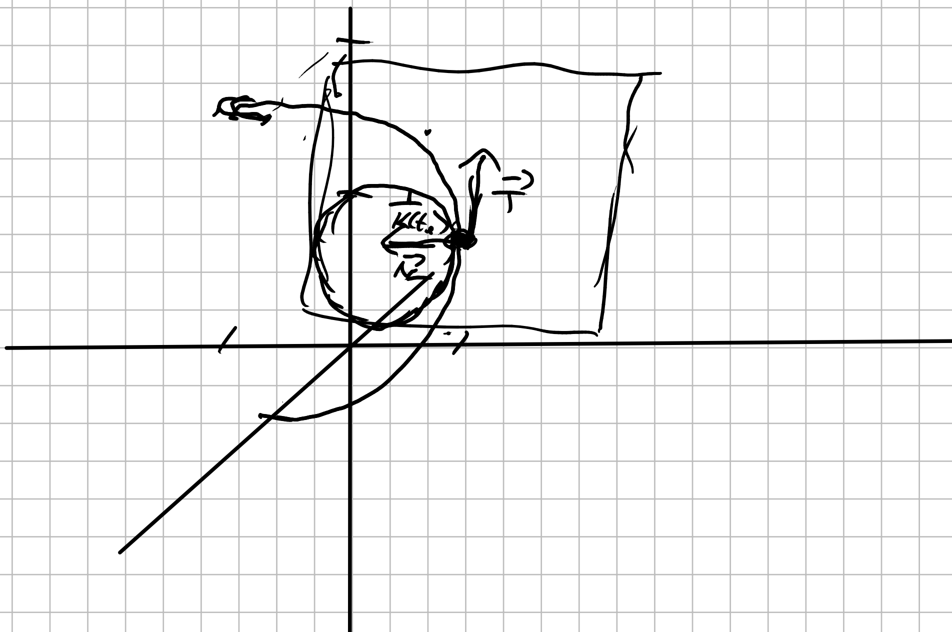


$\frac{1}{\kappa(t_0)}$ is called "radius of curvature"

Center of osculating circle is called "center of curvature"

$$\vec{r}(t_0) + \frac{1}{\kappa(t_0)} \vec{N}(t_0)$$

(2) In $\mathbb{R}^3$ Intuition is basically the same, but the circle lies in osculating plane: The plane generated by $\vec{T}$ & $\vec{N}$ at the point $\vec{r}(t_0)$



Center of curvature

$$\frac{1}{K(t_0)}\vec{N}(t_0) + \vec{r}(t_0)$$

Radius of curvature $\frac{1}{K(t_0)}$

To write eqn for osculating plane: use Binormal $\vec{B} = \vec{T} \times \vec{N}$

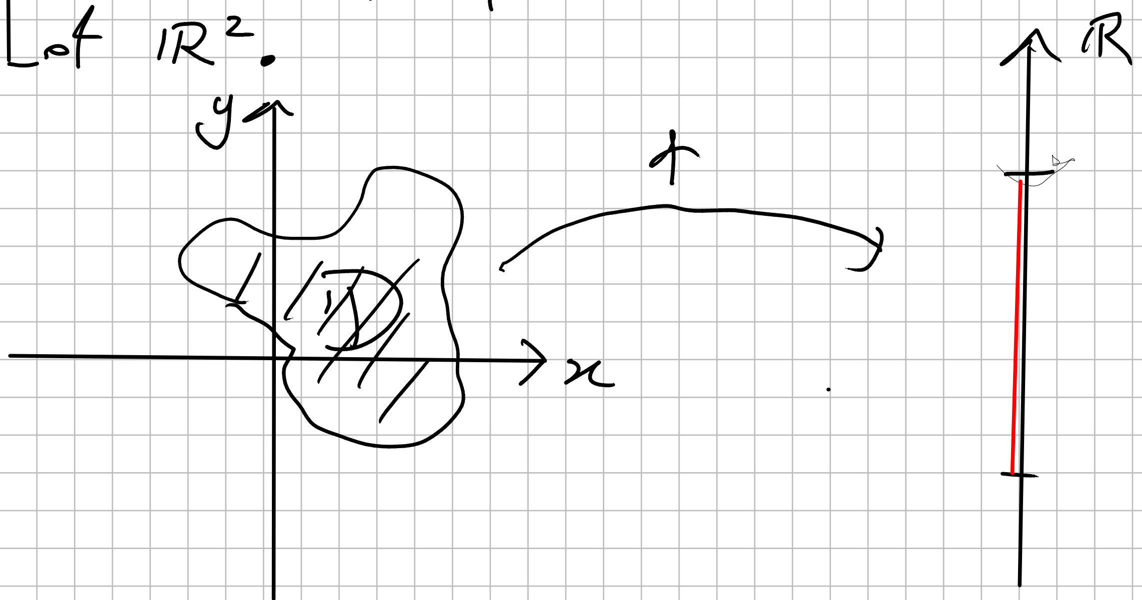
$\rightsquigarrow$ Osculating plane at $\vec{r}(t_0)$ is the plane normal to $\vec{B}(t_0)$ through $\vec{r}(t_0)$.

Class 7: Multivariable functions

Def$^{\underline{n}}$// A <u>function of two variables</u> is a function

$$f: D \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto f(x,y)$$

where D — called the <u>domain</u> of f — is a subset of $\mathbb{R}^2$.



The range (or image) of f is the set of all real numbers "hit" by f

$$\{f(x,y)\in\mathbb{R} \mid (x,y)\in D\}$$

Examples

Cartography

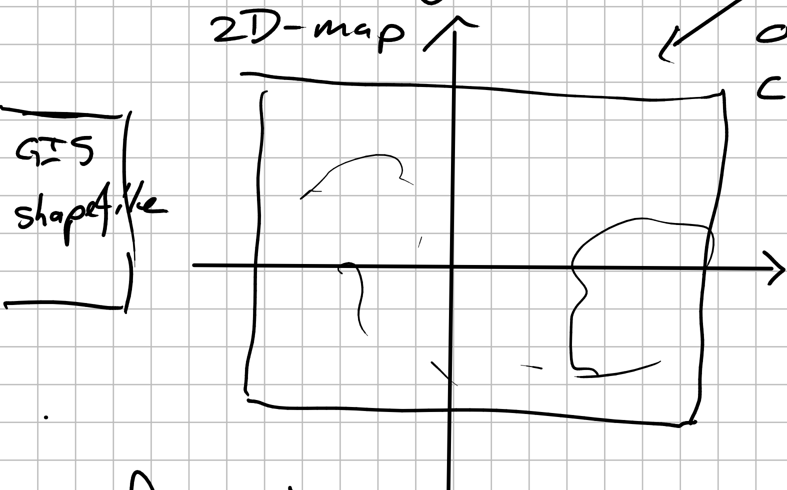
2D-map

D rectangle

$a \le x \le b$

$c \le y \le d$

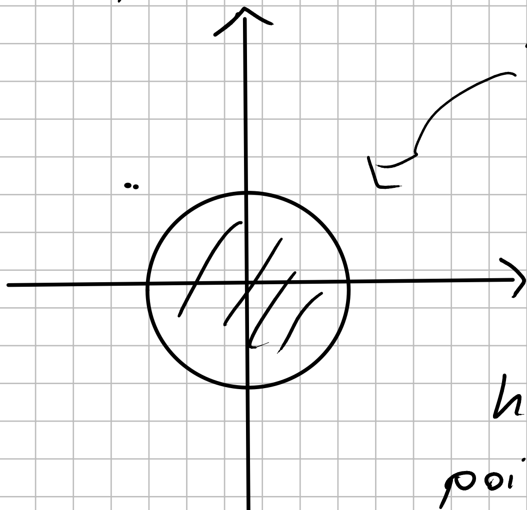
$[a,b]\times[c,d]$



$f(x,y)$ gives the elevation at the point (x,y)

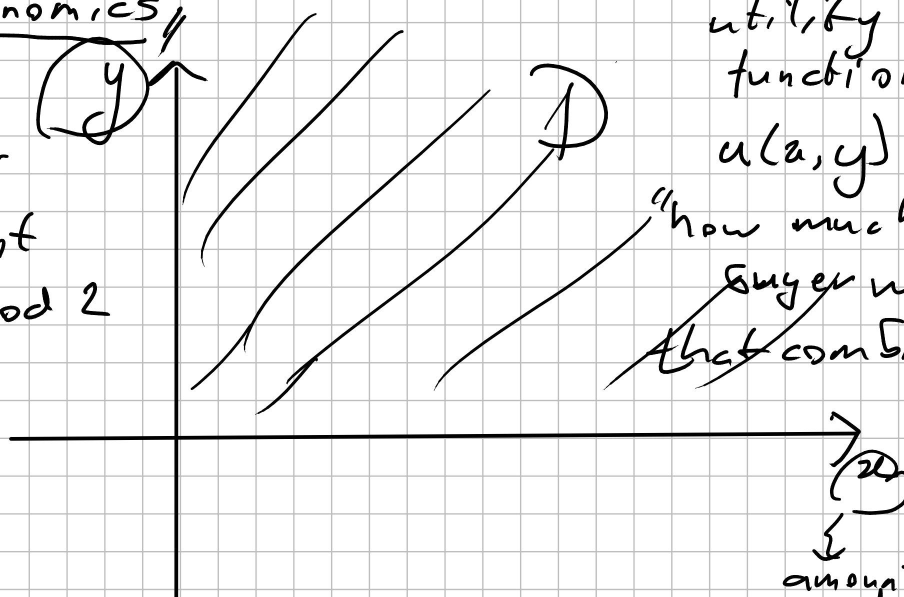
Physics// Heating element

$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$



$f(x,y)$ tells us how hot a given point on the disk is.

Economics//



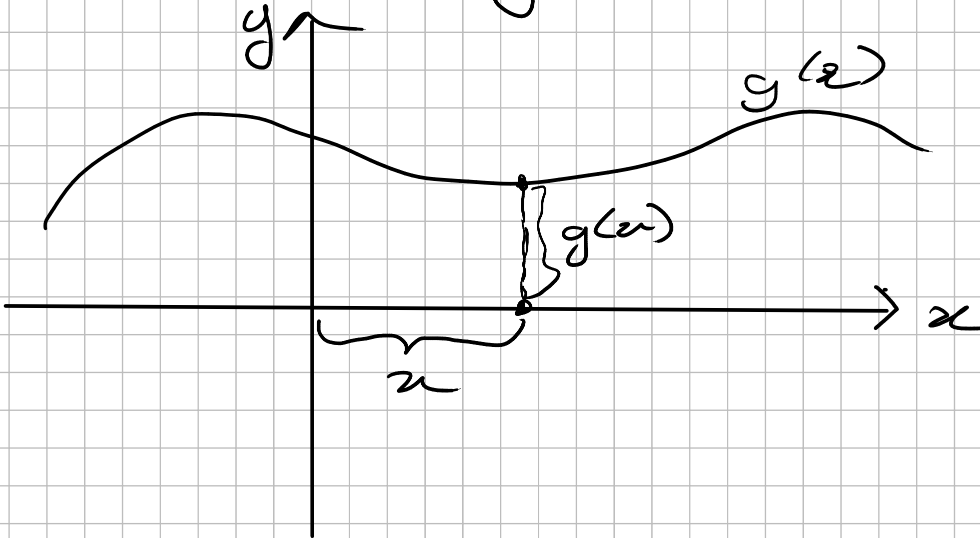
amount of good 2

utility function $u(x,y)$ "how much buyer wants that combination"

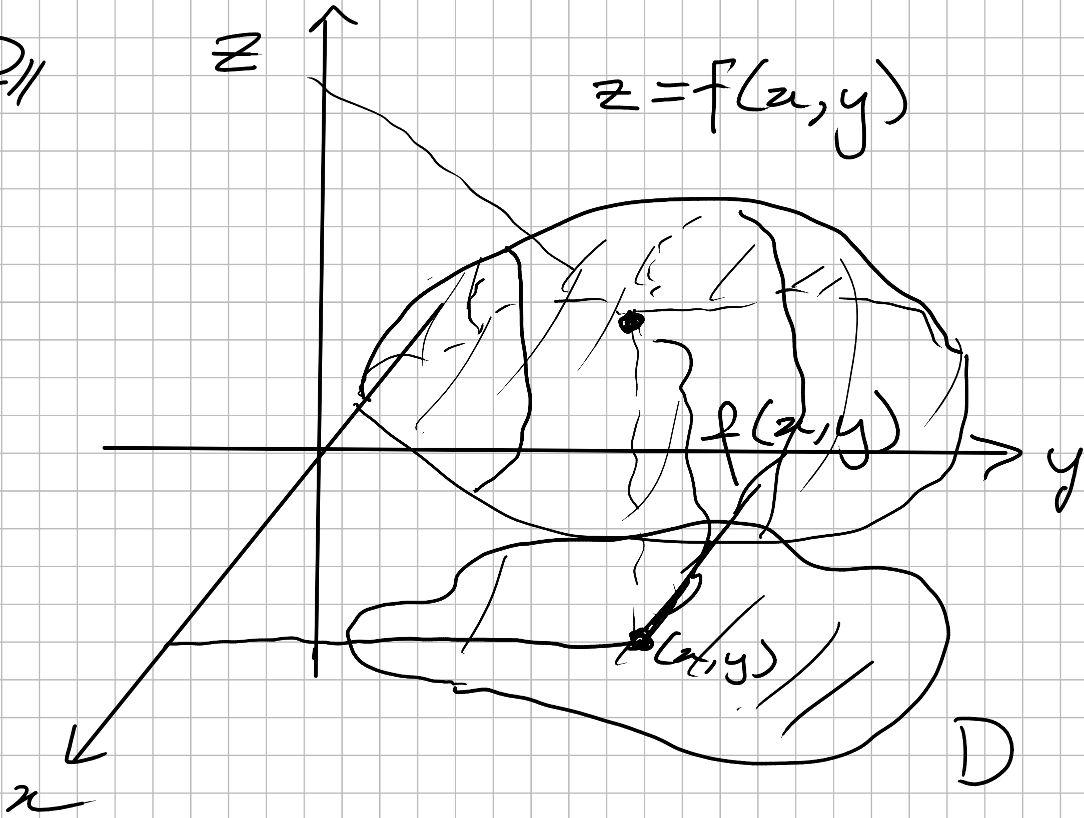
amount of good 1

How do we visualize $f(x,y)$?

think about $g(x)$



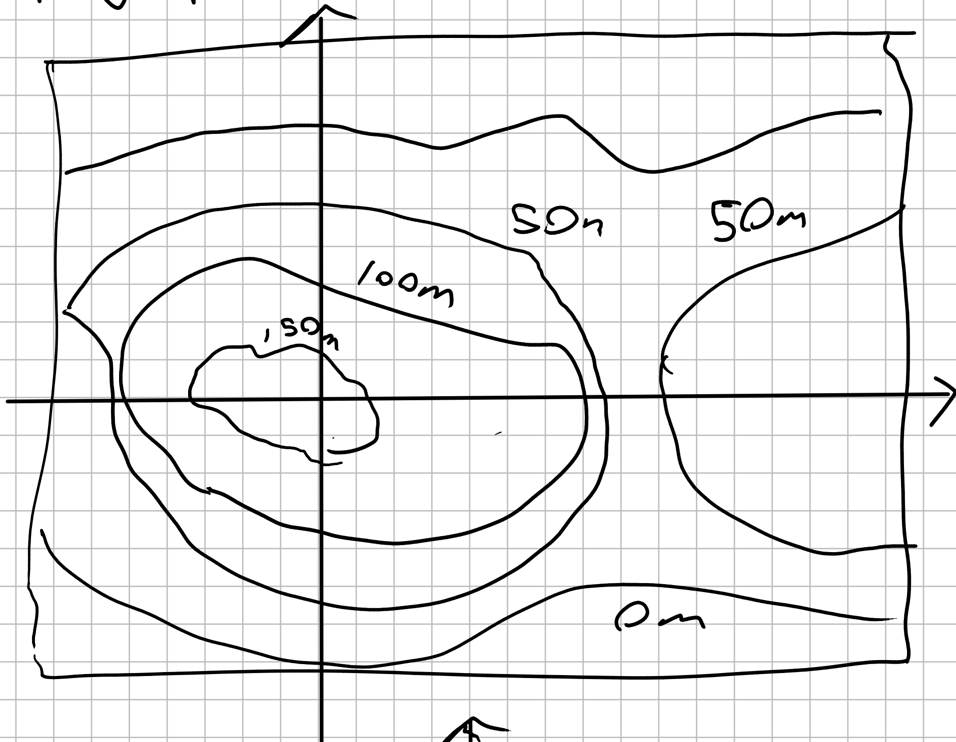
3D

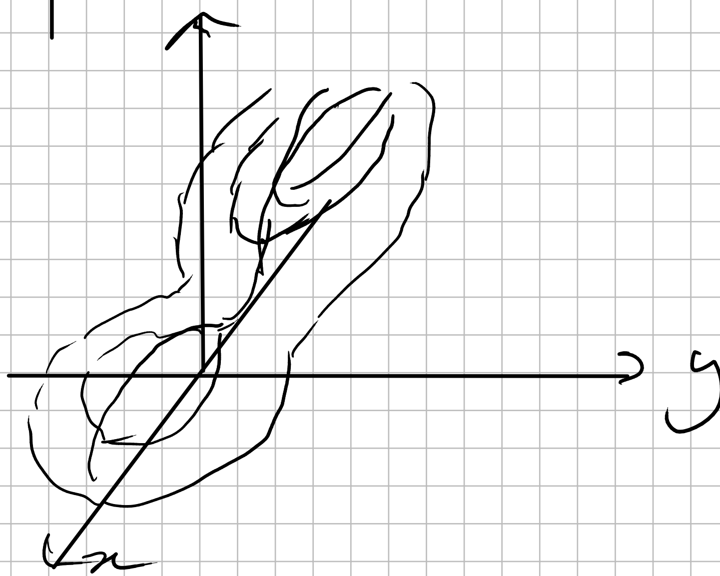


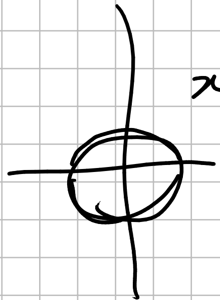
How do we find out what a graph looks like?

$\rightsquigarrow$ Contour map

Topographical Map







$x^2+y^2=1 \rightsquigarrow$ want $y(x)$

$y_1(x) = \sqrt{1-x^2}$

$y_2(x) = -\sqrt{1-x^2}$

Defn// A <u>level curve</u> of a fcn $f(x,y)$ on a domain D is the solution set to

$$f(x,y) = c$$

for a fixed constant $c \in \mathbb{R}$

in range of f

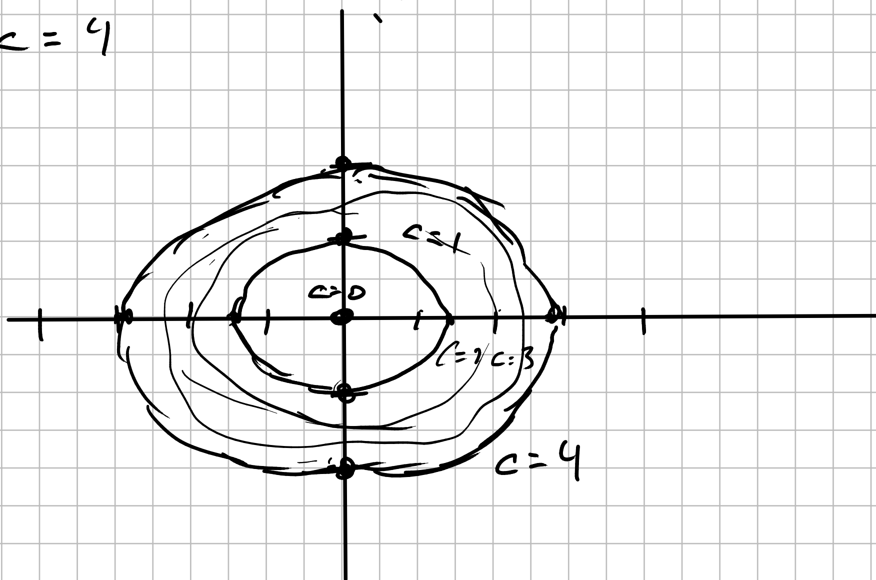
Eg// Level curves of

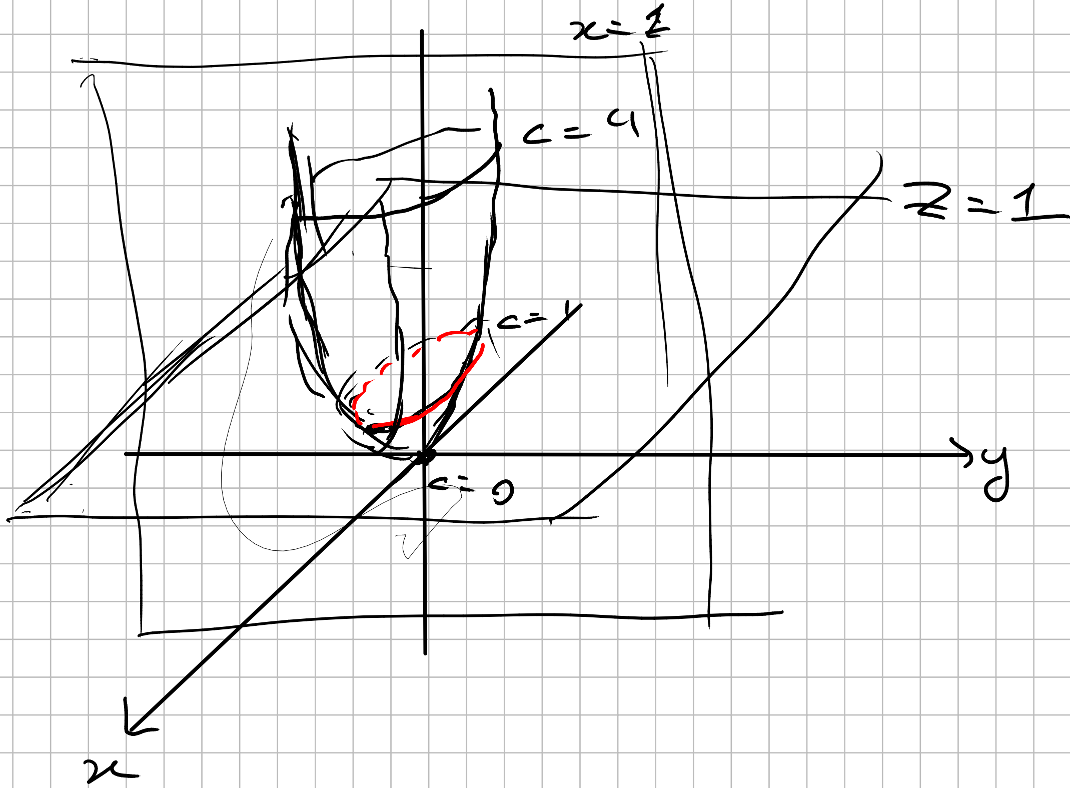
$f(x,y) = \frac{1}{2}x^2 + y^2$ domain $\mathbb{R}^2$

at $c=0 \rightarrow \frac{1}{2}x^2+y^2=0 \Rightarrow x=y=0$

$c=1$ $\frac{1}{2}x^2+y^2=1$

$c=4$





Defn// A <u>trace</u> of $f(x,y)$ at $c \in \mathbb{R}$ (where $\exists (c,d) \in D$) is the solution set to $f(c,y) = z$.

a y-trace at $d \in \mathbb{R}$

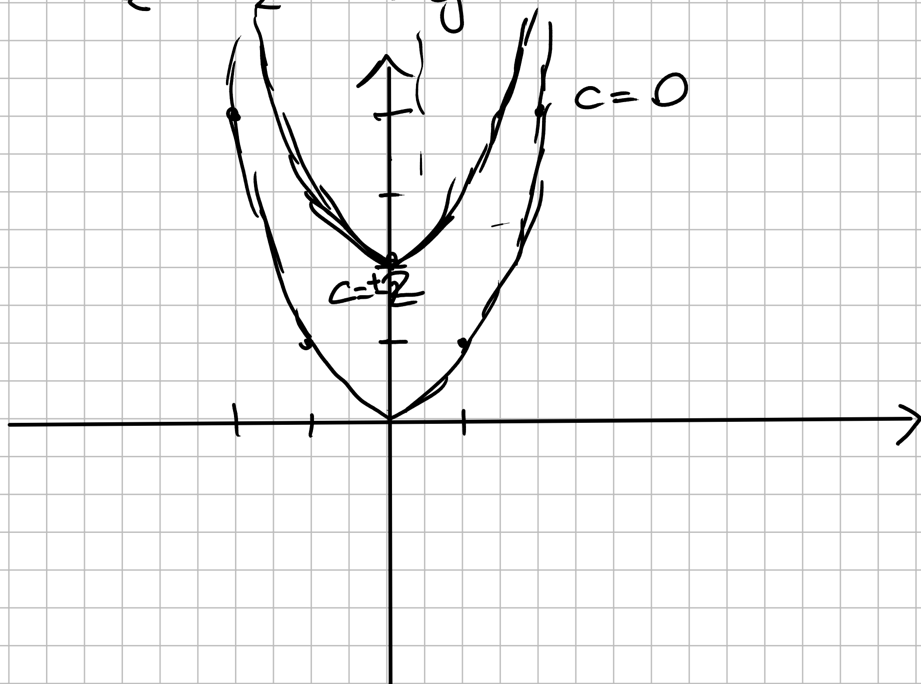
is solution set to

$$f(x, d) = z$$

$$f(x,y) = \frac{1}{2}x^2 + y^2$$

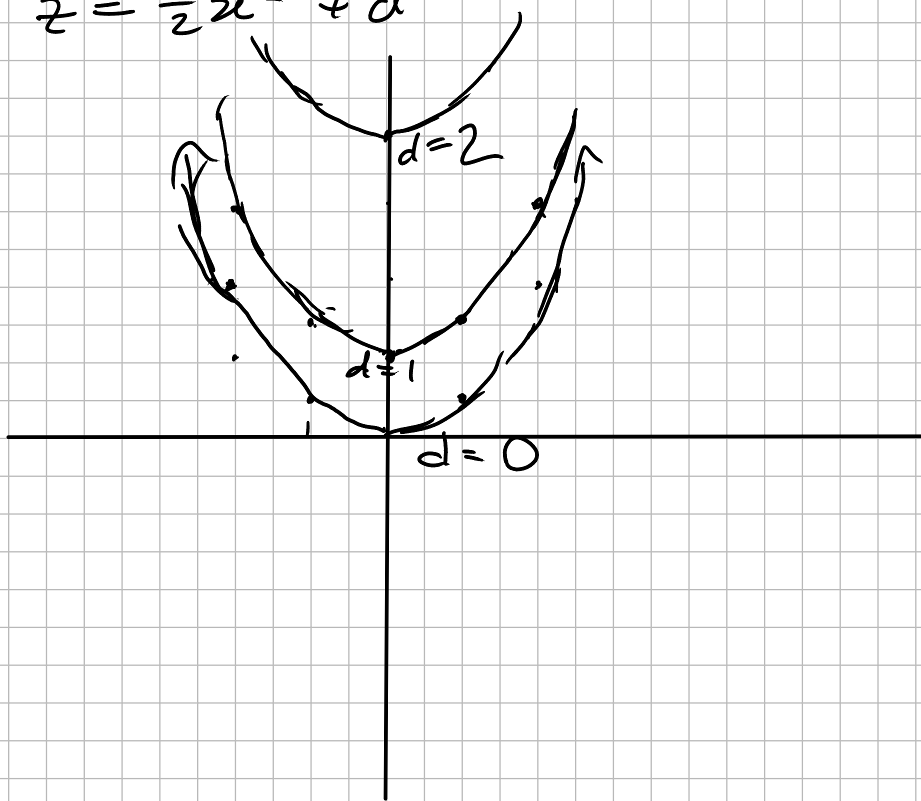
x-traces at c

$$z = \frac{1}{2}c^2 + y^2$$



y-traces $\quad d \in \mathbb{R}$

$$z = \frac{1}{2}x^2 + d^2$$



Can we use same techniques for surfaces that are not the graphs of fns?

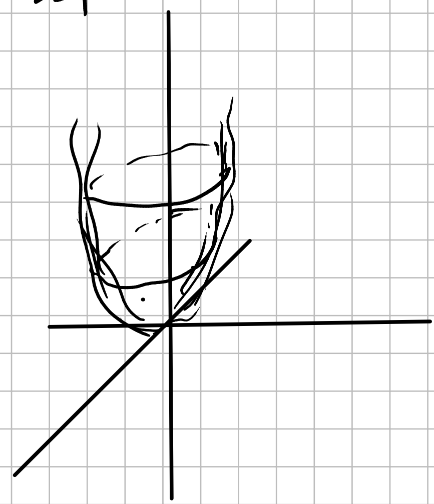
Quadric Surfaces

Defin// A quadric surface is the solution set to a degree 2 eqn in x, y, z

Seen// Elliptic paraboloids

solns to eqn of form

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



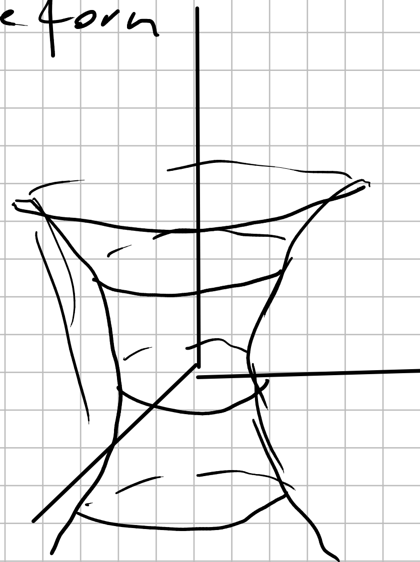
Hyperboloids

1-Sheet

Soln's to eqns of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$1 + \frac{z^2}{c^2}$$

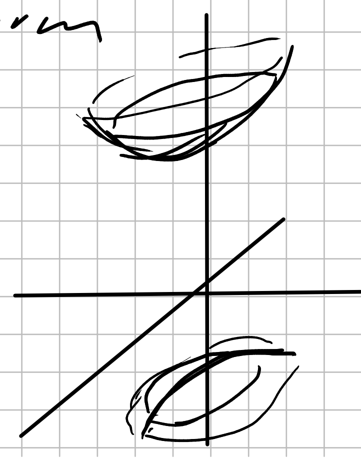


2-Sheets

Solns to eqns of form

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$1 - \frac{z^2}{c^2}$$



$$x^2+y^2-z^2=1$$

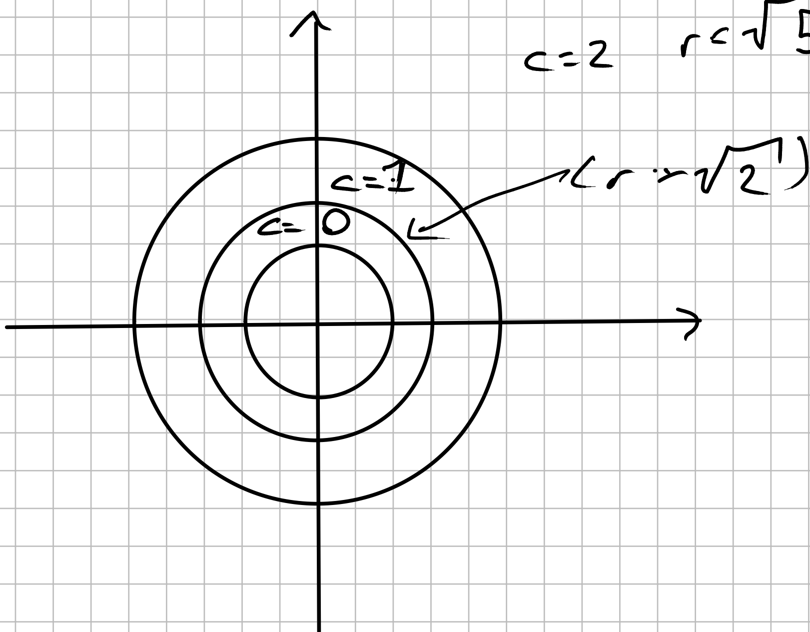
<u>Level curves</u>

$c \in \mathbb{R}$

$$x^2+y^2 = 1+c^2$$

$\rightsquigarrow$ circles of radius $\sqrt{1+c^2}$

$c=2 \quad r=\sqrt{5}$

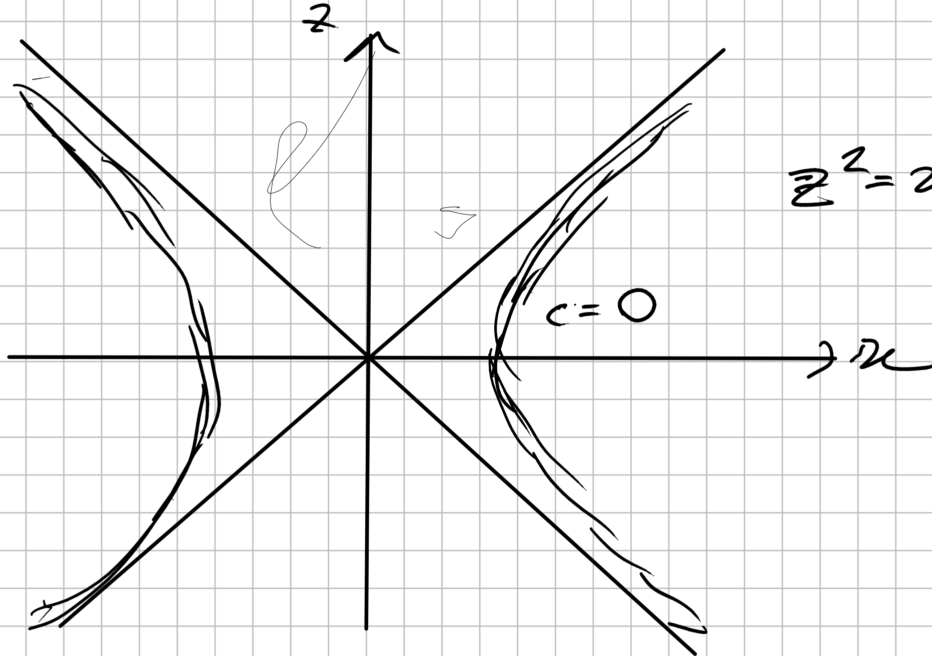


y-traces

$y = c \in \mathbb{R}$

$$x^2 - z^2 = 1 - c^2$$

$$z^2 = x^2 + c^2 - 1$$



$z^2 = x^2 - 1$

§2/

Defn// A <u>function of n variables</u> is a function

$$f: D \longrightarrow \mathbb{R}$$
$$(x_1, x_2, \dots, x_n) \longmapsto f(x_1, x_2, \dots, x_n)$$

where the <u>domain</u> $D \subset \mathbb{R}^n$

<u>Egs</u> • distance

$$d: \mathbb{R}^6 \longrightarrow \mathbb{R}$$
$$(a_1, a_2, a_3, b_1, b_2, b_3) \longmapsto d(\bar{a}, \bar{b})$$

• $f(x, y, z, w) = x^2 + \dfrac{y^2}{z^3 - w}$

Special case: 2-var, or 3-var

Let $f(x,y,z)$ be a fun of three variables on $D \subset \mathbb{R}^3$

A level surface of f at level $k \in \mathbb{R}$ is soln set to

$$f(x,y,z) = k$$

$\rightsquigarrow$ Surface in three dimensions

Eg $f(x,y,z) = \sqrt{(x-a_1)^2 + (y-a_2)^2 + (z-a_3)^2}$

then level surface at level k is a sphere of radius k around $\bar{a} = (a_1, a_2, a_3)$

Class 9

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{2x^2+y^2}} = 0$$

Pf// Want// Given $\varepsilon > 0$, can choose $\delta > 0$ s.t.

$$d((x,y),(0,0)) < \delta \Rightarrow \left| \frac{xy}{\sqrt{2x^2+y^2}} - 0 \right| < \varepsilon$$

$$\left| \frac{xy}{\sqrt{2x^2+y^2}} \right| = \frac{|xy|}{\sqrt{2x^2+y^2}}$$

Suppose $0 < \sqrt{x^2+y^2} < \delta$

Note// $(|x|-|y|)^2 = x^2+y^2-2|xy| \geq 0$

$\Rightarrow \quad x^2+y^2 \geq 2|xy|$

divide $\sqrt{x^2+y^2}$

$$\rightsquigarrow \sqrt{x^2+y^2} \geq \frac{2|xy|}{\sqrt{x^2+y^2}}$$

$$\sqrt{2x^2+y^2} \geq \sqrt{x^2+y^2}$$

$$\Downarrow$$

$$\frac{2|xy|}{\sqrt{x^2+y^2}} \geq \frac{2|xy|}{\sqrt{2x^2+y^2}}$$

$$\delta > \sqrt{x^2+y^2} \geq \frac{2|x,y|}{\sqrt{2x^2+y^2}}$$

$$\Downarrow$$

$$\frac{\delta}{2} > \frac{|xy|}{\sqrt{2x^2+y^2}}$$

Given any $\varepsilon > 0$, set $\delta = 2\varepsilon$

if $0 < \sqrt{x^2+y^2} < \delta$

then

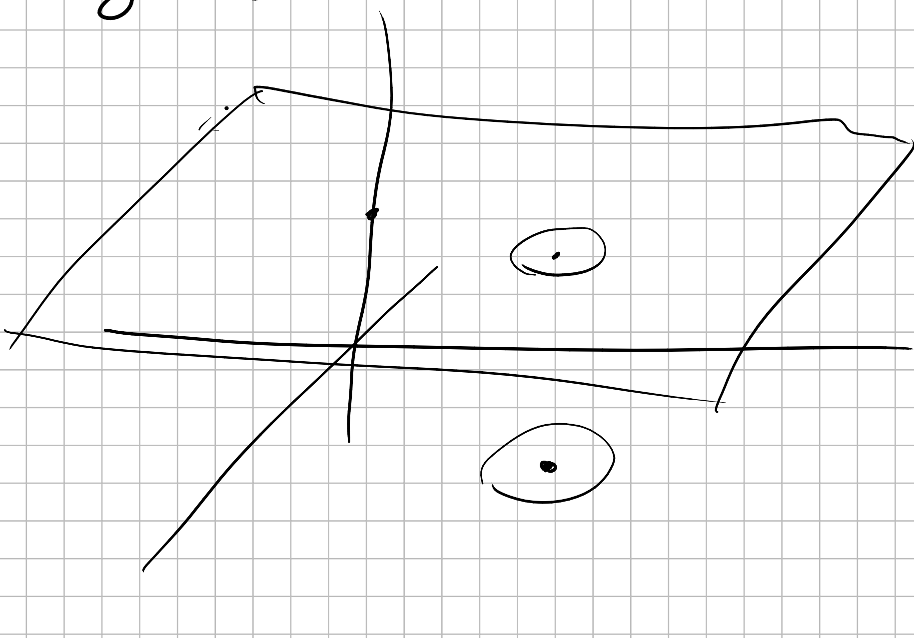
$$\frac{|xy|}{\sqrt{2x^2+y^2}} < \frac{\delta}{2} = \frac{2\varepsilon}{2} = \varepsilon \qquad \square$$

$f(x,y) = 4$

Want $\lim_{(x,y)\to(0,0)} 4 = 4$

Want// For any $\varepsilon > 0$ $\exists$ $\delta > 0$
s.t. if $d((x,y),(0,0)) < \delta$
then $\underbrace{|4-4|}_{=0} < \varepsilon$

given $\varepsilon > 0$, choose $\delta = 1000$



We can define

$$g(x,y) = \begin{cases} \frac{xy}{\sqrt{2x^2+y^2}} & (x,y) = (0,0) \\ 0 & \text{else} \end{cases}$$

We have shown is $g(x,y)$ is continuous at $(0,0)$

Infact $g(x,y)$ is continuous on $\mathbb{R}^2$

Lemma// Suppose

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto f(x,y)$$

is continuous on $\mathbb{R}^2$

and

$g : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$

& range of f is a subset of $[a,b]$

" $g(f(x,y))$ makes sense "

then $g(f(x,y))$ is continuous on $\mathbb{R}^2$

Pf// We show continuity at $(x_0, y_0) \in \mathbb{R}^2$ $\quad t_0 = f(x_0, y_0)$

Want// Given $\varepsilon > 0$ $\exists \delta > 0$ s.t. if $d((x,y),(x_0,y_0)) < \delta$, then

$$|g(f(x,y)) - g(f(x_0,y_0))| < \varepsilon$$

Given $\varepsilon > 0$, since g cont.,
$\exists$ $\delta > 0$ s.t.

$|t - t_0| < \delta$ $|g(t) - g(t_0)| < \varepsilon$

and since f is continuous
$\exists$ a $\gamma > 0$ s.t.

$d((x,y), (x_0,y_0)) < \gamma$
then

$$|f(x,y) - f(\overset{t_0}{\overset{\shortparallel}{x_0,y}})| < \delta$$

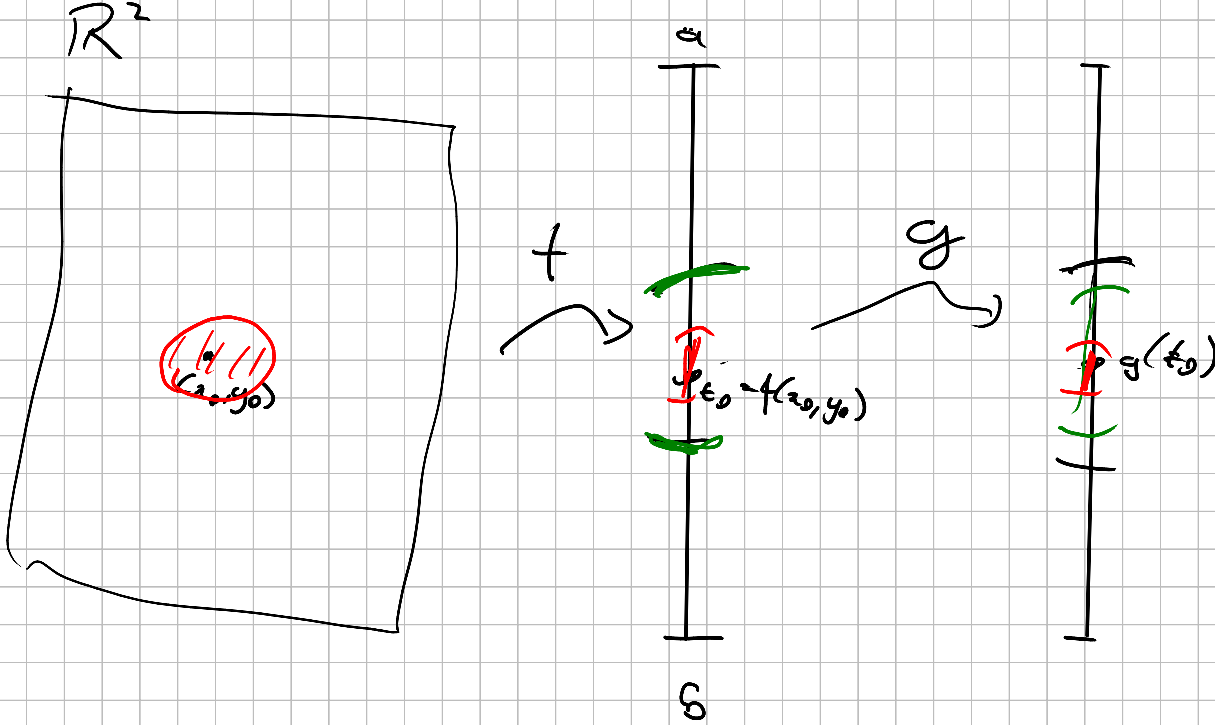
So if

$d((x,y), (x_0,y_0)) < \gamma$, then

$$|f(x,y) - f(x_0,y_0)| < \delta$$

and so

$$|g(f(x,y)) - g(f(x_0,y_0))| < \varepsilon \quad \square$$



Similarly,

$$f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \text{cont.}$$

$$\vec{v} : [a, b] \longrightarrow D \qquad \text{cont.}$$

$$t \longmapsto \langle v_1(t), v_2(t) \rangle$$

then

$$f(\vec{v}(t)) := f(v_1(t), v_2(t)) \quad \text{is cont. on } [a, b]$$

Eg//

$$f(x,y) = \begin{cases} \cos\left(\dfrac{xy}{\sqrt{2x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 1 & \text{else} \end{cases}$$

then $f(x,y)$ is continous on $\mathbb{R}^2$

Why?

$$g(x,y) = \begin{cases} \dfrac{xy}{\sqrt{2x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & \text{else} \end{cases}$$

is cont.

$\cos : \mathbb{R} \longrightarrow \mathbb{R}$ is cont.

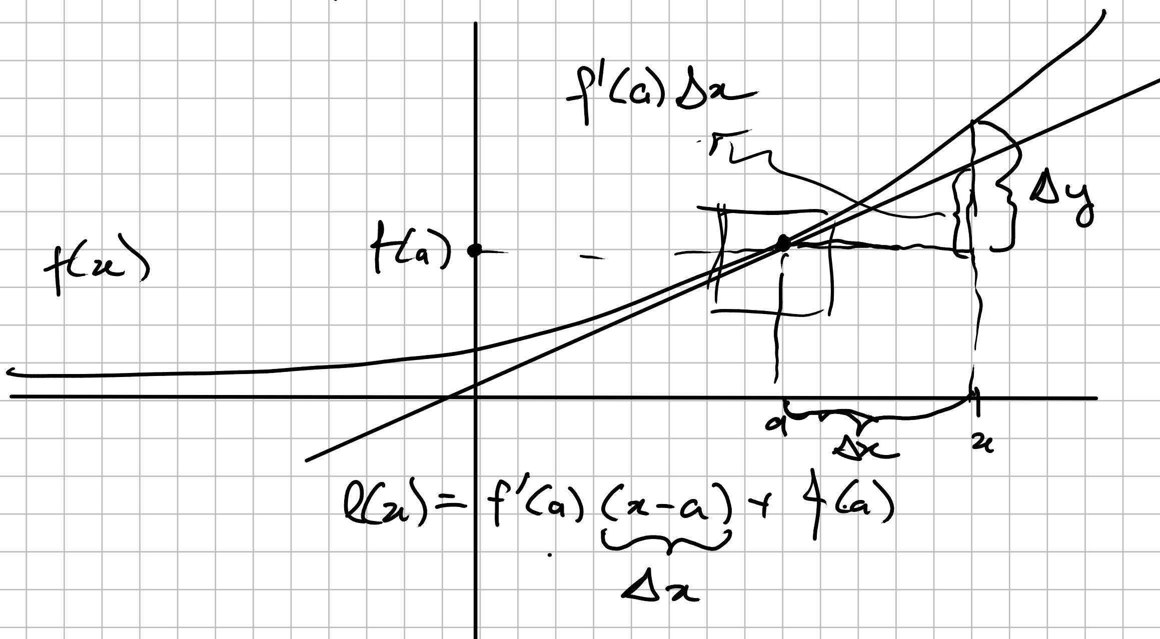
by lemma
$\Rightarrow$ composite

$f(x,y) = \cos(g(x,y))$ is cont.

Class 11

For 1-var funs

$\Delta y = f(a+\Delta x) - f(a)$



$$l(x) = f'(a)\underbrace{(x-a)}_{\Delta x} + f(a)$$

We say $f(x)$ is differentiable if

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} =: f'(a) \quad \text{exists}$$

$$\Updownarrow$$

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - \overbrace{f'(a)\Delta x - f(a)}^{l(a+\Delta x)}}{\Delta x} = 0$$

"Difference between $f(x)$ and $l(x)$ goes to zero quickly as $x \to a$"

We say that $l(x)$ is a <u>linear approximation</u> to $f(x)$ near a.

What do we mean by "linear"?

<u>1st</u> -order polynomial

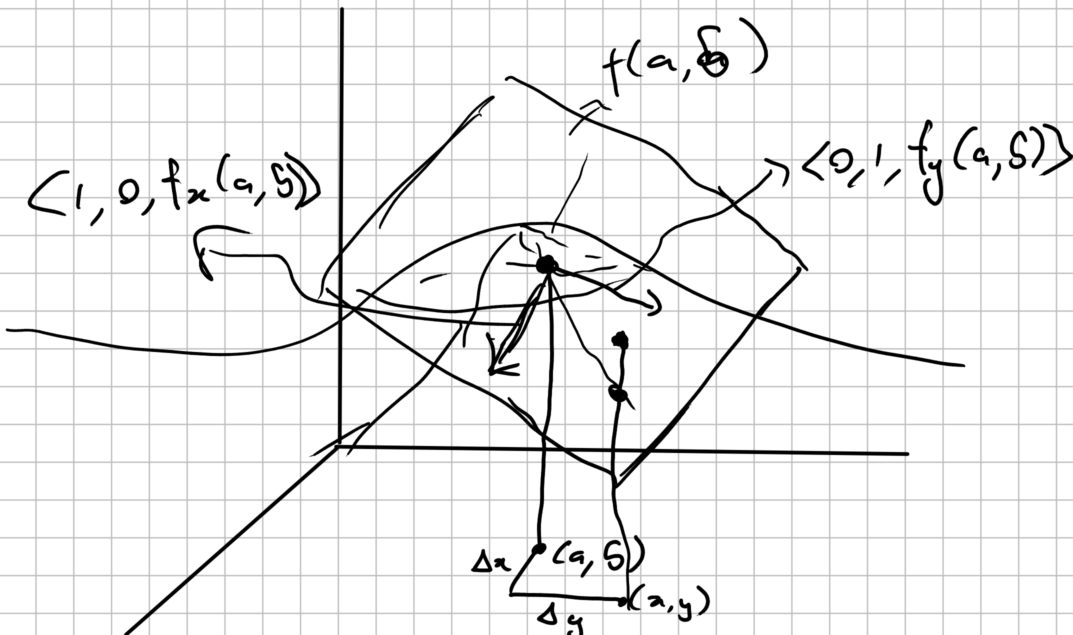
<u>line</u>: $l(x) = ax + b$

$\Big\}$ 2-vars

$\downarrow$

<u>plane</u> // $p(x,y) = ax + by + c$

Want: Linear approximation to a fxn $f(x,y)$ of 2-vars near (a,b).



Candidate for a linear approximation is tangent plane at (a,b)

Assume $\frac{\partial f}{\partial x} = f_x(x,y)$ & $\frac{\partial f}{\partial y} = f_y(x,y)$ exist near (a,b)

Function for tangent plane

$$p(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Want// "Differentiable" means

$f(x,y) - p(x,y) \longrightarrow 0$ quickly

as $(x,y) \to (a,b)$

Write $\Delta x = (x-a)$

$\Delta y = (y-b)$

Defn// A 2-variable fun $f(x,y)$ is differentiable at (a,b) if $f_x(a,b)$ & $f_y(a,b)$ exist and

$$\lim_{(\Delta x,\Delta y)\to 0} \frac{f(a+\Delta x, b+\Delta y) - p(a+\Delta x, b+\Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

Lemma // Let $f(x,y)$ be diff. at (a,b)
Then for any unit vector $\hat{u} \in \mathbb{R}^2$
$(D_{\hat{u}} f)(a,b)$ exists

Pf // Let $\vec{l}(t) = \langle a,b \rangle + t\,\hat{u}$ be the line thru (a,b) in the $\hat{u}$-direction

$$(D_{\hat{u}} f)(a,b) = \lim_{t \to 0} \frac{f(a+u_1 t,\, b+u_2 t) - f(a,b)}{t}$$

$$p(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

---

0 by diff. of $f(x,y)$

$$= \lim_{t \to 0} \left[\frac{f(a+u_1 t,\, b+u_2 t) - p(a+u_1 t,\, b+u_2 t)}{t} \right]$$

$$+ \lim_{t \to 0} \frac{f_x(a,b) u_1 \cancel{t} + f_y(a,b) u_2 \cancel{t}}{\cancel{t}}$$

$$= \lim_{t \to 0} \left[f_x(a,b) u_1 + f_y(a,b) u_2 \right] = f_x(a,b) u_1 + f_y(a,b) u_2$$

$\square$

comp.

$f(a+u_1t, b+u_2t) - p(a+u_1t, b+u_2t)$

$= \underbrace{f(a+u_1t, b+u_2t)} - f_x(a,b)(u_1t) - f_y(a,b)u_2t$

$\quad - \underbrace{f(a,b)}$

$\wr$

$f(a+u_1t, b+u_2t) - f(a,b)$

$= f(a+u_1t, b+u_2t) - p(a+u_1t, b+u_2t)$

$\quad + f_x(a,b)\,u_1t + f_y(a,b)\,u_2t)$

Cor. If $f(x,y)$ is differentiable at (a,b) then for any unit vector $\hat{u} \in \mathbb{R}^2$

$$(D_{\hat{u}}f)(a,b) = f_x(a,b)\,u_1 + f_y(a,b)\,u_2$$

$$= \langle f_x(a,b), f_y(a,b) \rangle \cdot \hat{u}$$

We can compute <u>all</u> dir derivs. from the partial derivatives!

Eg $f(x,y) = x^2 + 3y^2$ is diff. on <u>all</u> of $\mathbb{R}^2$

"Pf" Show diff. at (a,b)

$x = a + \Delta x$, $y = b + \Delta y$

$$f_x(x,y) = 2x \qquad f_y(x,y) = 6y$$

Tangent plane at (a,b)

$$p(x,y) = 2a(\Delta x) + 6b(\Delta y) + a^2 + 3b^2$$

<u>Want</u>

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(a+\Delta x, b+\Delta y) - p(a+\Delta x, b+\Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

Numerator

$$(a+\Delta x)^2 + 3(b+\Delta y)^2 -$$

$$(2a(\Delta x) + 6b(\Delta y) + a^2 + 3b^2)$$

$$= \cancel{a^2} + \cancel{2a\Delta x} + \Delta x^2 + \cancel{3b^2} + \cancel{6b\Delta y} + 3\Delta y^2$$

$$- \cancel{(2a(\Delta x)} + \cancel{6b(\Delta y)} + \cancel{a^2} + \cancel{3b^2)}$$

$$= \Delta x^2 + 3\Delta y^2$$

Want,

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{(\Delta x)^2 + 3\Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

$$0 \leq \frac{(\Delta x)^2 + 3\Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}} \leq \frac{3\Delta x^2 + 3\Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}} = 3\sqrt{\Delta x^2 + \Delta y^2}$$

$$\Rightarrow \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{(\Delta x)^2 + 3\Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}} = 0 \quad \checkmark$$

Thm// Let $f(x,y)$ be a cont. fxn near (a,b) s.t $f_x(x,y)$ & $f_y(x,y)$ exist near (a,b), and are continuous at (a,b). Then $f(x,y)$ is differentiable at (a,b)

If $f(x,y)$ is diff. at (a,b) the tangent plane $p(x,y)$ is a good approximation for $f(x,y)$ near (a,b).

<u>Historically</u>// this could be used to approximate fxn values

<u>Eg</u>// $f(x,y) = \sqrt{x^2+y^2}$

We can approximate $f(3.1, 3.9)$

close to $(3, 4)$

Compute $p(x,y)$

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \; ; \; f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_x(3,4) = \frac{3}{5} = 0.6 \; ; \; f_y(3,4) = \frac{4}{5} = 0.8$$

$$f(3,4) = 5$$

$$p(x,y) = 5 + 0.6(x-3) + 0.8(y-4)$$

Using $p(x,y)$ as approx. for $f(x,y)$

$$p(3.1, 3.9) = 5 + 0.06 - 0.08$$
$$= 4.98$$

From a calculator

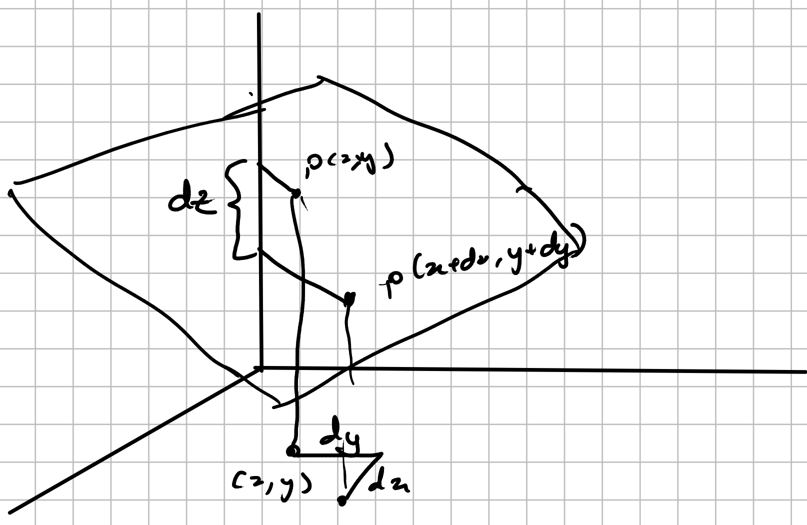
$$f(3.1, 3.9) = 4.98197\ldots$$

Differentials Define dx & dy to be new indep variables, and let $f(x,y)$ be a diff. fxn.

The **differential** of f $(z=f(x,y))$ is

$$dz = df = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

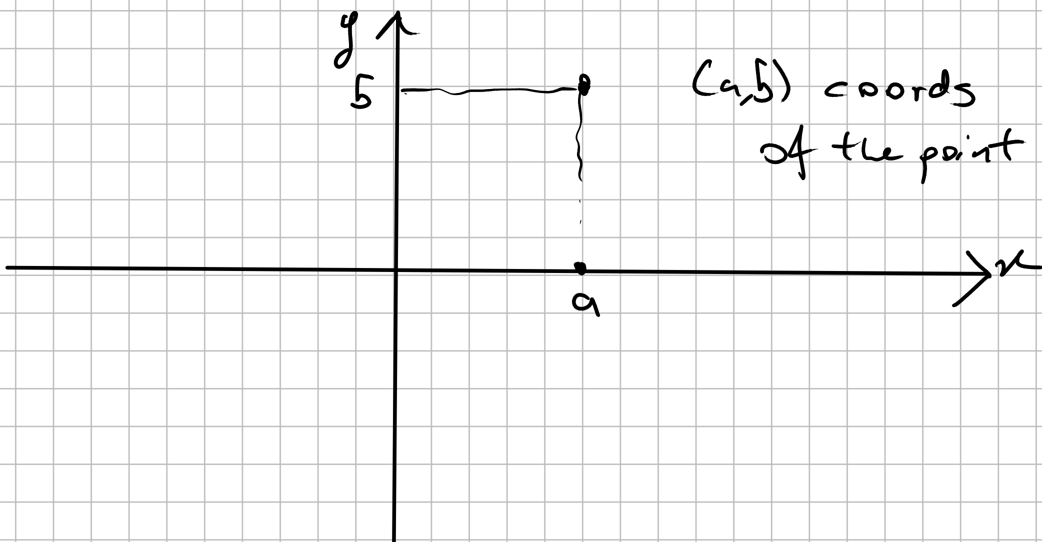
This is precisely how much the height of the tangent plane at (x,y) changes from (x,y) to $(x+dx, y+dy)$.



Lecture (n+1): Coordinates & Areas

Polar Coordinates

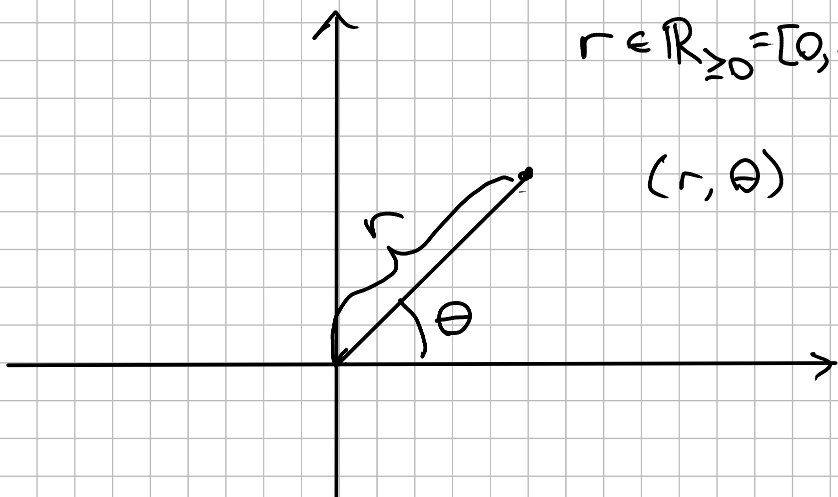
Usually describe pts in the plane using Cartesian coordinates



(a,b) coords of the point

Other method //

Polar coords



$r \in \mathbb{R}_{\geq 0} = [0, \infty)$

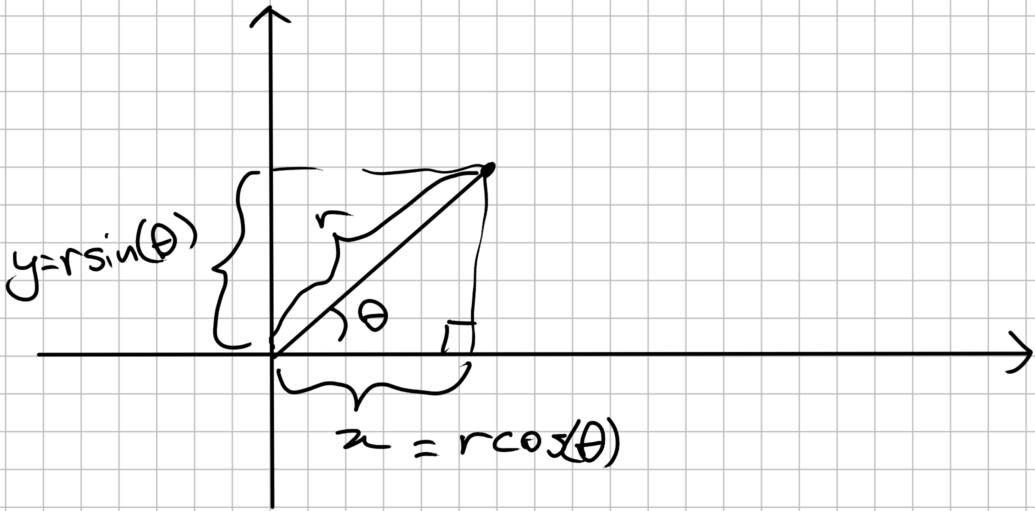
(r, θ)

We can take $\theta \in \mathbb{R}$, but sometimes restrict to $[0, 2\pi]$

Warning // Multiple coordinates can specify the same pt.

- $(0, \theta)$ specifies the origin for any θ
- $(r, \theta + 2\pi n)$ specifies the same pt. as (r, θ)

How to get from cart. to polar & vice-versa



$$\boxed{\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}}$$

<u>Pythagorean Thm</u>

$$\boxed{r^2 = x^2 + y^2}$$

<u>Trig</u>

$$\boxed{\tan(\theta) = \frac{y}{x}}$$

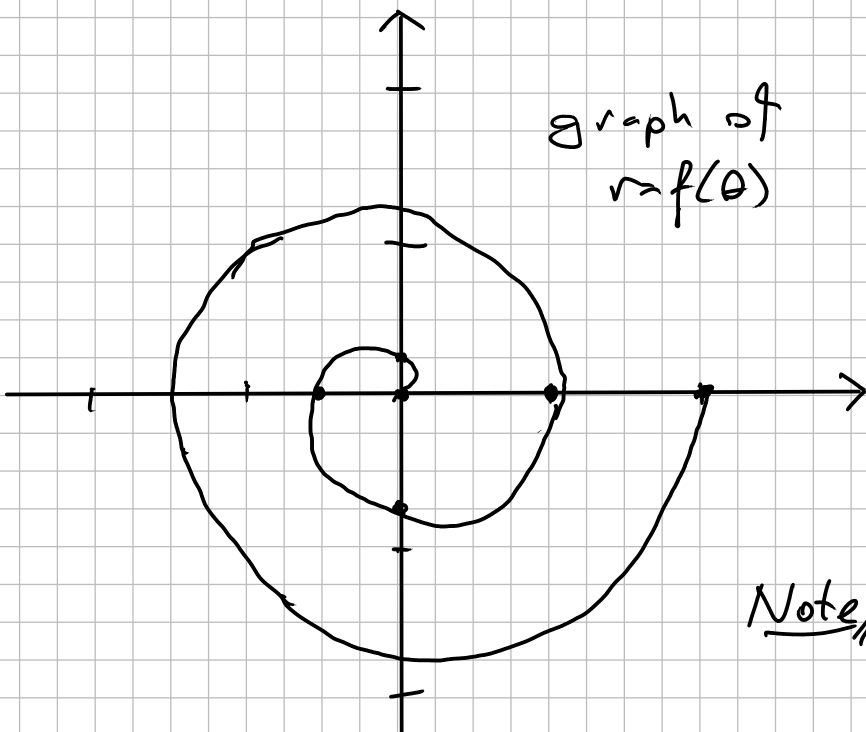
Curves in polar coordinates

Write $r = f(\theta)$ fxn

graph is set of points

w/ polar coords $(f(\theta), \theta)$

Eg// $r = \frac{\theta}{2\pi}$ $0 \le \theta \le 4\pi$



graph of

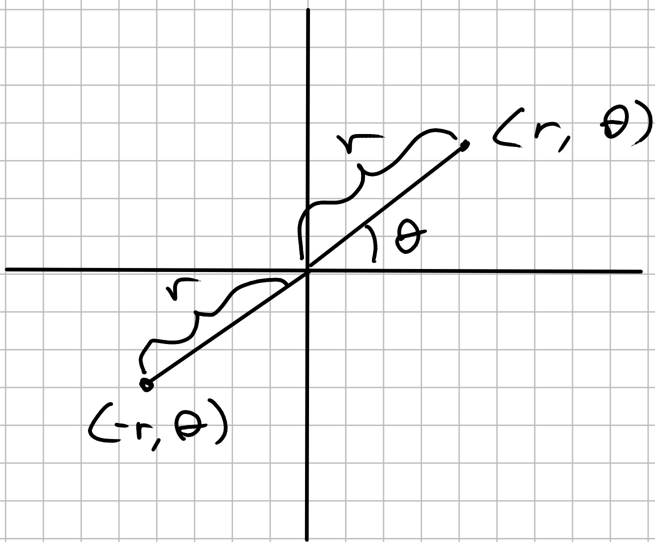
$r = f(\theta)$

Note// if

$r = f(\theta)$ is

negative

$\rightsquigarrow$ opposite direction



Calculus w/ polar coords?

Polar curve $r = f(\theta)$

$\Updownarrow$

Parametric curve

$x = \alpha_1(\theta) = f(\theta)\cos(\theta)$

$y = \alpha_2(\theta) = f(\theta)\sin(\theta)$

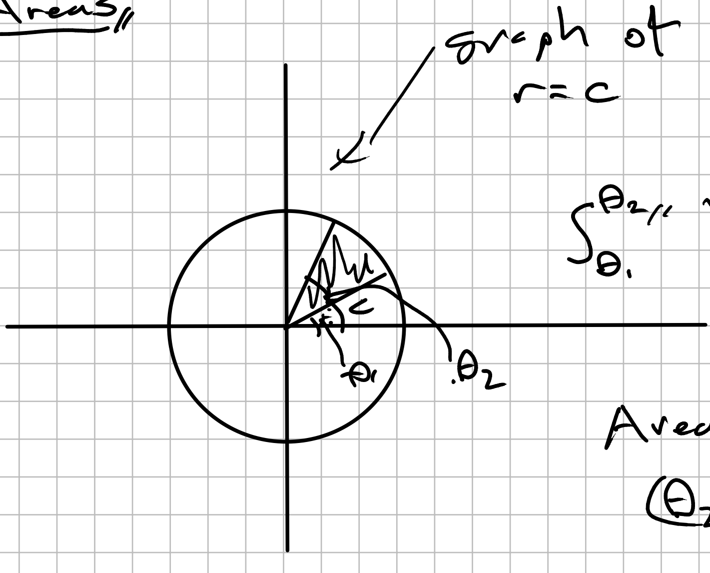
slope of the tangent is

$$\boxed{\frac{dy}{dx} = \frac{\frac{d\alpha_2}{d\theta}}{\frac{d\alpha_1}{d\theta}}}$$

dir.^{n} of tangent vector

$\vec{\alpha}'(\theta)$

<u>Areas</u>

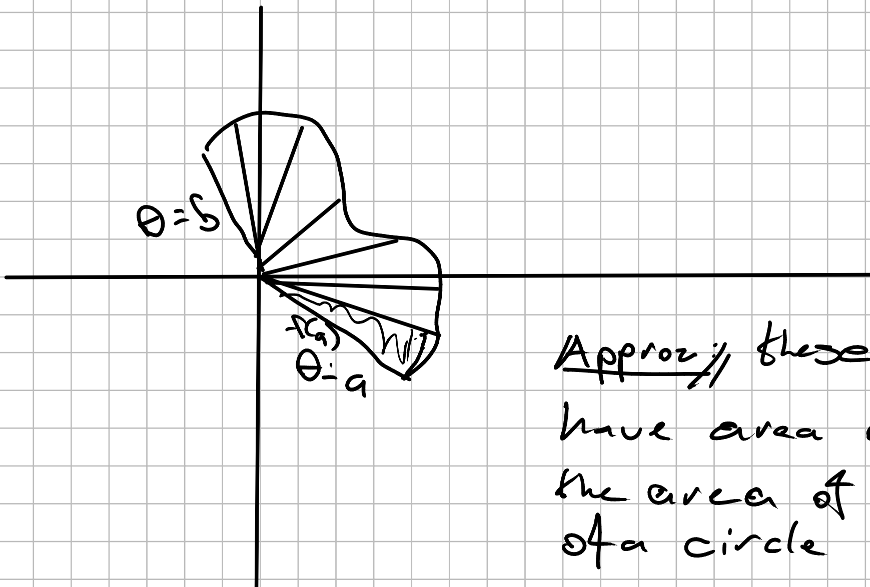


$\int_{\theta_1}^{\theta_2} \;\; d\theta$ is Area

Area is

$(\theta_2 - \theta_1)\frac{r^2}{2}$

Given $r=f(\theta)$ $\quad a \le \theta \le b$



Approx these slices have area approximately the area of a sector of a circle

$\rightsquigarrow$
$$\boxed{A=\int_a^b \frac{1}{2}(f(\theta))^2\, d\theta}$$

Length

for $\vec{\alpha}(t)$ $\quad a \leq t \leq b$

$$s = \int_a^b \sqrt{\left(\frac{d\alpha_1}{dt}\right)^2 + \left(\frac{d\alpha_2}{dt}\right)^2}\, dt$$

54 $\alpha_1(\theta) = f(\theta)\cos(\theta)$

$\alpha_2(\theta) = f(\theta)\sin(\theta)$

$$\sqrt{\left(\frac{d\alpha_1}{d\theta}\right)^2 + \left(\frac{d\alpha_2}{d\theta}\right)^2} = \sqrt{(f(\theta))^2 + \left(\frac{df}{d\theta}\right)^2}$$

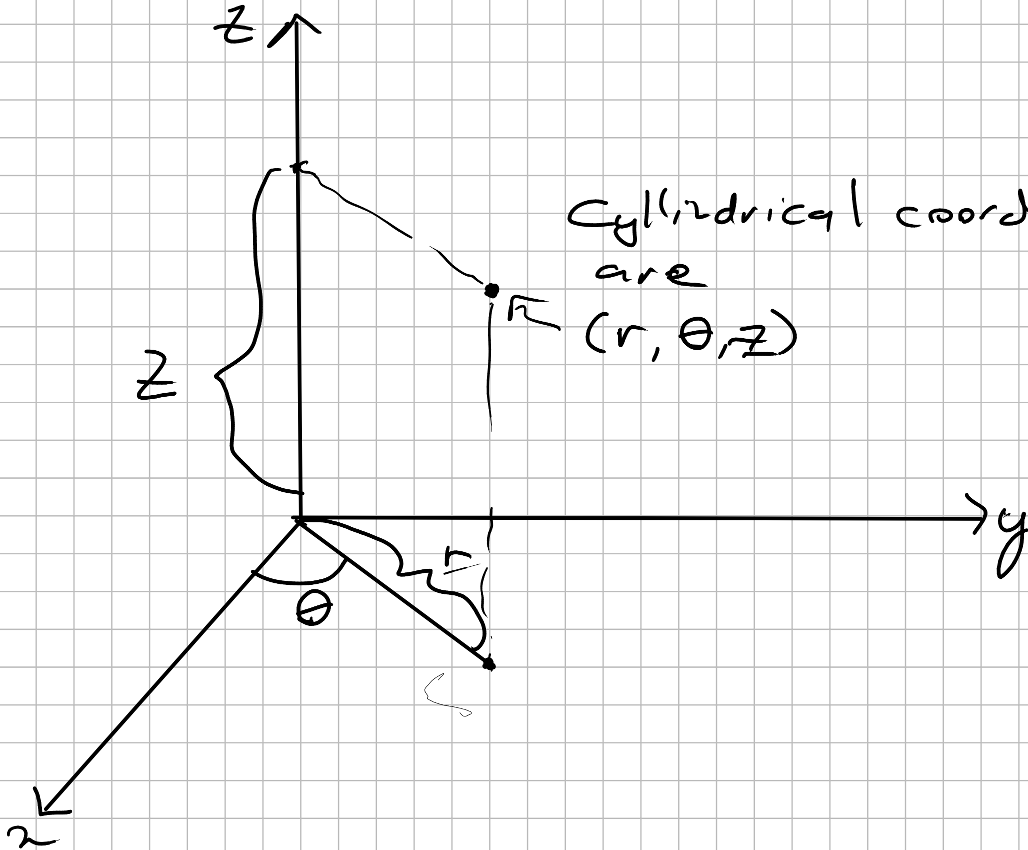
Arc length of polar curve $f(\theta)$ $\quad a \leq \theta \leq b$

$$\boxed{s = \int_a^b \sqrt{f(\theta)^2 + \left(\frac{df}{d\theta}\right)^2}\, d\theta}$$

Cyllindrical coords (On $\mathbb{R}^3$)

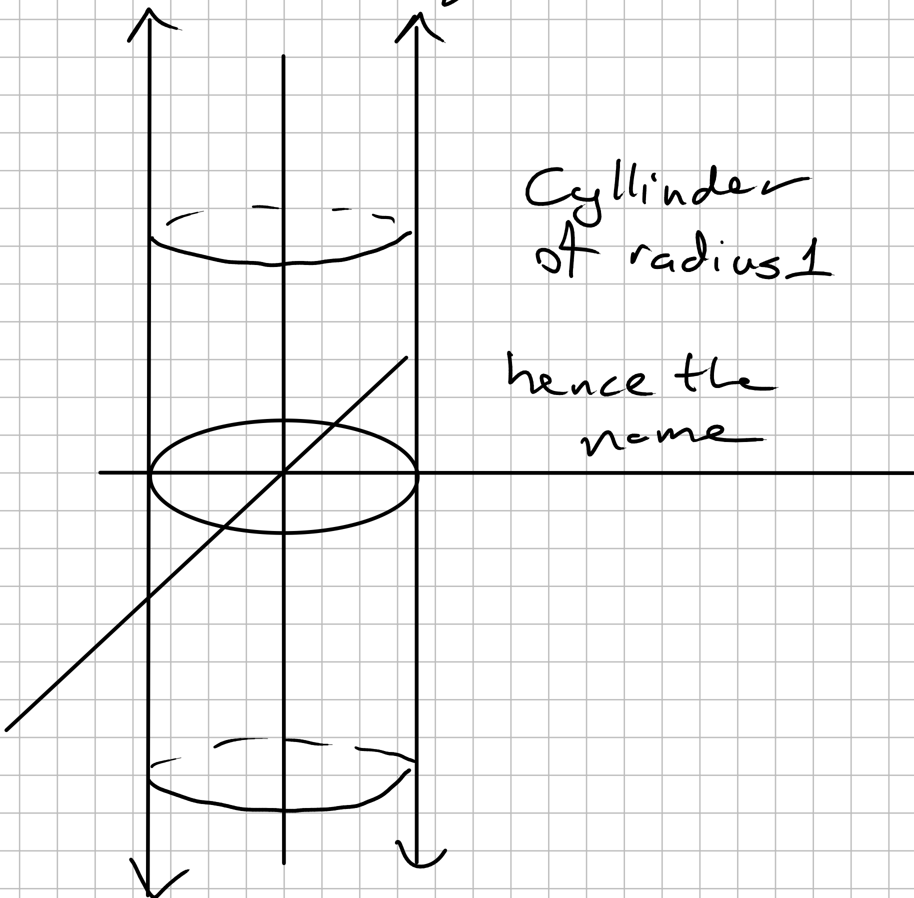
Idea: $\mathbb{R}^3$ consists of an x-y plane & a z-axis

Use polar coords for x-y plane & z-coord. for z-axis



Cyllindrical coords are (r, θ, z)

Eg What is the solution set in $\mathbb{R}^3$ to the eqn. $r=1$?



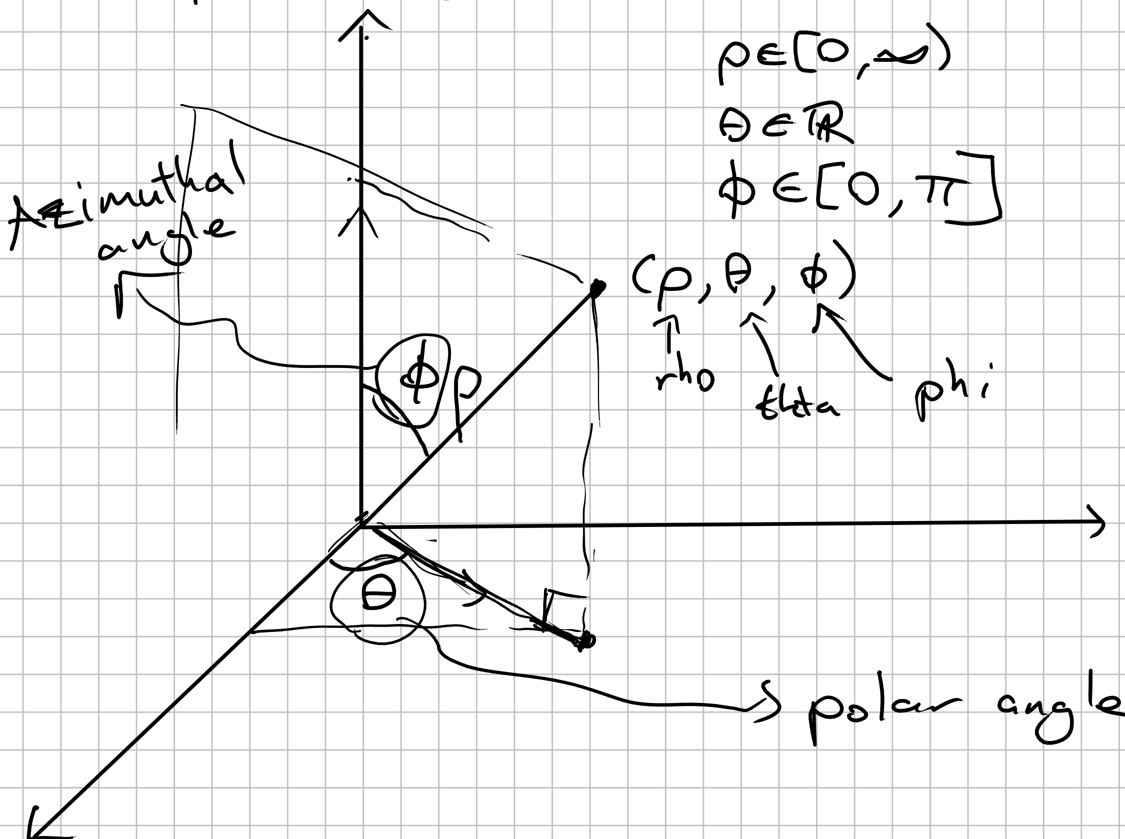
Cyllinder of radius 1

hence the name

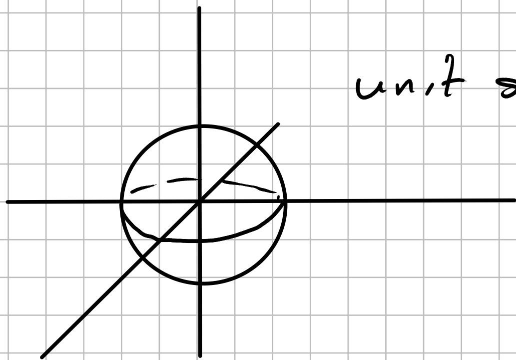
Spherical Coordinates

Idea: Like with polar coords specify a direction by angles & a radius

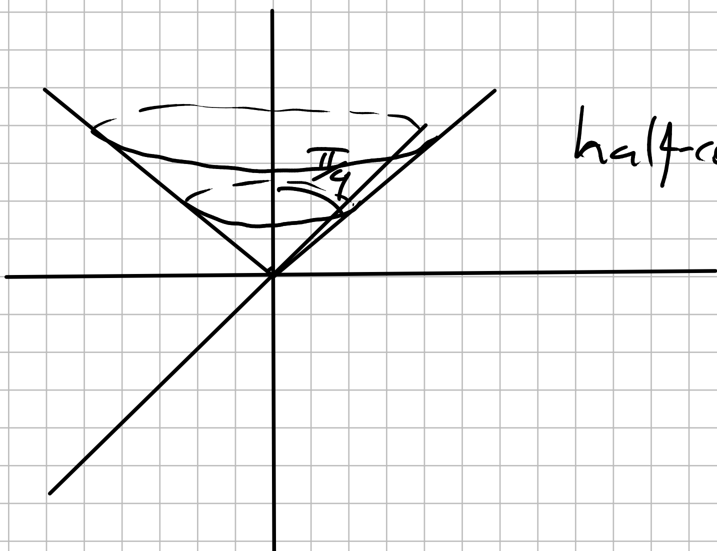
$\rho \in [0, \infty)$
$\theta \in \mathbb{R}$
$\phi \in [0, \pi]$

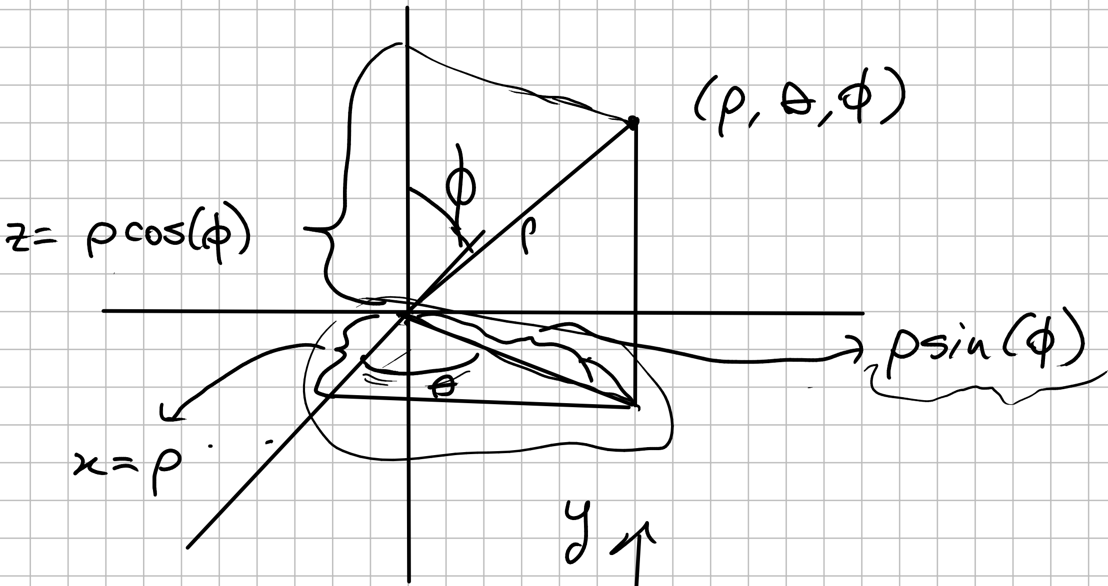


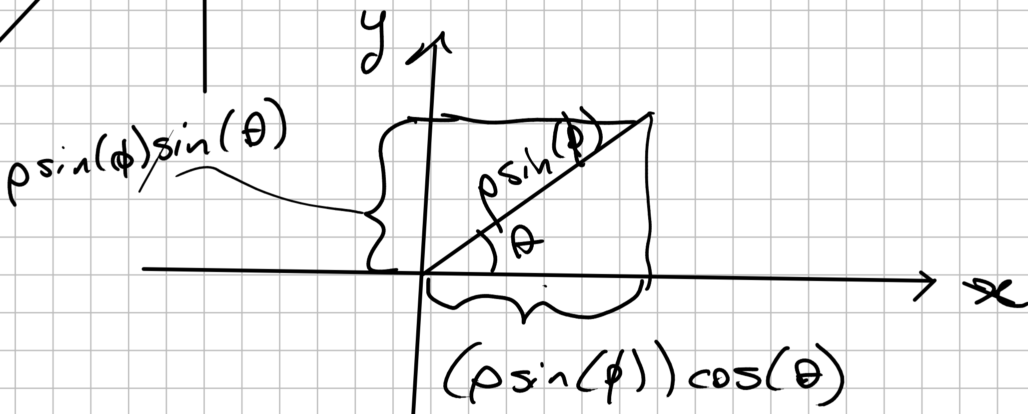
Eg (1) Solution set to $\rho = 1$



(2) Sol'n set to $\phi = \frac{\pi}{4}$

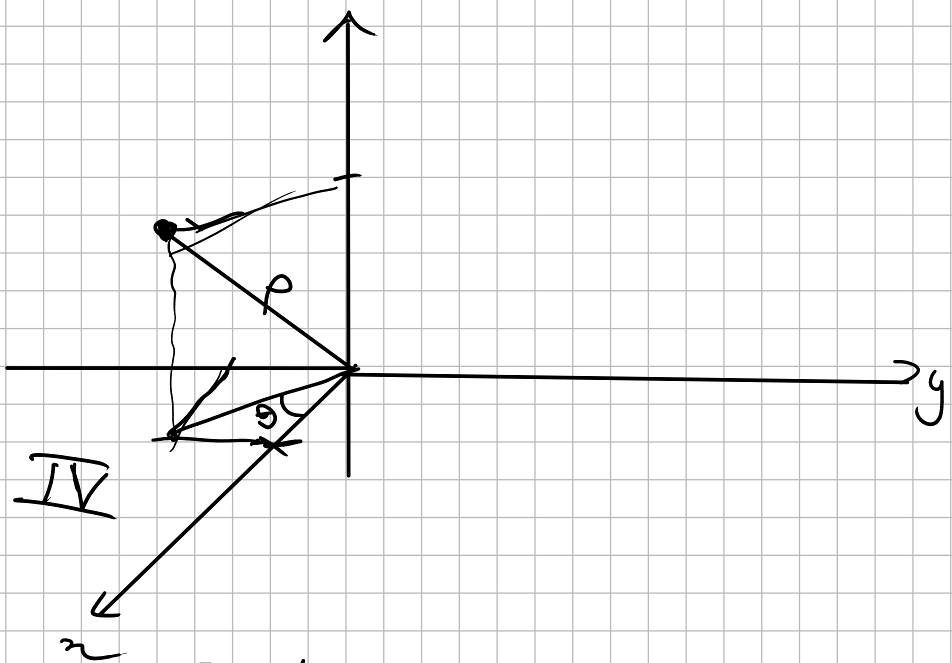






$$\boxed{\begin{aligned} z &= \rho\cos(\phi) \\ y &= \rho\sin(\phi)\sin(\theta) \\ x &= \rho\sin(\phi)\cos(\theta) \end{aligned}}$$

Eg// Find spherical coords for cartesian pt. $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$



Distance formula

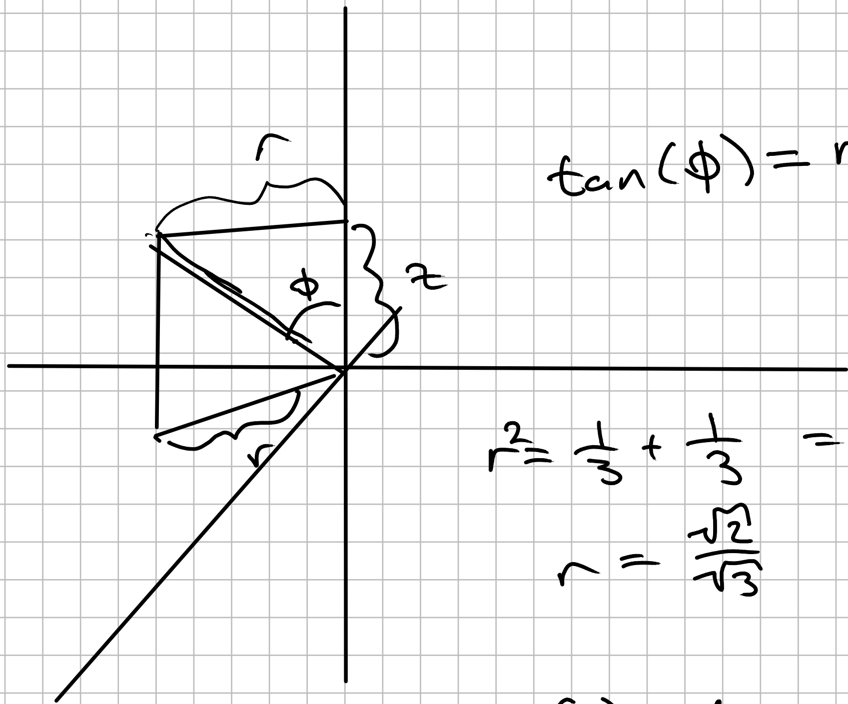
$\rightsquigarrow \boxed{\rho^2 = x^2 + y^2 + z^2}$

$\rho^2 = \frac{1}{3} + \frac{1}{3} + \frac{2}{3} = \frac{4}{3} \rightsquigarrow \rho = \frac{2}{\sqrt{3}}$

$\tan(\theta) = \frac{y}{x} = -1$

$\rightsquigarrow$ value of arctan in Quad IV

$\rightsquigarrow \theta = -\frac{\pi}{4}$



$\tan(\phi) = r/z$

$r^2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

$r = \frac{\sqrt{2}}{\sqrt{3}}$

$\tan(\phi) = 1$

$0 \le \phi \le \pi/2$

value of $\arctan(1)$ between 0 & $\pi/2$ is $\boxed{\pi/4 = \phi}$

spherical coords are

$$\boxed{\left(\frac{2}{\sqrt{3}}, -\pi/4, \frac{\pi}{4}\right)}$$

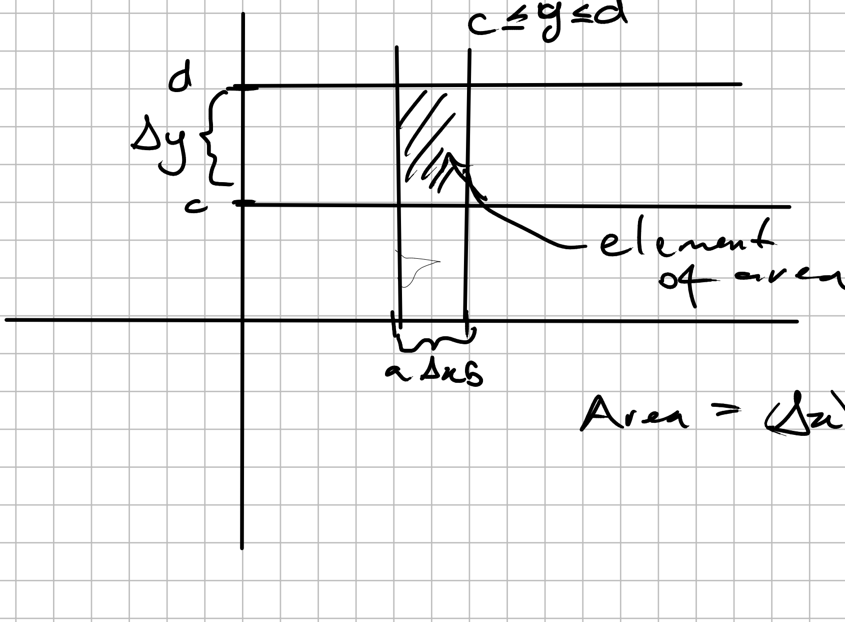
Elements of Area & Volume

We're going to study integration ⇝ need areas/vols of "small boxes" defined by coord. intervals.

Elements of area

Cartesian

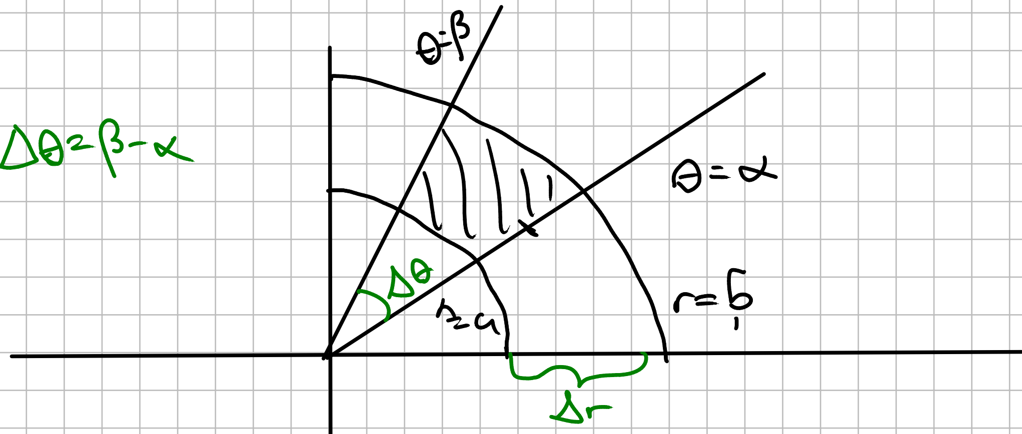
$a \le x \le b$
$c \le y \le d$

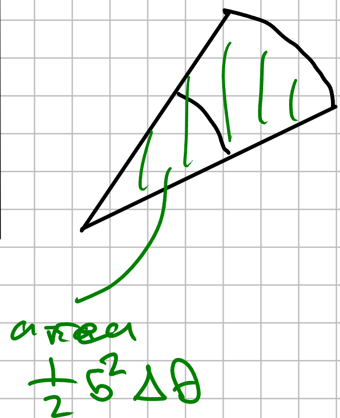


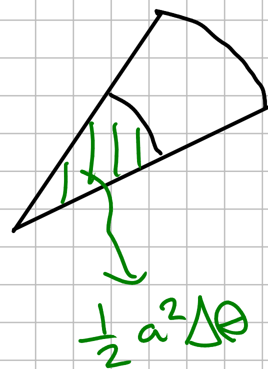
Area = $(\Delta x)(\Delta y)$

Polar $\quad a \le r \le b, \quad \alpha \le \theta \le \beta$

$\Delta\theta = \beta - \alpha$







Area of area element is

$$\frac{1}{2}(b^2-a^2)\Delta\theta = \frac{1}{2}(b+a)(b-a)\Delta\theta$$

$$= \frac{(b+a)}{2}\Delta r\,\Delta\theta$$

write $\bar{r} = \frac{b+a}{2}$

$$\boxed{A = \bar{r}\,\Delta r\,\Delta\theta}$$

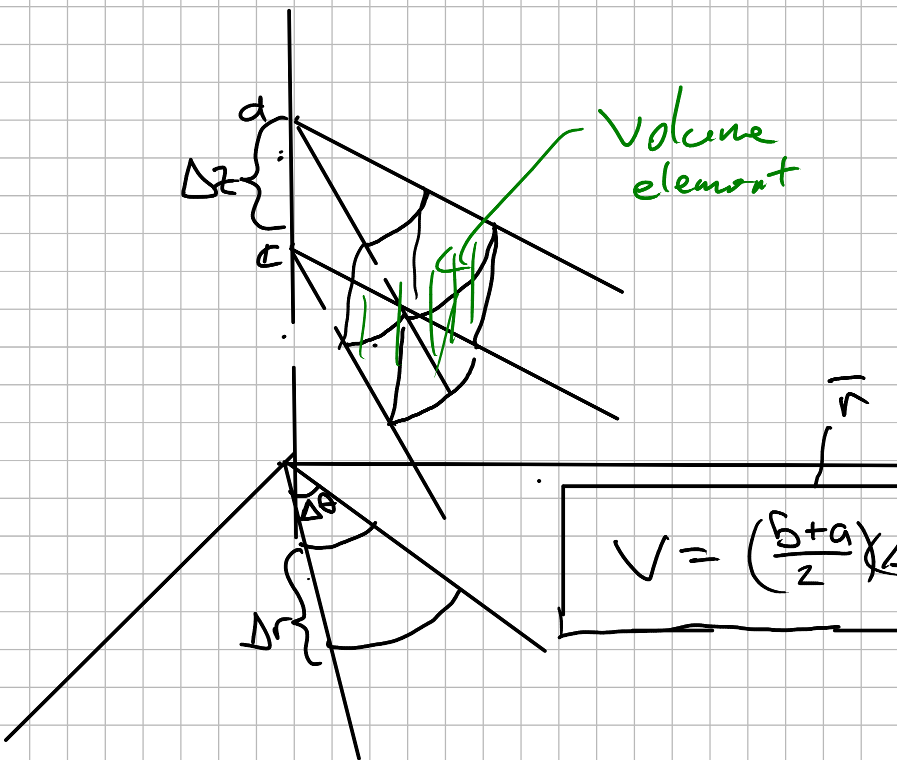
<u>Cyllindrical</u> Volume element

coord. box:

$a \le r \le b$

$\alpha \le \theta \le \beta$

$c \le z \le d$



$$\boxed{V = \overbrace{\left(\frac{b+a}{2}\right)}^{\bar{r}}(\Delta r)(\Delta\theta)(\Delta z)}$$

Class: 22 Oct.

Housekeeping
- Exam: 5 Nov.
- Election day
- Reflection
- Video Lecture

General Properties of double integrals

(1) $\iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA = \iint_D f(x,y)+g(x,y)\,dA$

(2) $\iint_D c f(x,y)\,dA = c \iint_D f(x,y)\,dA \quad c \in \mathbb{R}$

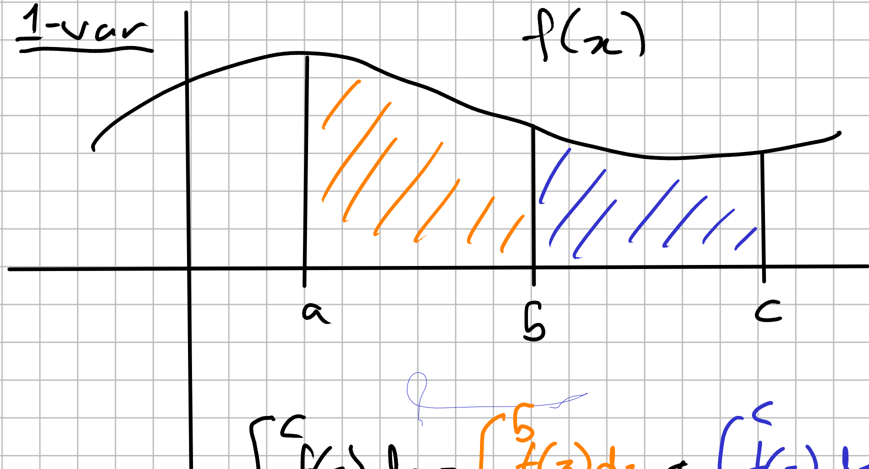
(3) $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

(4) $\iint_D 1\,dA = \text{Area}(D)$

Decomposing Integrals

1-var

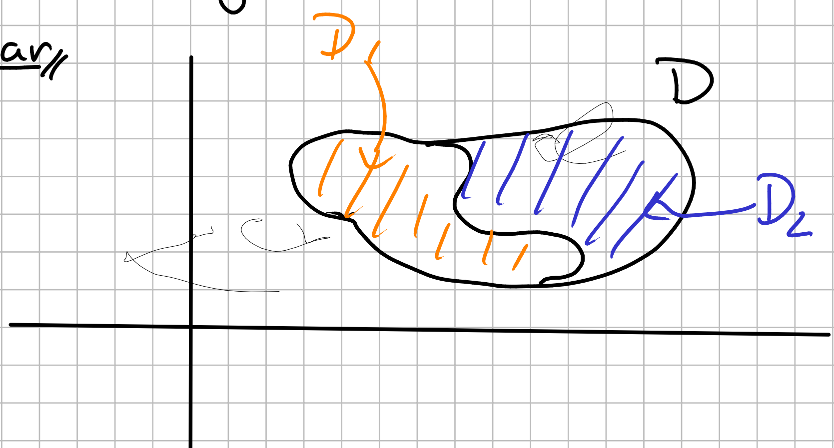


$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$[a,c] = [a,b] \cup [b,c]$

only intersect on their boundaries

2-var

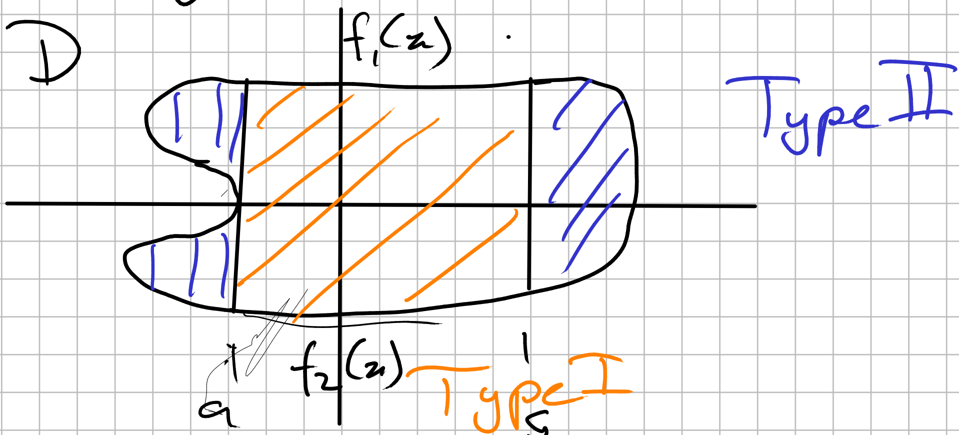


$D = D_1 \cup D_2$

intersect only on their boundaries

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

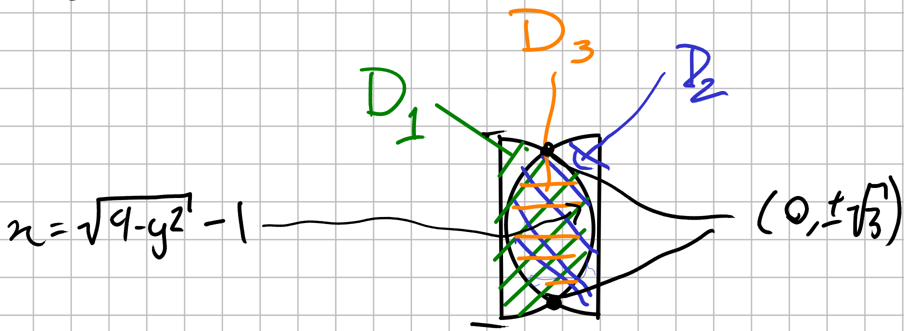
Use// Decompose domains into types I & II
⇝ apply Fubini's Thm.



Eg// Consider Domain

Union of two half-disks centered on (1,0), (-1,0) of radius 2

Integrate $(x+1)|y|$ on D

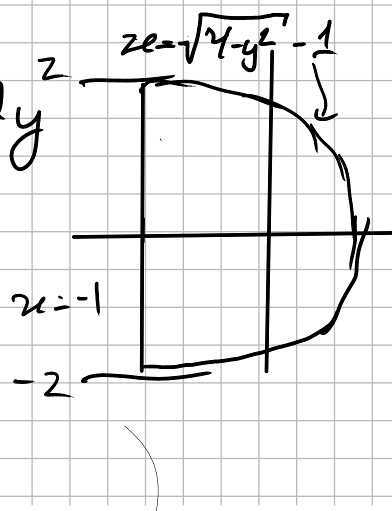


$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA - \iint_{D_3} f(x,y)\,dA$$

$$\iint_{D_1} (x+1)|y|\,dA \overset{FT}{=} \int_{-2}^{2}\int_{-1}^{\sqrt{4-y^2}-1} (x+1)|y|\,dx\,dy$$

comp

$$\overset{\sim}{=} \int_{-2}^{2} \left(2-\frac{y^2}{2}\right)|y|\,dy$$



$$\iint_{D_2 \text{ comp}} (x+1)|y|\,dA \overset{FT}{=} \int_{-2}^{2}\int_{1-\sqrt{4-y^2}}^{1} (x+1)|y|\,dx\,dy$$

$$\overset{?}{=} \int_{-2}^{2} \left(2\sqrt{4-y^2} + \frac{y^2}{2} - 2\right)|y|\,dy$$

$$\iint_{D_1} (x+1)|y|\,dA + \iint_{D_2} (x+1)|y|\,dA$$

$$= \int_{-2}^{2} \left(\cancel{2} - \cancel{\frac{y^2}{2}}\right)|y|\,dy + \int_{-2}^{2} \left(2\sqrt{4-y^2} + \cancel{\frac{y^2}{2}} - \cancel{2}\right)|y|\,dy$$

$$= \int_{-2}^{2} 2|y|\sqrt{4-y^2}\,dy$$

$$\iint_{D_3} (x+1)|y|\,dA = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{1-\sqrt{4-y^2}}^{\sqrt{4-y^2}-1} (x+1)|y|\,dx\,dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} -2|y| + 2|y|\sqrt{4-y^2}\,dy$$

$$\int_{-a}^{a} 2\sqrt{4-y^2}\,|y|\,dy = 2\int_{0}^{a} 2\sqrt{4-y^2}\,y\,dy$$

u-sub

$$= 2\int_{4-a^2}^{4} \sqrt{u}\,du$$

$$= \frac{4}{3}\left[8-(4-a^2)^{3/2}\right]$$

$$\underbrace{\iint_{D_1}(x+1)|y|\,dA + \iint_{D_2}(x+1)|y|\,dA}_{} - \iint_{D_3}(x+1)|y|\,dA$$

$$= \frac{4}{3}\left[8-(\cancel{4-2^2})^{3/2}\right] - \frac{4}{3}\left[8-(\cancelto{1}{4-3})^{3/2}\right] + 6$$

$$= \boxed{\frac{22}{3}}$$

Integration in Polar coords

Idea: Some fns & domains have nicer descriptions in polar coords

Want: Be able to compute double integrals using these descriptions ie to write

$\iint_D f(x,y)\,dA$ in terms of

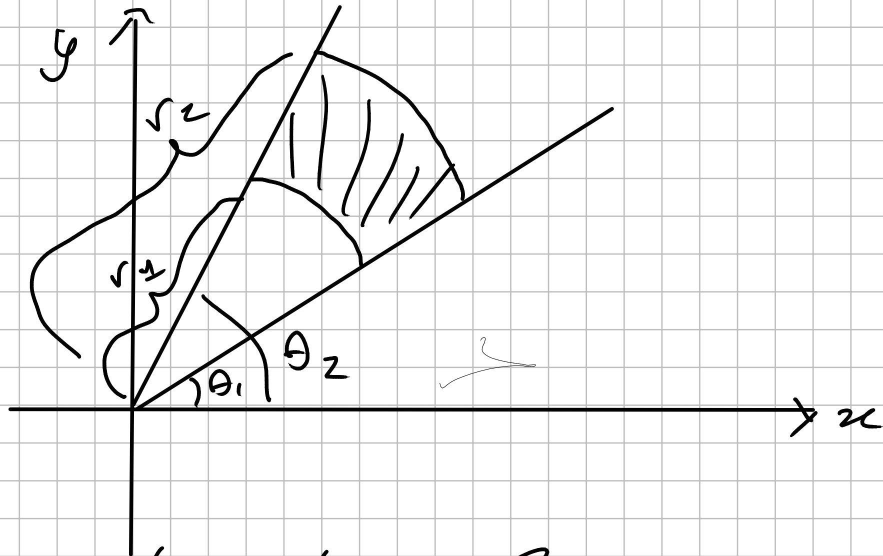
$f(r\cos\theta, r\sin\theta)$ & a polar desc. of D.

How?

Consider a polar rectangle

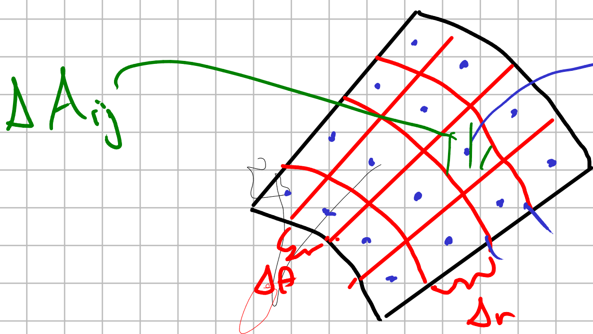
$r_1 \le r \le r_2 \qquad \theta_1 \le \theta \le \theta_2$

$$P = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \text{ for some } \begin{array}{l} r_1 \le r \le r_2 \\ \theta_1 \le \theta \le \theta_2 \end{array} \right\}$$



How to integrate?

(1) subdivide & pick test pts



test pt is center of little rectangle in r & θ coords

(r_i^*, θ_j^*)

$$r_i^* = \frac{r_{i-1}+r_i}{2}$$

$$\theta_j^* = \frac{\theta_{j-1}+\theta_j}{2}$$

(2) Note, Area of polar rectangle ij

$$\Delta A_{ij} = \underbrace{\left(\frac{r_{i-1}+r_i}{2}\right)}_{r_i^*} \Delta r \, \Delta\theta$$

(3) Approximate volume under the graph of $f(x,y)$ by

$$\sum_{j=1}^{m}\sum_{i=1}^{n} f(r_i^*\cos(\theta_j^*), r_i^*\sin(\theta_j^*))\,\Delta A_{ij}$$

$$= \sum_{j=1}^{m}\sum_{i=1}^{n} f(r_i^*\cos(\theta_j^*), r_i^*\sin(\theta_j^*))\, r_i^*\,\Delta r\,\Delta\theta$$

(4) Volume is

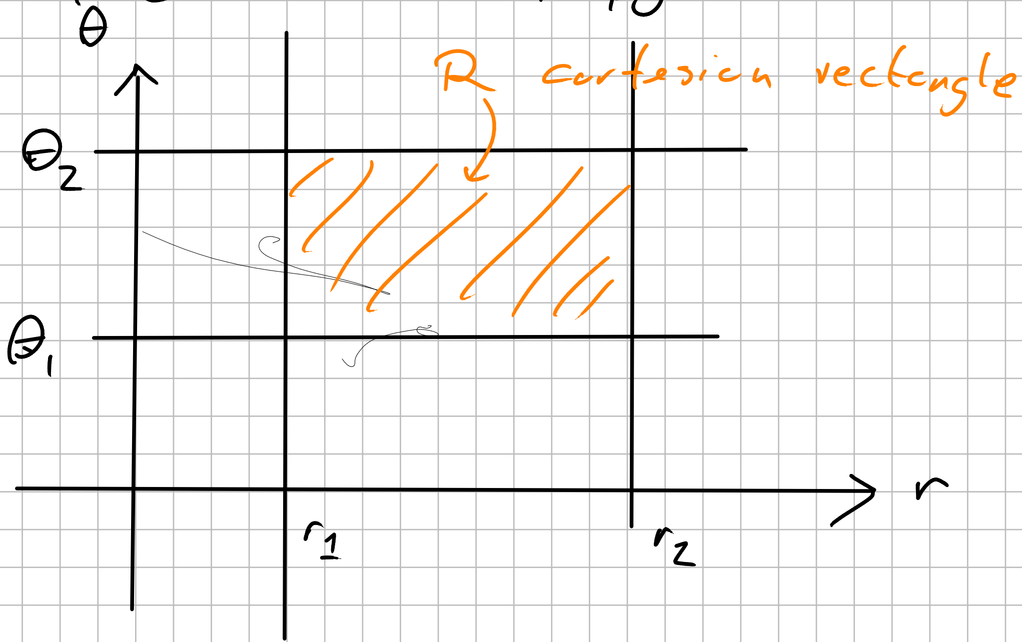
$$\lim_{n,m\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(r_i^*\cos(\theta_j^*), r_i^*\sin(\theta_j^*))\, r_i^*\,\Delta r\,\Delta\theta$$

$$= \iint_P f(x,y)\,dA$$

$$\lim_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{f(r_i^* \cos(\theta_j^*), r_i^* \sin(\theta_j^*))\, r_i^*}_{=: g(r_i^*, \theta_j^*)} \Delta r \Delta \theta$$

Notice,,

This looks like a formula for a cartesian integral



$$\iint_R g(r,\theta)\, dA$$

$\rightsquigarrow$ Fubini's Thm.

Thm [Change of coords for polar rectangles]

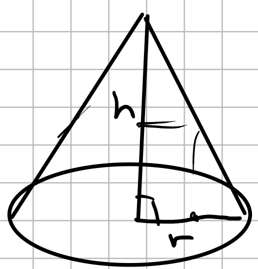
If $f(x,y)$ is continuous on a polar rectangle

$$P = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \text{ s.t. } \begin{matrix} r_1 \le r \le r_2 \\ \theta_1 \le \theta \le \theta_2 \end{matrix} \right\}$$

then

$$\boxed{\iint_P f(x,y)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta}$$

Eg Use a Polar integral to compupute the volume of a cone w/ height h & circular base of radius a



Exercise

$$f(x,y) = \left| h - \frac{h}{a}\sqrt{x^2 + y^2} \right|$$

$$P = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x = r\cos\theta & 0 \le \theta \le 2\pi \\ y = r\sin\theta & 0 \le r \le a \end{matrix} \right\}$$

$$\iint_P f(x,y)\,dA = \int_0^{2\pi}\int_0^a \left| h - \frac{h}{a}\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \right| r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^a \left| h - \frac{h}{a} r \right| r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^a hr - \frac{h}{a} r^2 \,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{h}{2} r^2 - \frac{h}{3a} r^3 \right]_{r=0}^{a} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} ha^2 - \frac{1}{3} ha^2 \,d\theta = \int_0^{2\pi} \frac{1}{6} ha^2 \,d\theta$$

$$= 2\pi \left(\frac{1}{6} ha^2 \right) = \boxed{\frac{\pi}{3} ha^2}$$