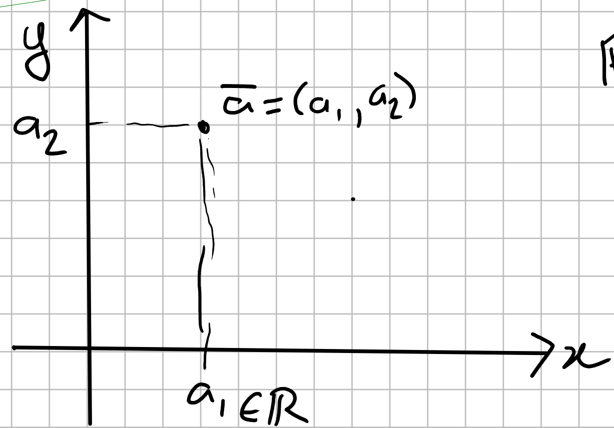
Calculus III: Lecture 1a

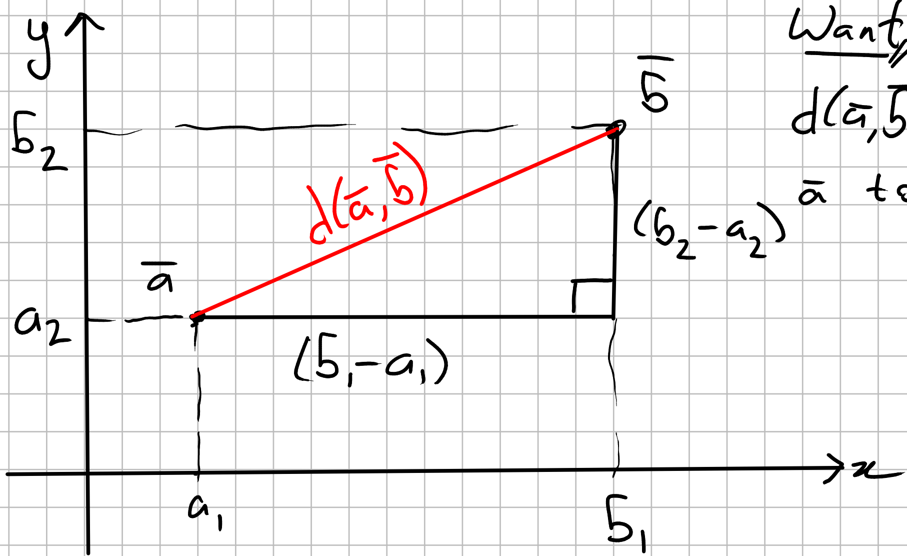
Today: Distances & coordinates in 3d

Warm-up: Distances & coordinates in 2d

$\mathbb{R}^2$



Want: Distance $d(\bar{a}, \bar{b})$ from $\bar{a}$ to $\bar{b}$



Pythagorean Thm

$\leadsto$

$$d(\bar{a},\bar{b})^2 = (b_1-a_1)^2 + (b_2-a_2)^2$$

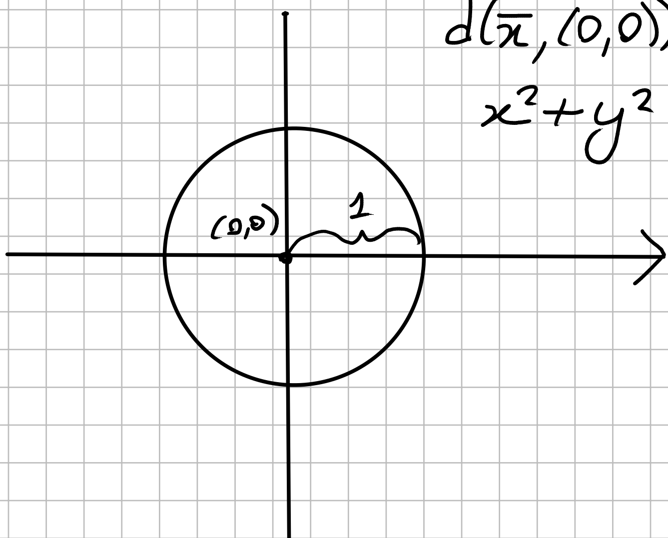
$$\boxed{d(\bar{a},\bar{b}) = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2}}$$

2d <u>Distance Formula</u>

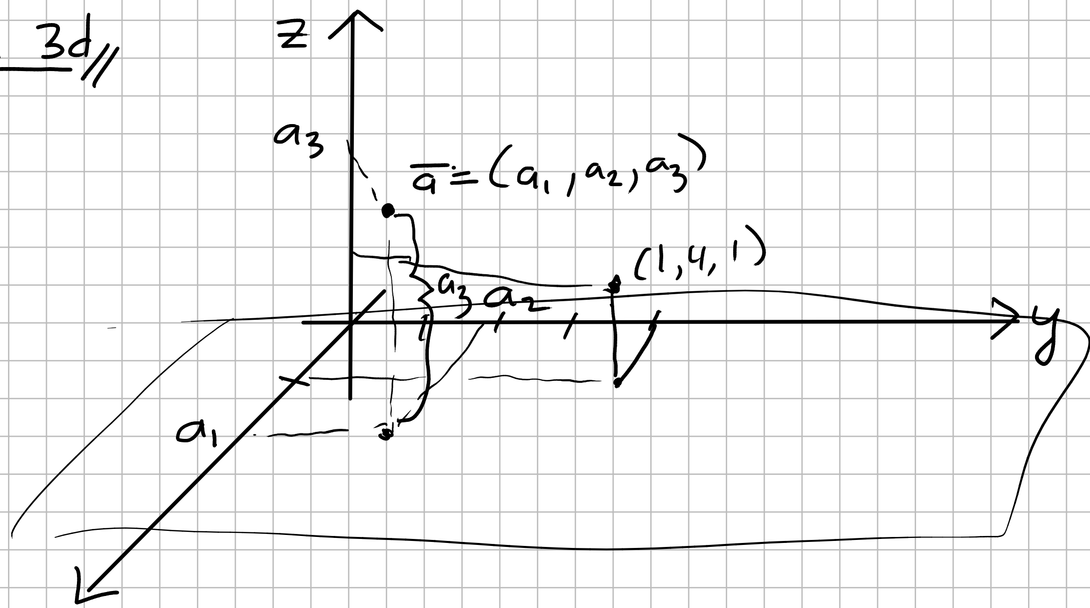
Eg// Unit circle

$$d(\bar{x},(0,0)) = 1$$

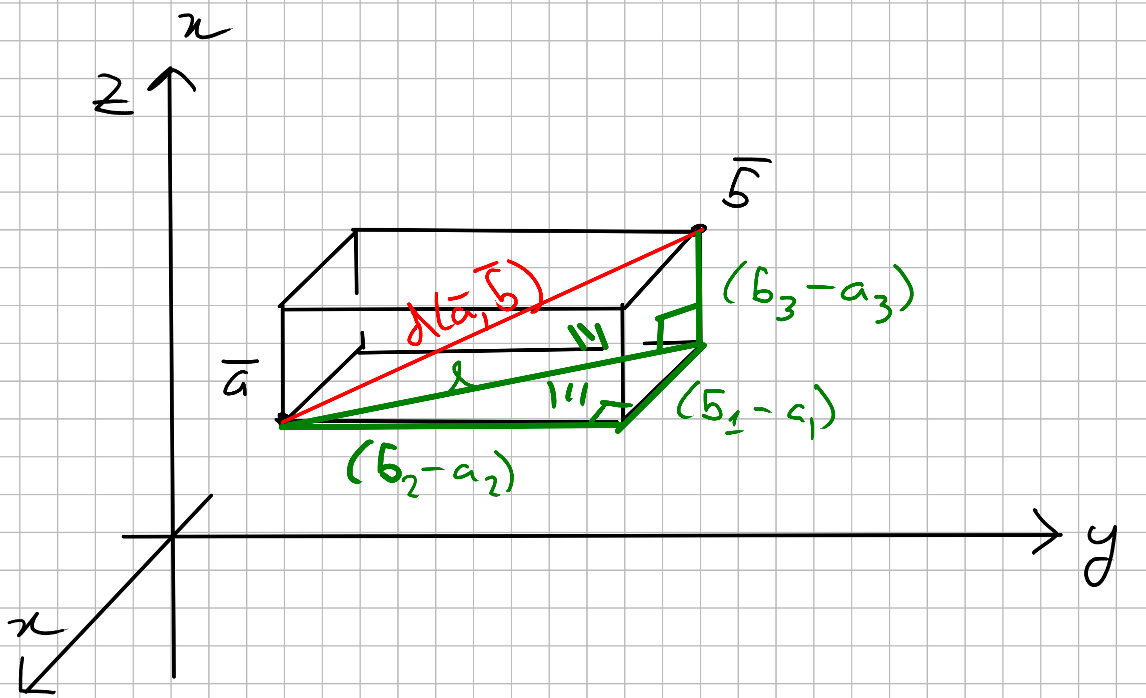
$$x^2+y^2 = 1^2 = 1$$



In 3d



z
a_3
$\bar{a} = (a_1, a_2, a_3)$
(1, 4, 1)
a_3
a_2
y
a_1
x



z
$\bar{b}$
$(b_3 - a_3)$
$d(\bar{a}, \bar{b})$
$\bar{a}$
ℓ
$(b_1 - a_1)$
$(b_2 - a_2)$
y
x

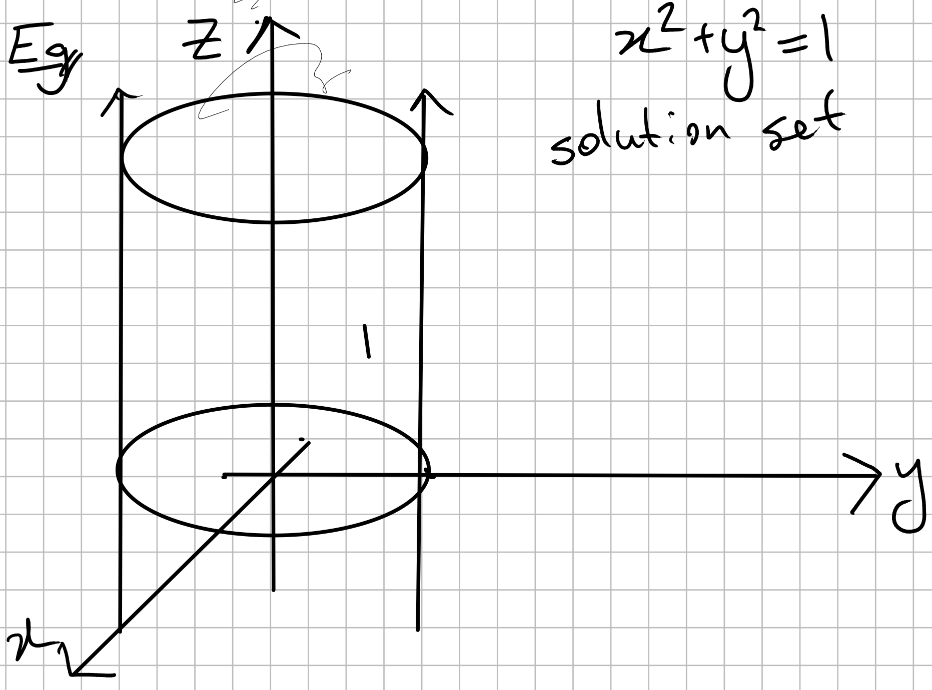
$l^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2$

$d(\bar{a}, \bar{b})^2 = l^2 + (b_3 - a_3)^2$

$$\boxed{d(\bar{a}, \bar{b}) = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}}$$

1 3d Distance formula

Eg

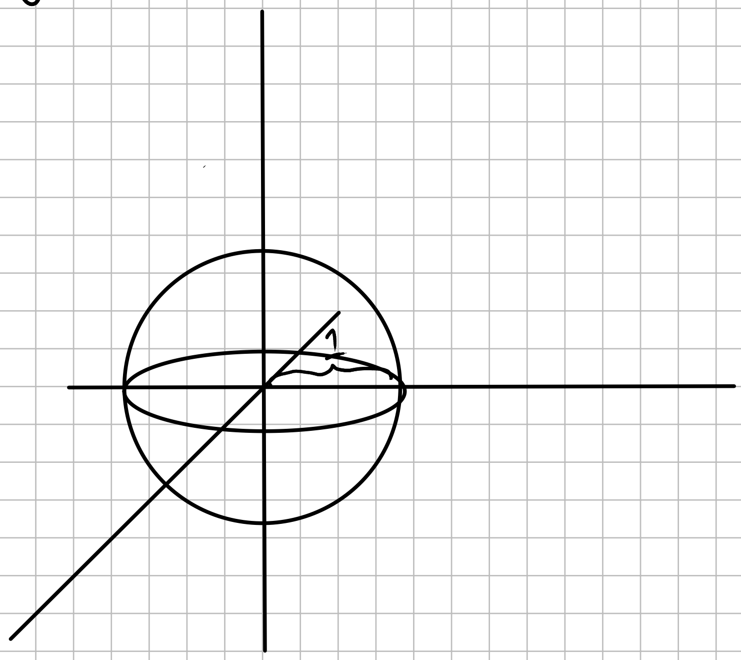


$x^2 + y^2 = 1$

solution set

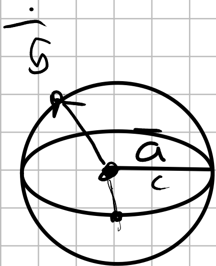
$d(\bar{x}, (0,0,0))^2 = 1^2 \rightsquigarrow$ points distance 1 from $\mathcal{O} = (0,0,0)$

$x^2 + y^2 + z^2 = \underline{1}$



Lecture 15: Vectors

<u>Problem</u>// distances don't say much about the relation of points in 3d.

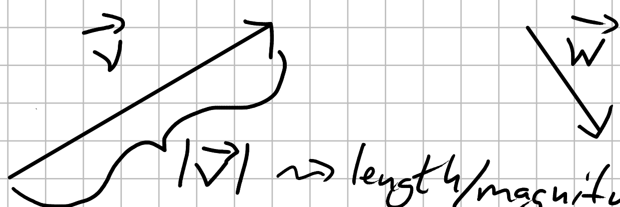


$$d(\bar{a}, \bar{b}) = c$$

<u>Solution</u>// Remembering a <u>direction</u>

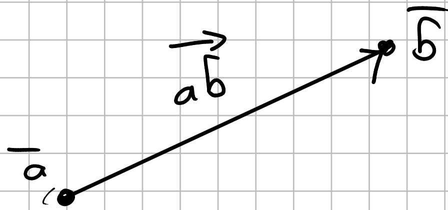
<u>Defn</u>// A <u>vector</u> is a mathematical object with a <u>magnitude</u> & a <u>direction</u>

$\underbrace{\text{magnitude}}_{\text{length}}$



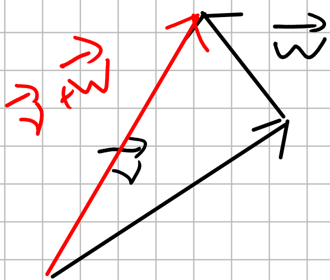
$|\vec{v}| \rightsquigarrow$ length/magnitude/norm of $\vec{v}$

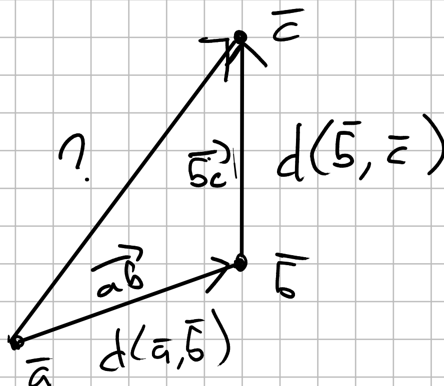
Eg// Given points



<u>Displacement vector</u> from $\bar a$ to $\bar b$

<u>Vector addition</u>





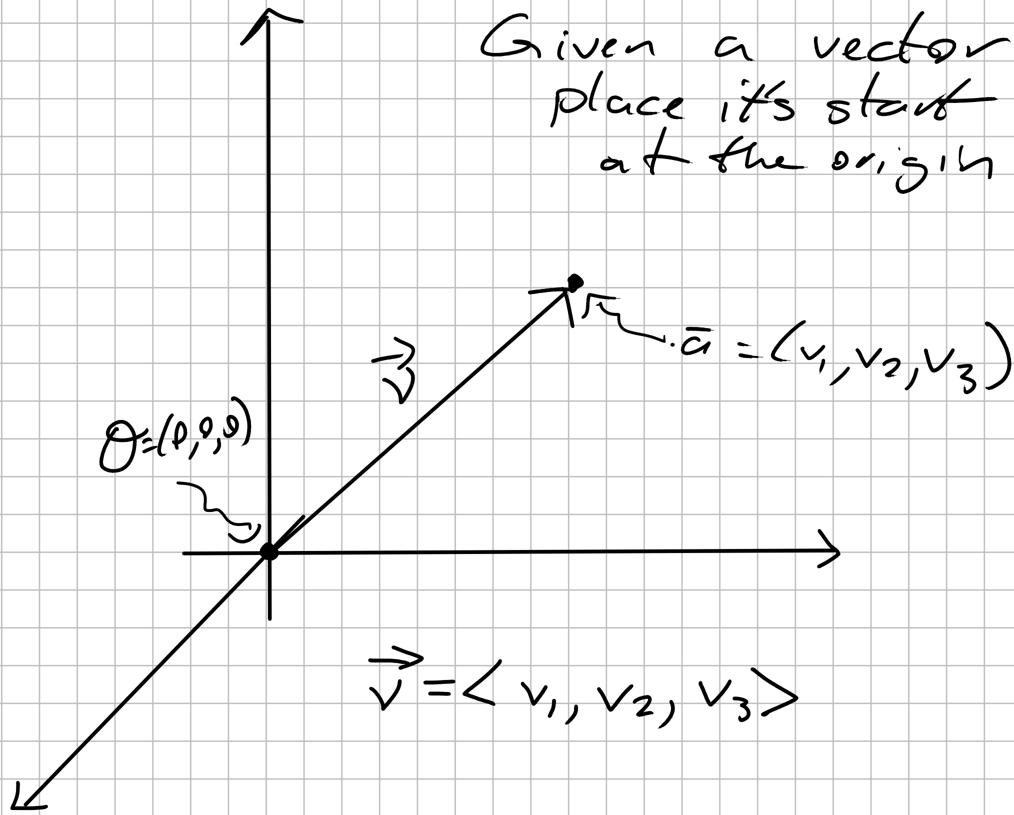
$$d(\bar a, \bar c) \le d(\bar a, \bar b) + d(\bar b, \bar c)$$

$$\vec{ac} = \vec{ab} + \vec{bc}$$

$$|\vec v + \vec w| \le |\vec v| + |\vec w|$$

How do we specify a vector?

Given a vector place it's start at the origin



$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

Slogan:

"Points & vectors are the same information, viewed differently"

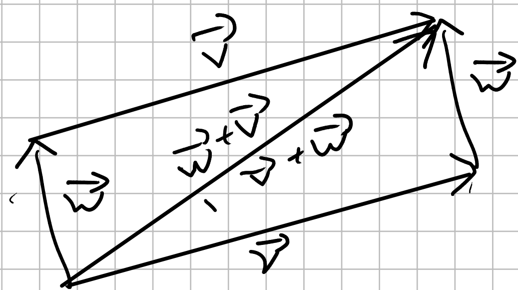
Properties of vector addition

$\vec{v} = \langle v_1, v_2, v_3 \rangle \qquad \vec{w} = \langle w_1, w_2, w_3 \rangle$

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$$

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

"commutative"



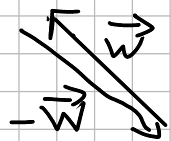
$$\vec{v} + (\vec{w} + \vec{u}) = (\vec{v} + \vec{w}) + \vec{u}$$

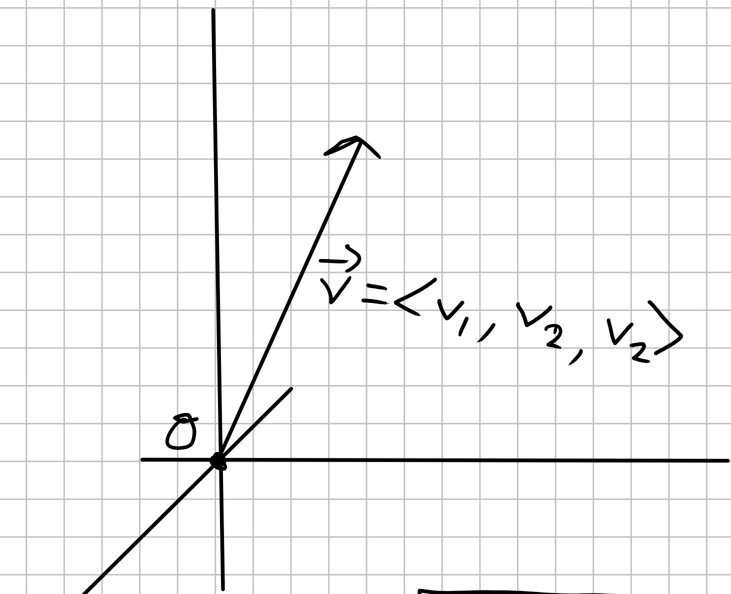
Note: Special vector $\quad \vec{0} = \langle 0, 0, 0 \rangle$

$$\vec{0} + \vec{v} = \vec{v}$$

$$\vec{v} - \vec{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$$

Geometric Interpretation of $-\vec{w}$





$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Change length

$2 \cdot \vec{v} := \langle 2v_1, 2v_2, 2v_3 \rangle$

"scalar"
=
real number

$$|2 \cdot \vec{v}| = \sqrt{(2v_1)^2 + (2v_2)^2 + (2v_3)^2}$$
$$= 2\sqrt{v_1^2 + v_2^2 + v_3^2} = 2|\vec{v}|$$

Generally, $\lambda \in \mathbb{R} \rightsquigarrow$

$$\lambda\vec{v} = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle$$

Same direction

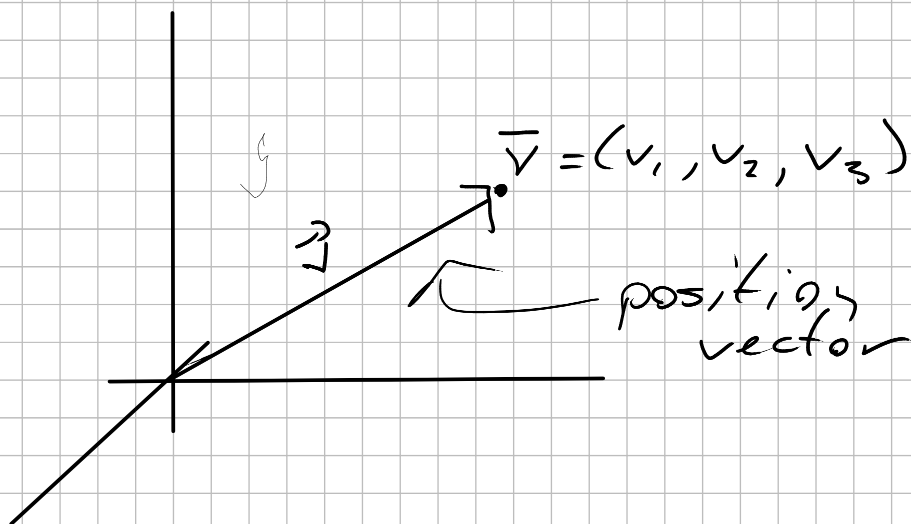
$$|\lambda\vec{v}| = |\lambda| \cdot |\vec{v}$$

Lecture 2a: Dot Products

Recall
- 3d coordinates
- vectors

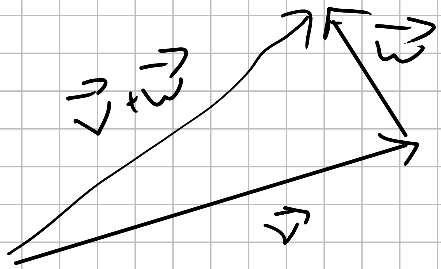
- Vectors in coords

$$\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$$



- Vector operations $(\vec{v}, \vec{w} \in \mathbb{R}^3)$

- $\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$



- $\lambda \in \mathbb{R}$ scalar

$$\lambda \vec{v} = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle$$

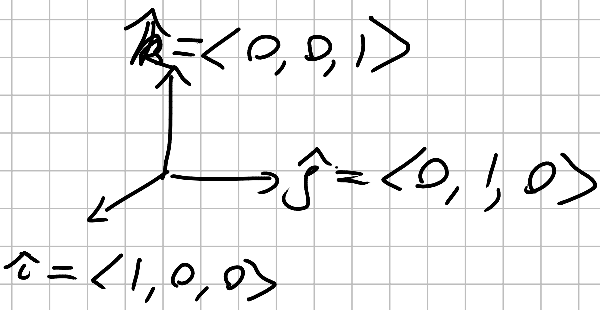
$$|\lambda \vec{v}| = |\lambda|\,|\vec{v}|$$

- Unit vectors $|\hat{v}| = 1$

$\vec{w} \in \mathbb{R}^3$ $\quad \vec{w} \neq \vec{0}$

get unit vector $\rightsquigarrow$
$$\hat{w} = \frac{\vec{w}}{|\vec{w}|} = \frac{1}{|\vec{w}|}\vec{w}$$

- Standard unit vectors



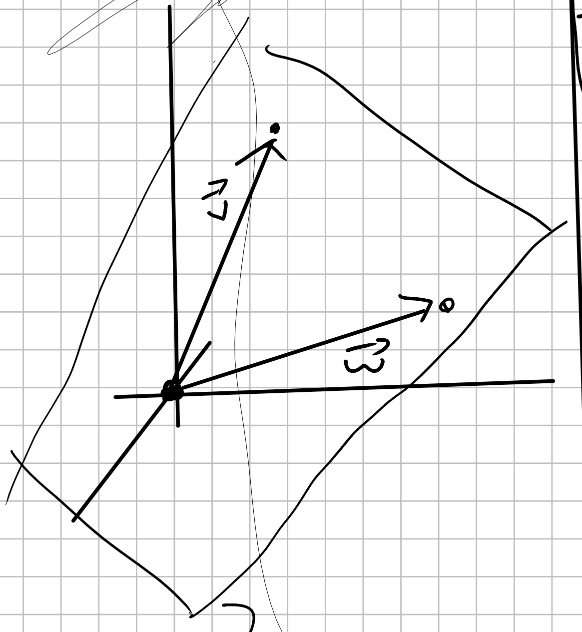
$$\vec{v} = \langle v_1, v_2, v_3 \rangle \iff \vec{v} = v_1 \hat{\imath} + v_2 \hat{j} + v_3 \hat{k}$$

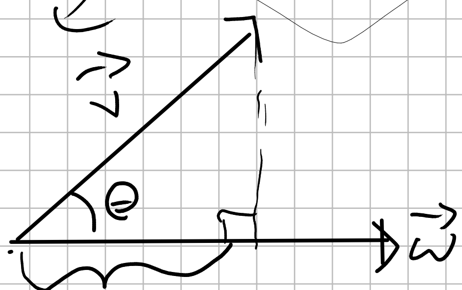
"v_1 is the amount of $\vec{v}$ in the $\hat{\imath}$-direction"

[We say v_1 is the $\hat{\imath}$- (or x-) component of $\vec{v}$]

<u>Q</u>// Is $\hat{\imath}$ special? or can we find a component of $\vec{v}$ in the $\vec{w}$-direction for any vector $\vec{w}$?

Geometry





$$\mathrm{comp}_{\vec{w}}(\vec{v}) = |\vec{v}|\cos(\theta)$$

Algebra

Defn// For $\vec{v},\vec{w}\in\mathbb{R}^3$ we define the **dot product**

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

Rmk// (1) $\vec{v}\cdot\vec{v} = |\vec{v}|^2$

(2) $(\lambda\vec{v})\cdot\vec{w} = \lambda(\vec{v}\cdot\vec{w})$
$= \vec{v}\cdot(\lambda\vec{w})$

(3) $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$

(4) $(\vec{w}+\vec{v})\cdot\vec{u} = (\vec{w}\cdot\vec{u}) + (\vec{v}\cdot\vec{u})$

(5) $\vec{v}\cdot(\vec{w}+\vec{u})$
$= \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{u}$

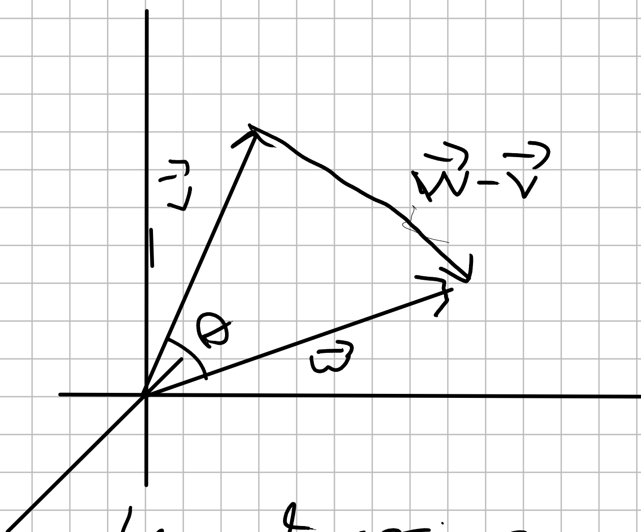
(6) $\vec{v}\cdot\vec{0} = 0$.

Prop For $\vec{v}, \vec{w} \in \mathbb{R}^3$ non-zero

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\theta)$$

θ ← angle between $\vec{v}$ & $\vec{w}$

Pf



Law of cosines

$$|\vec{w}-\vec{v}|^2 = |\vec{w}|^2 + |\vec{v}|^2 - 2|\vec{v}||\vec{w}|\cos(\theta)$$

$$|\vec{w}-\vec{v}|^2 = (\vec{w}-\vec{v}) \cdot (\vec{w}-\vec{v})$$

$$= |\vec{w}|^2 + |\vec{v}|^2 - 2\vec{v} \cdot \vec{w}$$

$$\cancel{|\vec{w}|^2} + \cancel{|\vec{v}|^2} - 2\vec{v}\cdot\vec{w}$$

$$= \cancel{|\vec{w}|^2} + \cancel{|\vec{v}|^2} - 2|\vec{v}||\vec{w}|\cos(\theta)$$

$$\Rightarrow \quad \vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\theta) \qquad \square$$

<u>Cor</u>// For non-zero $\vec{v},\vec{w}\in\mathbb{R}^3$

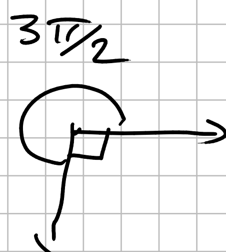
$(\vec{v}\cdot\vec{w}=0) \Rightarrow \vec{v}$ is perpendicular to $\vec{w}$

orthogonal

$$\cancel{|\vec{v}|}\,\cancel{|\vec{w}|}\cos(\theta)=0$$

$$\cos(\theta)=0$$

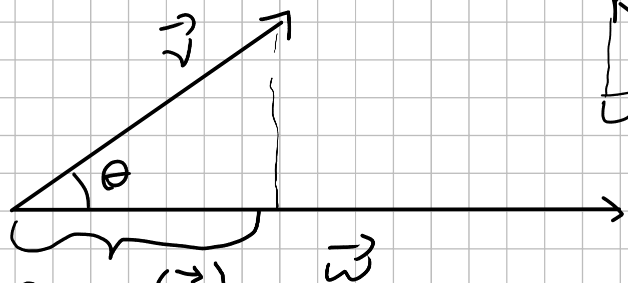
$$\theta = n\frac{\pi}{2} \qquad n \text{ odd}$$

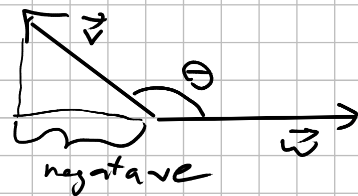


Lecture 25: Cross Product

We just showed

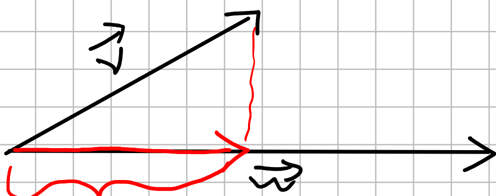
$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\theta)$$





$$\text{comp}_{\vec{w}}(\vec{v}) = |\vec{v}|\cos(\theta) = \frac{1}{|\vec{w}|}(\vec{v}\cdot\vec{w}) = \vec{v}\cdot\left(\frac{\vec{w}}{|\vec{w}|}\right) = \vec{v}\cdot\hat{w} \quad \text{(\underline{scalar})}$$

DA 1// The projection of $\vec{v}$ onto $\vec{w}$ is the vector



$$\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v}\cdot\hat{w})\hat{w}$$

Cross Products

Determinants

A 2x2 matrix is an array of real #'s

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the determinant of A is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

A 3x3 matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

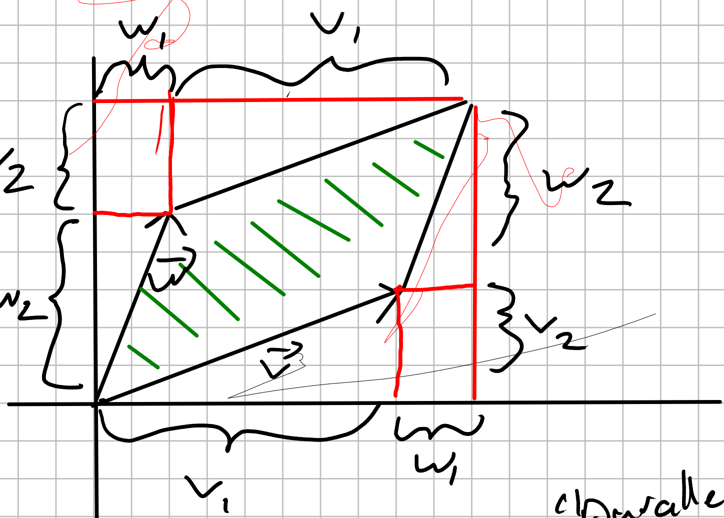
$$+ a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Prop// $\vec{v} = \langle v_1, v_2 \rangle$, $\vec{w} = \langle w_1, w_2 \rangle$ vectors in $\mathbb{R}^2$

then

$$\left| \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \right| \text{ is area of}$$

"Pf//"

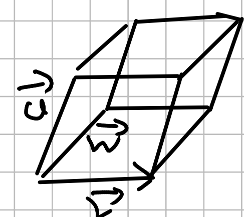


$\square$

Claim// $\vec{v}, \vec{w}, \vec{u} \in \mathbb{R}^3$

$$\left| \det \begin{pmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{pmatrix} \right| = \text{volume of}$$

"parallelpiped defined by $\vec{v}, \vec{w}, \& \vec{u}$"



Defn// $\vec{v}, \vec{w} \in \mathbb{R}^3$, the <u>cross product</u> of $\vec{v}$ w/ $\vec{w}$ is

$$\vec{v} \times \vec{w} = \left\langle \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle$$

$$= \langle v_2 w_3 - w_2 v_3, -v_1 w_3 + w_1 v_3, v_1 w_2 - w_1 v_2 \rangle$$

Rmk// $\vec{v} \cdot \vec{w}$ is a <u>scalar</u>

$\vec{v} \times \vec{w}$ is a <u>vector</u>

Rmk// (1) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

(2) $\vec{v} \times (\vec{w} + \vec{u}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{u}$

(3) $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$

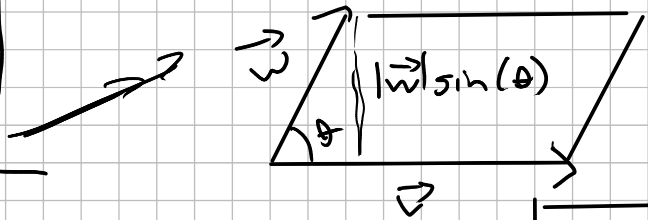
Lemma// $\vec{v}\times\vec{w}$ is orthogonal to $\vec{v}$ & $\vec{w}$.

Pf// $$(\vec{v}\times\vec{w})\cdot\vec{w} = w_1\cancel{v_2 w_3} - w_1\cancel{w_2}v_3 - w_2\cancel{v_1 w_3} + w_2\cancel{v_3}v_1 + w_3\cancel{v_1}w_2 - w_3\cancel{v_2}w_1 = 0$$
$\square$

Lemma// $\vec{v},\vec{w}\in\mathbb{R}^3$, θ angle between
$$|\vec{v}\times\vec{w}| = |\vec{v}|\,|\vec{w}|\sin(\theta)$$
choose $0\le\theta\le\pi$

Pf// $$|\vec{v}\times\vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v}\cdot\vec{w})^2$$
$$= |\vec{v}|^2|\vec{w}|^2 - |\vec{v}|^2|\vec{w}|^2\cos^2(\theta)$$
$$= |\vec{v}|^2|\vec{w}|^2(1-\cos^2(\theta)) = |\vec{v}|^2|\vec{w}|^2\sin^2(\theta)$$
$\square$

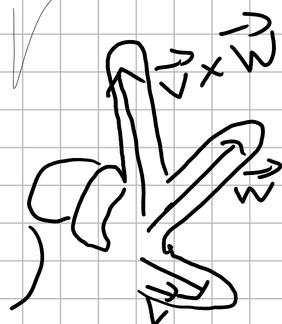
Cor/ $|\vec{v} \times \vec{w}|$ is the area of the parallelogram



$A = |\vec{v}|\,|\vec{w}|\sin(\theta)$

Cor// $\vec{v}, \vec{w}$ are parallel ($\vec{v} = \lambda \vec{w}$) if and only if $\vec{v} \times \vec{w} = \vec{0}$

Right-hand Rule



Return to determinants & Area/Volume

$\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$

Defn The triple product of $\vec{u}, \vec{v}$, & $\vec{w}$ is

$$\vec{u} \cdot (\vec{v} \times \vec{w}) \in \mathbb{R}$$

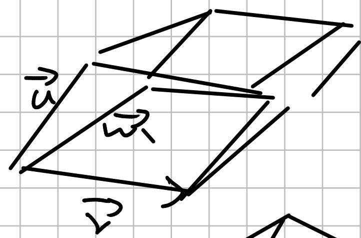
Rmk By def

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{u} \cdot \left\langle \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle$$

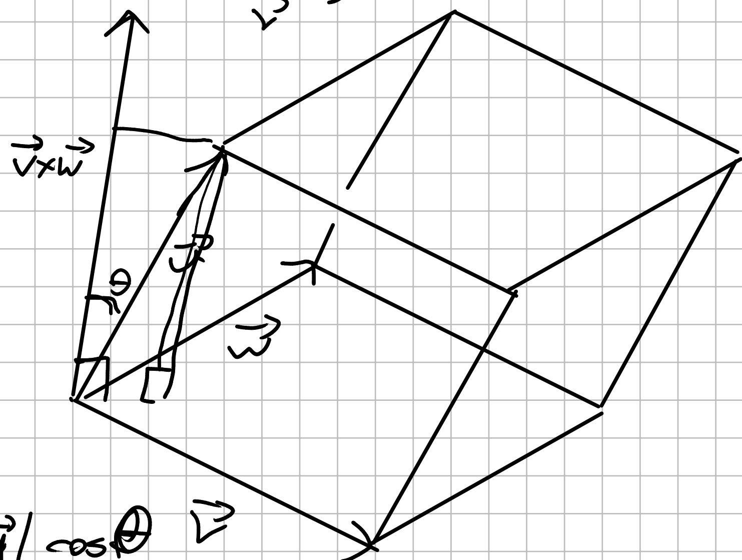
$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

Prop// $|\vec{u}\cdot(\vec{v}\times\vec{w})|$ is the volume of parallelpiped



Pf//



h = height $= |\vec{u}|\cos\theta$

A = Area of the base is $|\vec{v}\times\vec{w}|$

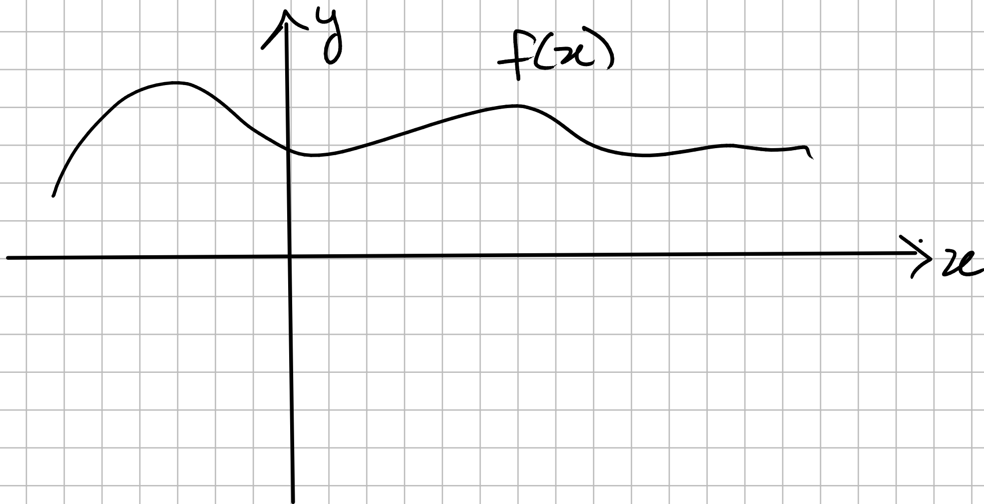
$\Rightarrow \text{Vol} = A\cdot h = |\vec{u}|\,|\vec{v}\times\vec{w}|\cos(\theta)$

$= |\vec{u}\cdot(\vec{v}\times\vec{w})|$ $\square$

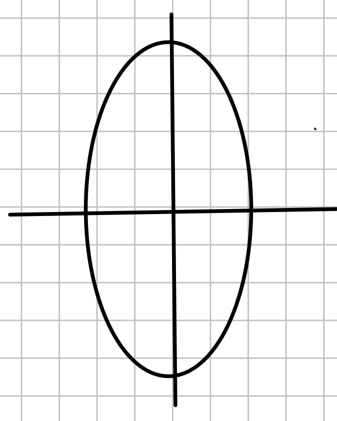
Lecture 3a: Curves in $\mathbb{R}^2$

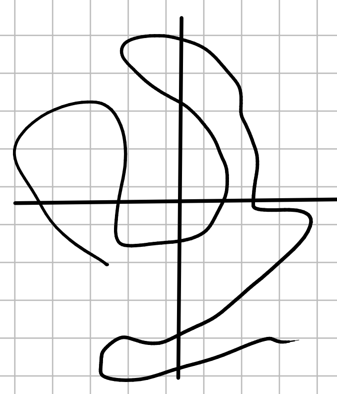
<u>Goal</u>: Describe curves in 2D

<u>Already</u>: graphs of fxns (function)



What about weirder things?



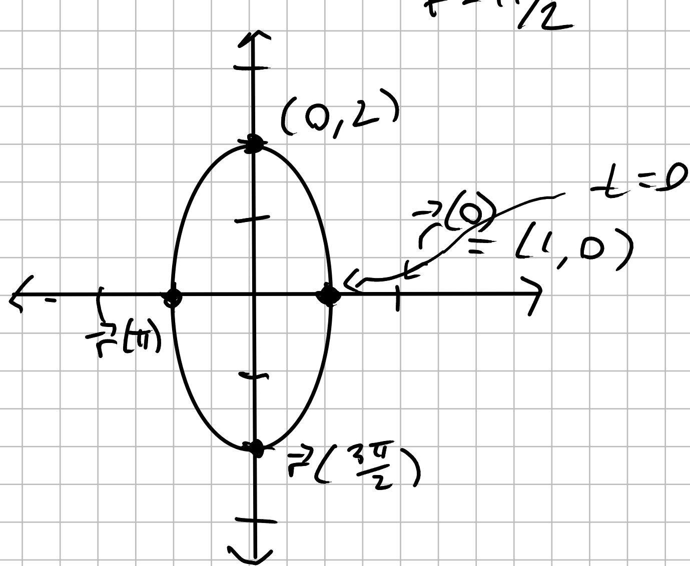


give a parameterization: a fun

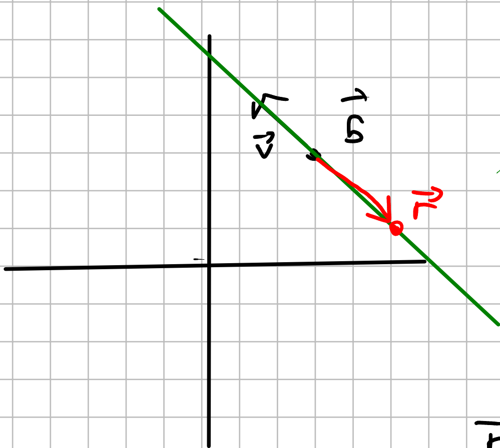
$$\vec{r} : [a,b] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \langle r_1(t), r_2(t) \rangle$$

view as a position vector

Eg $\vec{r}(t) = (\cos(t), 2\sin(t))$
$t \in [0, 2\pi]$



Eg given a vector $\vec{v} \in \mathbb{R}^2$ and a point $\vec{b} \in \mathbb{R}^2$



$\vec{r} = (x, y)$

$\vec{r} - \vec{b}$ parallel to $\vec{v}$

$\rightsquigarrow$ hence:

$\vec{r} - \vec{b} = t\vec{v}$ for some $t \in \mathbb{R}$

solve for $\vec{r}$

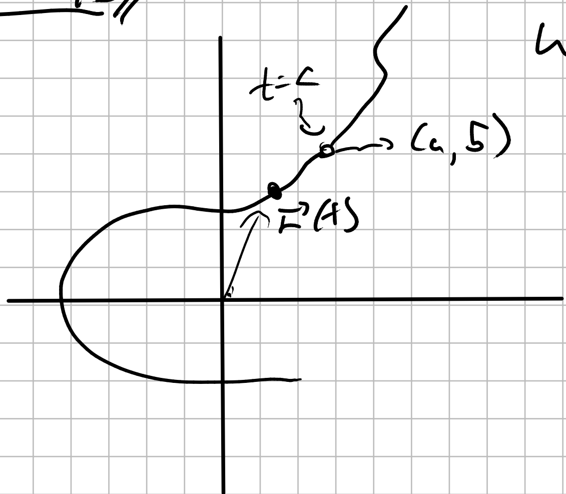
$$\boxed{\vec{r} = \vec{b} + t\vec{v}}$$

this is a parameterization

$$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$t \longmapsto (\underbrace{b_1 + tv_1}_{r_1(t)}, \underbrace{b_2 + tv_2}_{r_2(t)})$$

Limits



Want

"$\lim_{t \to c} \vec{r}(t) = (a, b)$"

Note limit of a vector-valued fuction is a vector.

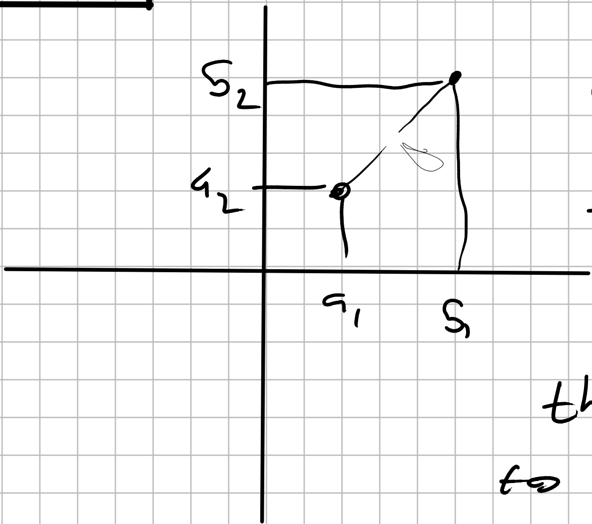
Note Idea of a limit doesn't change

slogan "If $\vec{\ell} = \lim_{t \to c} \vec{r}(t)$ this means

that if t gets arbitrarily close to c, then $\vec{r}(t)$ gets arbitrarily $\underbrace{\text{close to } \vec{\ell}}$"

$d(\vec{r}(t), \vec{\ell})$ is small.

Note//



if b_1 gets close to a_1 & b_2 gets close to a_2 then $\bar{b}$ gets close to $\bar{a}$ & vice-versa

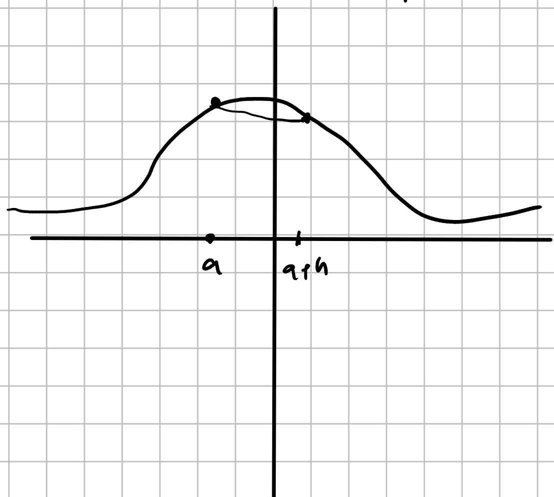
Defn// let $\vec{r}:[a,b] \longrightarrow \mathbb{R}^2$ be a 2D curve the <u>limit</u> of $\vec{r}(t)$ as $t \to c$, $c \in [a,b]$, is

$$\lim_{t \to c} \vec{r}(t) = \left\langle \lim_{t\to c} r_1(t), \lim_{t \to c} r_2(t) \right\rangle$$

Lecture 36: Calculus with Curves

Derivatives

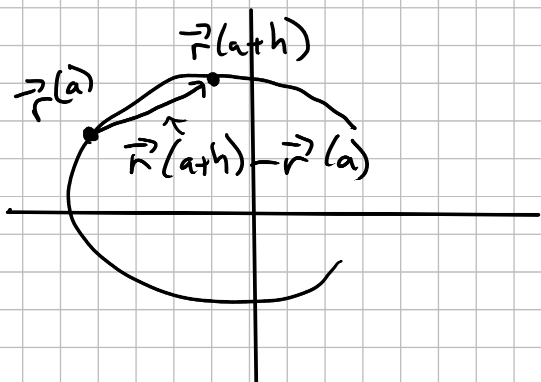
For fxn $f(x)$



limit of averge rates of change

$$\frac{df}{dx}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For a parametric curve $\vec{r} : [a,b] \longrightarrow \mathbb{R}^2$



$\vec{r}(a+h) - \vec{r}(a)$

displacement vector

"vector traveled in time h"

$$\frac{\vec{r}(a+h) - \vec{r}(a)}{h} = \text{average velocity}$$

Defn// The *derivative* of $\vec{r}(t)$ at a t-value a is the limit

$$\frac{d\vec{r}}{dt}(a) := \vec{r}\,'(a) := \lim_{h\to 0} \frac{1}{h}\left(\vec{r}(a+h) - \vec{r}(a)\right)$$

note that the derivative is a vector!

If this limit exists, say $\vec{r}$ is ~~diff~~ *differentiable at a*.

If $\vec{r}$ is differentiable at every value in its domain, we say it is differentiable.

Lem// If $\vec{r}(t) = \langle r_1(t), r_2(t)\rangle$ and if $r_1(t)$ & $r_2(t)$ are differentiable then

(1) $\vec{r}(t)$ is differentiable

(2) $\dfrac{d\vec{r}}{dt} = \left\langle \dfrac{dr_1}{dt}, \dfrac{dr_2}{dt} \right\rangle$

$\cancel{\text{Pf}}$ $$\frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{1}{h} \langle r_1(t+h) - r_1(t), r_2(t+h) - r_2(t) \rangle$$

$$= \left\langle \lim_{h \to 0} \frac{1}{h} (r_1(t+h) - r_1(t)), \lim_{h \to 0} \frac{1}{h} (r_2(t+h) - r_2(t)) \right\rangle$$

$$= \left\langle \frac{dr_1}{dt}, \frac{dr_2}{dt} \right\rangle \quad \square$$

Interpreting $\frac{d\vec{r}}{dt}$

(1) $\vec{r}(a+h) - \vec{r}(a)$ is the "vector travelled in time h" so that $\frac{\vec{r}(a+h) - \vec{r}(a)}{h}$ is the average velocity vector from $t=a$ to $t=a+h$

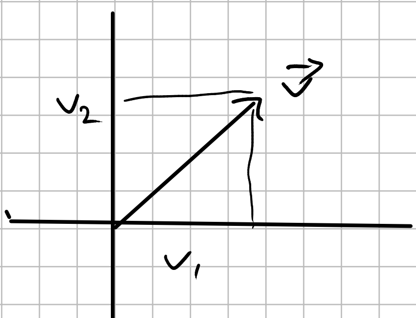
$\rightsquigarrow$ $\frac{d\vec{r}}{dt}(a)$ is the instantaneous velocity vector a time a

- $\left|\frac{d\vec{r}}{dt}\right|$ is the speed of the particle
- direction of $\frac{d\vec{r}}{dt}$ is the direction of motion
- the components of $\frac{d\vec{r}}{dt}$ are the rates of change in the coordinate directions

---

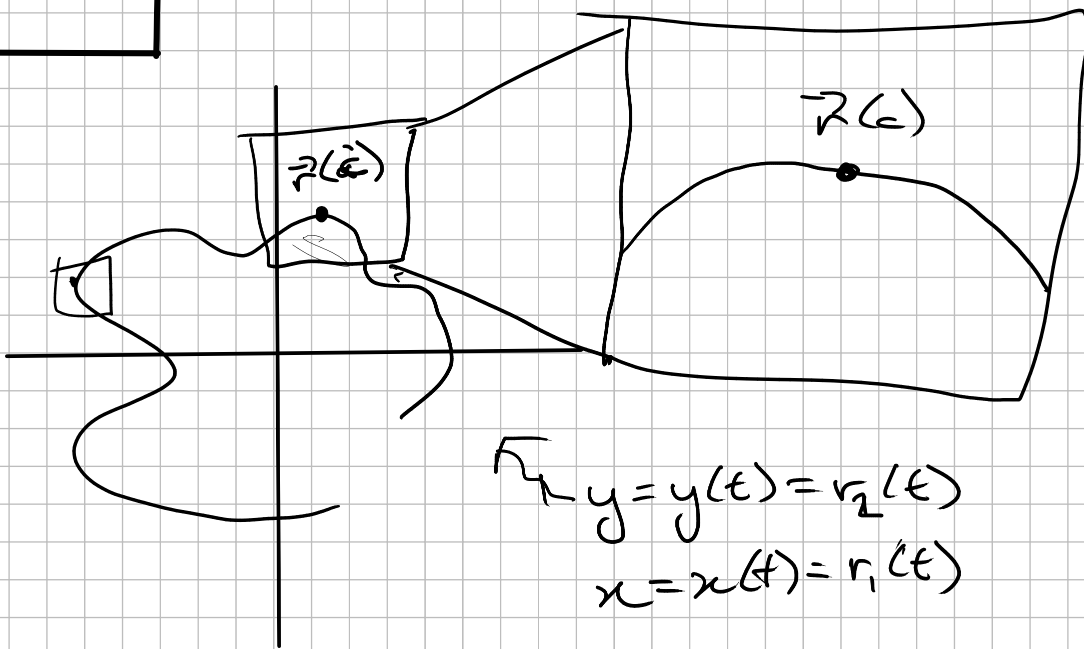
(2)

<u>Slope of a Vector</u>



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{v_2}{v_1}$$

$\vec{r} : [a, b] \longrightarrow \mathbb{R}^2$



$y = y(t) = r_2(t)$

$x = x(t) = r_1(t)$

(*) Assume $y(t) = y(x(t))$

slope tangent line

Chain rule:

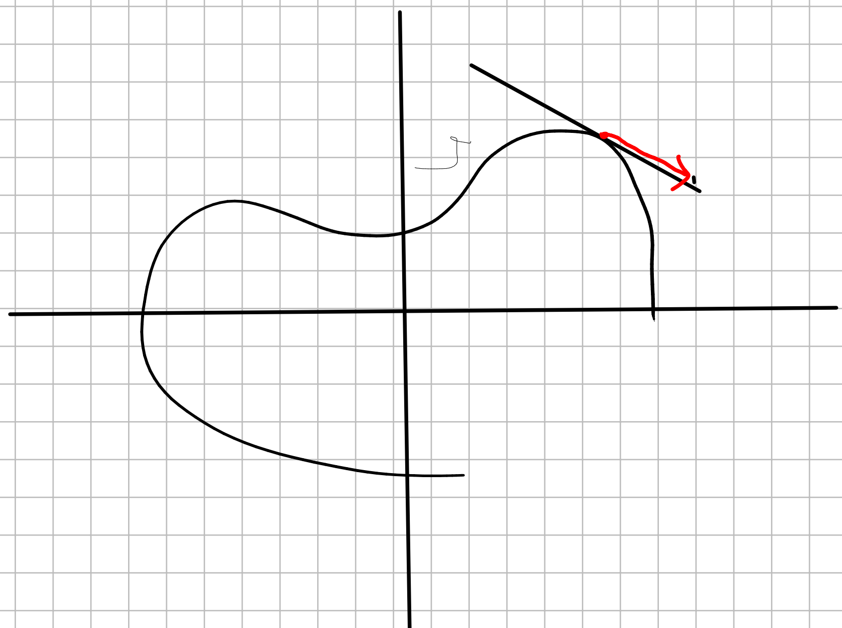
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \rightsquigarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{\frac{dr_2}{dt}}{\frac{dr_1}{dt}}}$$

slope of $\frac{d\vec{r}}{dt}$

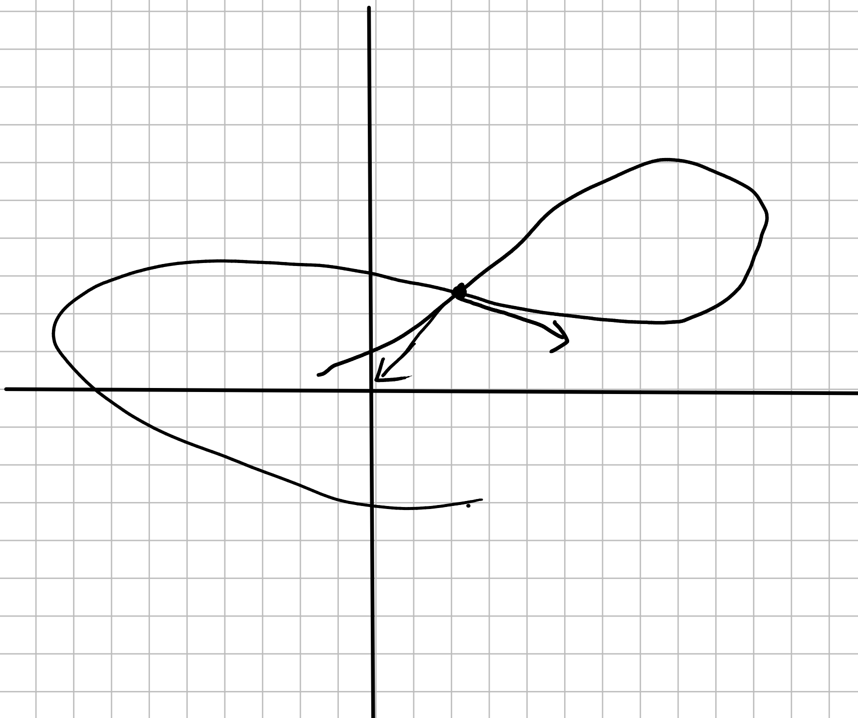
We find that

slope of $\frac{d\vec{r}}{dt}(c)$ is the slope of the tangent line to $\vec{r}(t)$ at c

$\leadsto$ Can view $\frac{d\vec{r}}{dt}(c)$ as a tangent vector to $\vec{r}$ at c.



<u>Warning</u>// Curve $\vec{r}(t)$ can hit same pt. multiple times w/ multiple different tangent vectors



Lecture 4a: Curves in $\mathbb{R}^3$

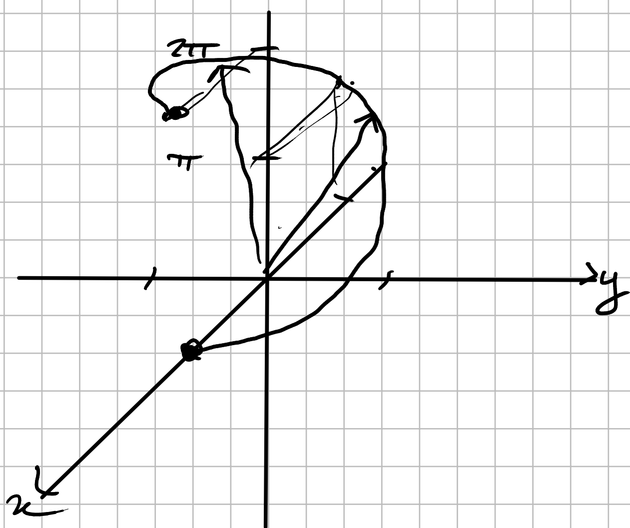
We call a fxn

$$\vec{r}: \mathbb{R} \longrightarrow \mathbb{R}^3 \qquad \left(\vec{r}: [a,b] \longrightarrow \mathbb{R}^3\right)$$

a <u>parametric curve</u> in $\mathbb{R}^3$ or a <u>vector fxn</u>. As in $\mathbb{R}^2$ view $\vec{r}(t)$ as position vectors of points in $\mathbb{R}^3$

<u>Eg</u> $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle \qquad t \in [0, 2\pi]$

<u>helix</u>



Defn// If $\vec{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$, $t \in [a, b]$ and r_1, r_2, & r_3 are continuous, we call the set

$$C = \{ (x,y,z) \in \mathbb{R}^3 \mid \exists t \in [a,b) \quad (x,y,z) = (r_1(t), r_2(t), r_3(t)) \}$$

a space curve.

Then the vector fun $\vec{r}(t)$ is called a parameterization (or parametric eqns) for the curve C.

Vector fun $\vec{r}(t)$	Curve C
"trajectory" it moves in time	"Geometric figure"

Warning: A curve C can have multiple different parameterizations.

Eg. $\vec{r}: [0, 2\pi] \longrightarrow \mathbb{R}^3$

$t \longmapsto \langle \cos(t), \sin(t), t \rangle$

helix

$\vec{s}: [0, \pi] \longrightarrow \mathbb{R}^3$

$t \longmapsto \langle \cos(2t), \sin(2t), 2t \rangle$

Both parametrize the helix but $\vec{s}$ traces through it twice as quickly.

As with param. curves in the plane, can define

$$\lim_{t \to c} \vec{r}(t) = \langle \lim_{t \to c} r_1(t), \lim_{t \to c} r_2(t), \lim_{t \to c} r_3(t) \rangle$$

and

$$\frac{d\vec{r}}{dt}(a) := \vec{r}\,'(a) := \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}$$

Lem For $\vec{r}(t)$ a vector fun,
$\vec{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$, then

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dr_1}{dt}, \frac{dr_2}{dt}, \frac{dr_3}{dt} \right.$$

Pf Same as in dim 2. $\square$

We again have 2 interpretations

- If view $\vec{r}(t)$ as trajectory of a particle $\rightsquigarrow$ $\frac{d\vec{r}}{dt}$ is velocity vector
- Geometrically ~~Pl~~: $\frac{d\vec{r}}{dt}$ is a tangent vector to the curve parametrized by $\vec{r}(t)$.

Defn// For $\vec{r}(t)$ a param. space curve the <u>unit tangent vector</u> at $t=a$ is

$$\vec{T}(a) := \frac{1}{|\vec{r}\,'(a)|}\vec{r}\,'(a)$$

It is defined when $|\vec{r}\,'(t)| \neq 0$

Eg// Helix $\vec{r} : [0, 2\pi] \longrightarrow \mathbb{R}^3$

$$t \longmapsto \langle \cos(t), \sin(t), t \rangle$$

$$\vec{r}(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

so $$|\vec{r}\,'(t)|^2 = \sin^2(t) + \cos^2(t) + 1$$

$$= 2$$

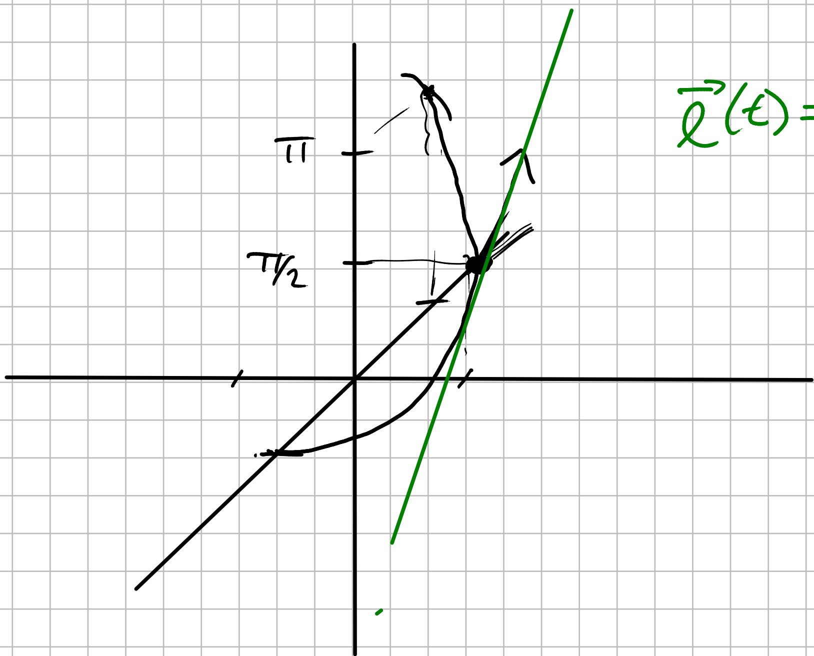
$$\Rightarrow |\vec{r}\,'(t)| = \sqrt{2}$$

$$\Rightarrow \vec{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1 \rangle$$

at $t = \frac{\pi}{2}$ can find eqn for tangent line

$$\vec{T}\left(\frac{\pi}{2}\right) = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{r}\left(\frac{\pi}{2}\right) = \left\langle 0, 1, \pi/2 \right\rangle$$



$$\vec{\ell}(t) = \left\langle -\frac{t}{\sqrt{2}}, 1, \frac{\pi}{2} + \frac{t}{\sqrt{2}} \right\rangle$$

Lecture 46, Computations,

Thm. $\vec{r}(t)$, $\vec{s}(t)$ vector fnns, $c \in \mathbb{R}$ scalar, $f(t)$ scalar-valued fnn.

- $\frac{d}{dt}(\vec{r}(t) + \vec{s}(t)) = \vec{r}\,'(t) + \vec{s}\,'(t)$
- $\frac{d}{dt}(c\vec{r}(t)) = c\,\vec{r}\,'(t)$

product rules

- $\frac{d}{dt}(f(t)\,\vec{r}(t)) = f'(t)\,\vec{r}(t) + f(t)\,\vec{r}\,'(t)$
- $\frac{d}{dt}(\vec{r}(t) \cdot \vec{s}(t)) = \vec{r}\,'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}\,'(t)$
- $\frac{d}{dt}(\vec{r}(t) \times \vec{s}(t)) = \vec{r}\,'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}\,'(t)$
- $\frac{d}{dt}(\vec{s}(f(t))) = f'(t)\,\vec{s}\,'(f(t))$ } chain rule

Defn// Let $\vec{r}(t)$ be a param of a space curve, and suppose $\vec{T}(t)$ is defined on $[a, b]$. The **unit normal** of $\vec{r}$ at t is

$$\vec{N}(t) := \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

Lem// $\vec{N}(t) \cdot \vec{T}(t) = 0$ whenever both are defined

Pf// It suffices to show

$$\vec{T}'(t) \cdot \vec{T}(t) = 0$$

$$|\vec{T}(t)|^2 = 1 \rightsquigarrow \frac{d}{dt} |\vec{T}(t)|^2 = 0$$

$$\frac{d}{dt}\left(\vec{T}(t) \cdot \vec{T}(t)\right) = \vec{T}'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}'(t)$$

$$= 2\vec{T}(t) \cdot \vec{T}'(t)$$

$$= \vec{T}(t) \cdot \vec{T}'(t) = 0$$

$\square$

Geometrically, $\vec{N}$ points "in the direction $\vec{r}(t)$ is sending"

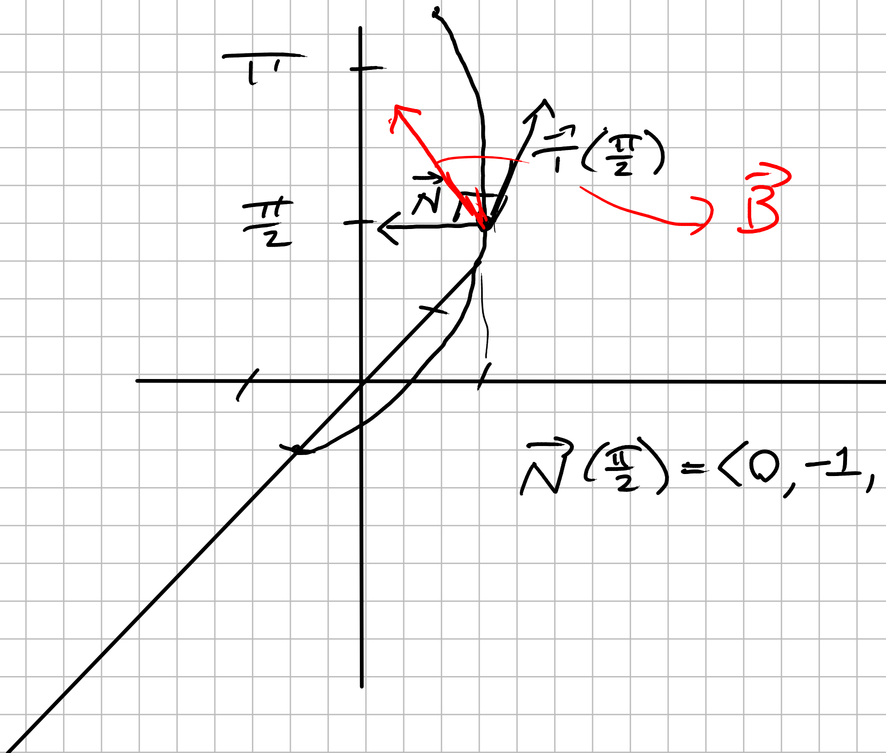
Eg// Helix $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$$\vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$$

$$\vec{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle$$

$$|\vec{T}'(t)| = \frac{1}{\sqrt{2}}$$

$$\vec{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$



$$\vec{N}\left(\tfrac{\pi}{2}\right) = \langle 0, -1, 0 \rangle$$

<u>Defn</u>// $\vec{r}(t)$ be a param. of a curve $\vec{T}(t)$ & $\vec{N}(t)$ both defined, we define the <u>Binormal vector</u>

$$\vec{B}(t) := \vec{T}(t) \times \vec{N}(t)$$

Together we $\vec{T}, \vec{N}, \vec{B}$ the <u>Frenet-Serre vectors</u>.

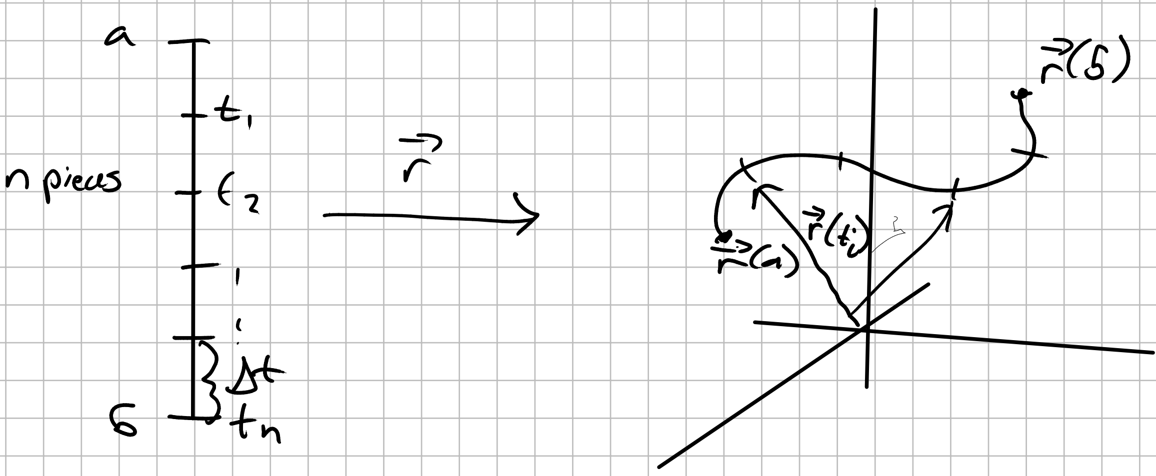
Lecture 5a: Integration, arclength

Given a vector fun

$$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$$

$$t \longmapsto \langle r_1(t), r_2(t), r_3(t) \rangle$$

We integrate as usual w/ Riemann sums
but w/ vector addition
Integrate $\vec{r}(t)$ from a to b.



Riemann sum:

$$\sum_{j=1}^{n} \vec{r}(t_j)\,\Delta t$$

Defn// The integral of $\vec{r}(t)$ from a to b is

$$\int_a^b \vec{r}(t)\,dt := \lim_{n\to\infty} \sum_{j=1}^{n} \vec{r}(t_j)\Delta t$$

$$= \left\langle \lim_{n\to\infty} \sum_{j=1}^{n} r_1(t_j)\Delta t,\ \lim_{n\to\infty} \sum_{j=1}^{n} r_2(t_j)\Delta t,\ \lim_{n\to\infty} \sum_{j=1}^{n} r_3(t_j)\Delta t \right\rangle$$

$$= \left(\int_a^b r_1(t)\,dt\right)\hat{\imath} + \left(\int_a^b r_2(t)\,dt\right)\hat{\jmath} + \left(\int_a^b r_3(t)\,dt\right)\hat{k}$$

<u>Fund Thm</u>// $\vec{R}(t)$ is antideriv. of $\vec{r}(t)$ (i.e. $\vec{R}'(t) = \vec{r}(t)$), then

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$$

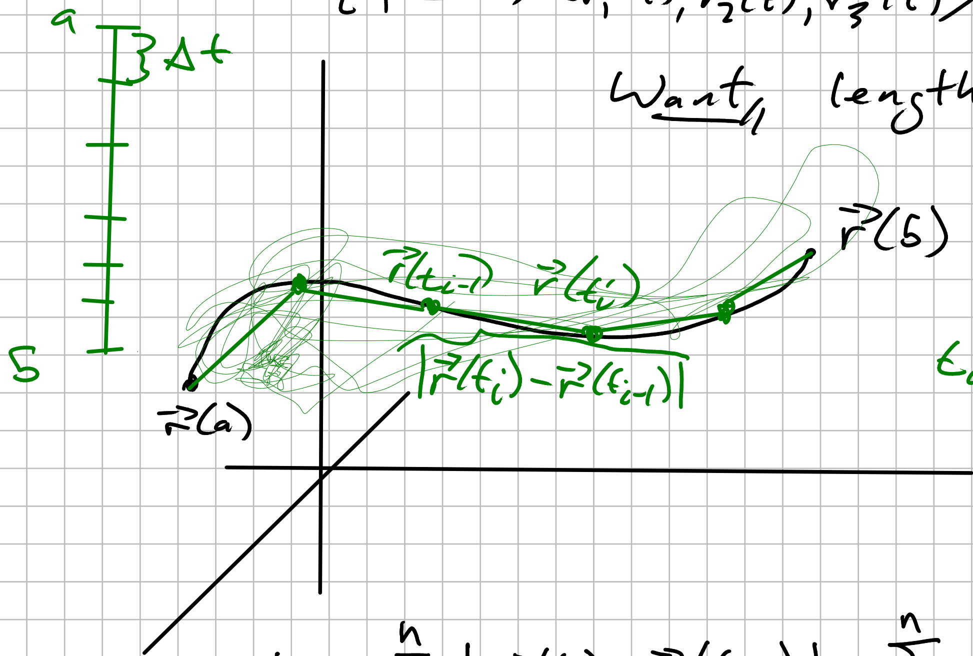
Arc length

Let C be the curve parametrized by

$$\vec{r} : [a, b] \longrightarrow \mathbb{R}^3$$

$$t \longmapsto \langle r_1(t), r_2(t), r_3(t) \rangle$$

Want length of curve



$t_i = t_{i-1} + \Delta t$

$$L \approx \sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})| = \sum_{i=1}^{n} |\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})|$$

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})|$$

$$\frac{\Delta t}{\Delta t} \times \rightsquigarrow$$

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \boxed{\frac{|\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})|}{\Delta t}} \Delta t$$

as $\Delta t \to 0$
$(n \to \infty)$

$$|\vec{r}\,'(t)|$$

$\Rightarrow$

$$\boxed{\begin{aligned} L &= \int_a^b |\vec{r}\,'(t)| \, dt \\ &= \int_a^b \sqrt{\left(\frac{dr_1}{dt}\right)^2 + \left(\frac{dr_2}{dt}\right)^2 + \left(\frac{dr_3}{dt}\right)^2} \, dt \end{aligned}}$$

Claim// If $f:[c,d]\to[a,b]$, is a fxn which is differentiable, onto, & $f'(s)>0$, then
$$\int_a^b |\vec{r}\,'(t)|\,dt = \int_c^d \left|\frac{d}{ds}(\vec{r}(f(s)))\right| ds$$

Pf// $$\int_c^d \left|\frac{d}{ds}(\vec{r}(f(s)))\right| ds$$
$$= \int_c^d |\vec{r}\,'(f(s))|\,|f'(s)|\,ds$$
$$= \int_c^d |\vec{r}\,'(f(s))|\,\overbrace{f'(s)\,ds}^{dt}$$
$t=f(s) \rightsquigarrow dt = f'(s)\,ds$
$$= \int_a^b |\vec{r}\,'(t)|\,dt \qquad \square$$

Thm// Arc length of a curve C does not depend on the parameterization of C

If we have C w/ param. $\vec{r}(t), a \le t \le b$
if $\vec{r}'(t)$ is cont.
we can define <u>arc length function</u>

$$\boxed{s(t) = \int_a^t |\vec{r}'(u)|\, du}$$

"length along C from $\vec{r}(a)$ to $\vec{r}(t)$"

Can solve $s = s(t)$ for t to get

$$t = t(s)$$

$\rightsquigarrow$ reparameterize,

$\vec{r}(t(s))$, the <u>parameterization of C by arc length</u>

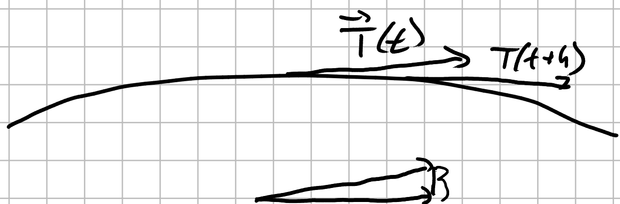
eg $\vec{r}(t(4))$ is the point in $\mathbb{R}^3$ on C 4 units of length along C.

Lecture 56// Curvature

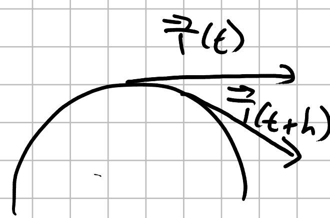
Want// A measure of "how much a curve C is curved at a point $\bar{p} \in C$" $\quad \vec{r}(t)$ a param of C

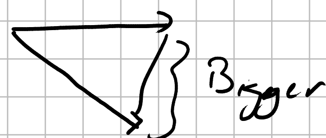
Idea// $\vec{T}(t)$ is the "direction of $\vec{r}(t)$ at time t". If $\vec{T}(t)$ changes rapidly then C must be more curved.

less curved



more curved





Problem// $\frac{d\vec{T}}{dt}$ depends on $\vec{r}(t)$

not really geometric.

Soln// Parameterize C by arclength so that $\vec{T}(s)$ depends on the arclength s.

Defn// The curvature of a curve C is

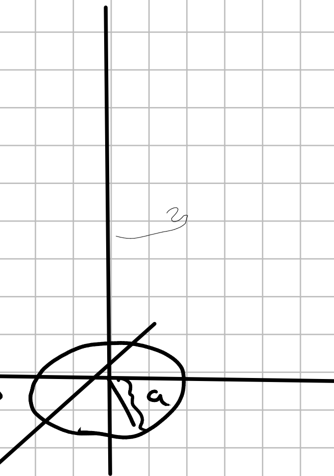
$$\kappa(s) := \left| \frac{d\vec{T}}{ds} \right|$$

Eg $\vec{r} = \langle a\cos(t), a\sin(t), 0\rangle \qquad 0 \le t < 2\pi$

circle in the plane $z=0$
radius $a>0$.

$$s(t) = \int_0^t a\,du = at$$

$$\rightsquigarrow \quad t(s) = \frac{s}{a}$$



Param by arclength

$$\vec{r}(s) = \vec{r}(t(s)) = \langle a\cos(\tfrac{s}{a}), a\sin(\tfrac{s}{a}), 0\rangle$$

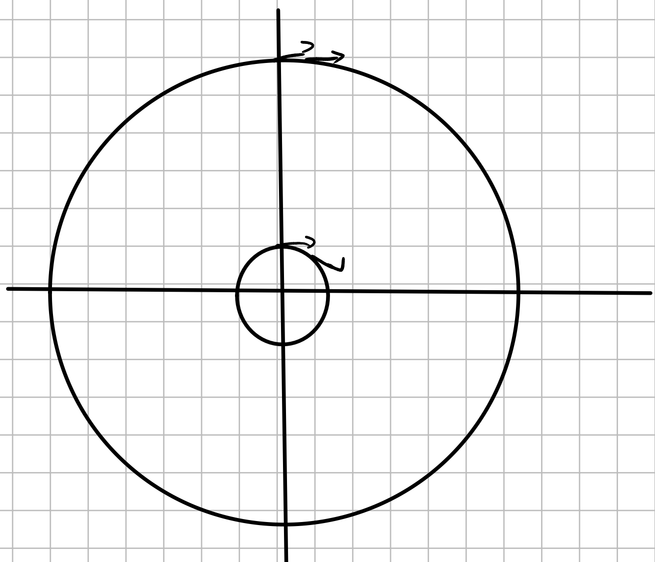
$$\vec{T}(s) = \langle -\sin(\tfrac{s}{a}), \cos(\tfrac{s}{a}), 0\rangle$$

$$\frac{d\vec{T}}{ds} = \left\langle -\frac{1}{a}\cos(\tfrac{s}{a}), -\frac{1}{a}\sin(\tfrac{s}{a}), 0\right\rangle$$

$\Rightarrow$

$$\kappa(s) = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{a}$$

Does this make sense?



yes smaller turning radius $\Rightarrow$ bigger curvature.

Eg// $\vec{r} = \langle a\cos(t), a\sin(t), t\rangle \qquad 0 \le t \le 2\pi$

$$\vec{r}\,'(t) = \langle -a\sin(t), a\cos(t), 1\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{a^2+1}$$

arc length

$$s(t) = \int_0^t \sqrt{a^2+1}\, du = \sqrt{a^2+1}\; t$$

$$t(s) = \frac{s}{\sqrt{a^2+1}}$$

Param. C by arclength

$$\vec{r}(t(s)) = \left\langle a\cos\left(\frac{s}{\sqrt{a^2+1}}\right),\ a\sin\left(\frac{s}{\sqrt{a^2+1}}\right),\ \frac{t}{\sqrt{a^2+1}} \right\rangle$$

$$\frac{d\vec{T}}{ds} = \left\langle -\frac{a}{a^2+1}\cos\left(\frac{s}{\sqrt{a^2+1}}\right),\ -\frac{a}{a^2+1}\sin\left(\frac{s}{\sqrt{a^2+1}}\right),\ 0 \right\rangle$$

$$K(s) = \left|\frac{d\vec{T}}{ds}\right| = \frac{a}{a^2+1}$$

Lecture 6: Computing curvature

C a curve in $\mathbb{R}^3$, param. by $\vec{r}(t)$

$a \leq t \leq b$

By def $\quad K(s) = \left| \dfrac{d\vec{T}}{ds} \right|$

$\vec{T}(s(t))$

chain rule $\rightsquigarrow$ $\quad \dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}}{ds} \dfrac{ds}{dt}$

$\dfrac{ds}{dt} \qquad\qquad s(t) = \int_a^t |\vec{r}\,'(u)| \, du$

Fund. Thm $\rightsquigarrow$ $\dfrac{ds}{dt} = |\vec{r}\,'(t)|$ $\quad \dfrac{ds}{dt}$ is speed of $\vec{r}(t)$!

$$\boxed{\frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \left|\frac{d\vec{T}}{ds}\right| = K(s)}$$

$$\rightsquigarrow \quad \kappa(s) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

Thm// The curvature of C is

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Pf// $$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} = \frac{\vec{r}'}{\frac{ds}{dt}}$$

$$\rightsquigarrow \quad \vec{r}' = \frac{ds}{dt}\vec{T} \quad \leftarrow$$

Chain rule

$$\vec{r}''(t) = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}' \quad \leftarrow$$

$$\rightsquigarrow \quad \vec{r}' \times \vec{r}'' = \frac{d^2s}{dt^2}\cancelto{0}{\vec{T}\times\vec{T}}\,\frac{ds}{dt} + \left(\frac{ds}{dt}\right)^2 \vec{T}\times\vec{T}'$$

$$= \left(\frac{ds}{dt}\right)^2 \vec{T}\times\vec{T}'$$

$$\vec{r}\,' \times \vec{r}\,'' = \left(\frac{ds}{dt}\right)^2 \vec{T} \times \vec{T}\,' \qquad |\vec{r}\,'(t)|^2$$

$$|\vec{r}\,' \times \vec{r}\,''| = \left(\frac{ds}{dt}\right)^2 \cancel{|\vec{T}|} \times |\vec{T}\,'| = \overbrace{\left(\frac{ds}{dt}\right)^2} |\vec{T}\,'|$$

divide by $|\vec{r}\,'(t)|^3$

$$\kappa(t) = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} \qquad \square$$

Eg// Helix $a > 0$, $\vec{r}(t) = \langle a\cos(t), a\sin(t), t \rangle$

$$\vec{r}\,'(t) = \langle -a\sin(t), a\cos(t), 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{a^2 + 1}$$

$$\vec{r}\,''(t) = \langle -a\cos(t), -a\sin(t), 0 \rangle$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin(t) & a\cos(t) & 1 \\ -a\cos(t) & -a\sin(t) & 0 \end{vmatrix}$$

$$= a\sin(t)\,\hat{i} - a\cos(t)\,\hat{k} + (\underbrace{a^2\sin^2(t) + a^2\cos^2(t)}_{a^2})\,\hat{k}$$

$$|\vec{r}\,' \times \vec{r}\,''| = \sqrt{a^2\sin^2(t) + a^2\cos^2(t) + a^4}$$

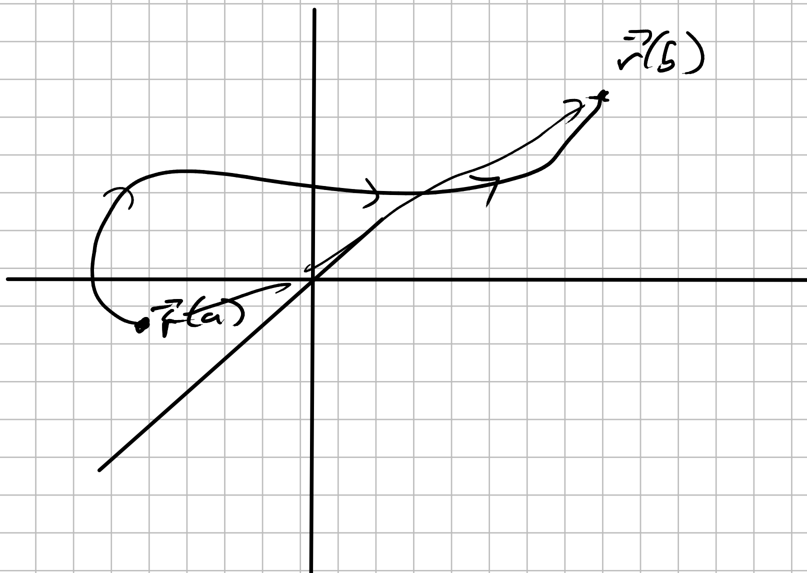
$$= \sqrt{a^2 + a^4} = a\sqrt{1+a^2}$$

$$\boxed{K(t) = \frac{a\sqrt{\cancel{1+a^2}}}{(\sqrt{a^2+1})^{\cancel{3}}} = \frac{a}{a^2+1}}$$

Lecture 6b: Recap

Vector fns

$$\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$$
$$\underset{\text{parameter}}{t} \longmapsto \langle r_1(t), r_2(t), r_3(t) \rangle$$



limits

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} r_1(t), \lim_{t\to a} r_2(t), \lim_{t\to a} r_3(t) \rangle$$

Cont. at a if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

Derivatives

$$\frac{d}{dt} \vec{r}(t) = \langle r_1'(t), r_2'(t), r_3'(t) \rangle$$

- $\frac{d\vec{r}}{dt}$ ⇝ velocity vector
- $\frac{d^2\vec{r}}{dt^2}$ ⇝ acceleration vector

Integrals

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b r_1(t)\,dt\right)\hat{\imath} + \left(\int_a^b r_2(t)\,dt\right)\hat{\jmath} + \left(\int_a^b r_3(t)\,dt\right)\hat{k}$$

Frenet–Serret vectors

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ unit tangent

$\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$ unit normal

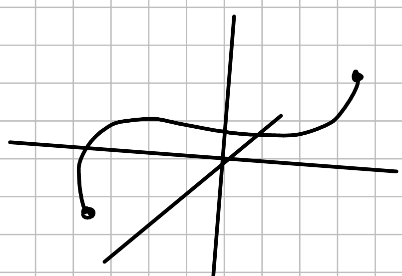
$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ unit Binormal

Curves

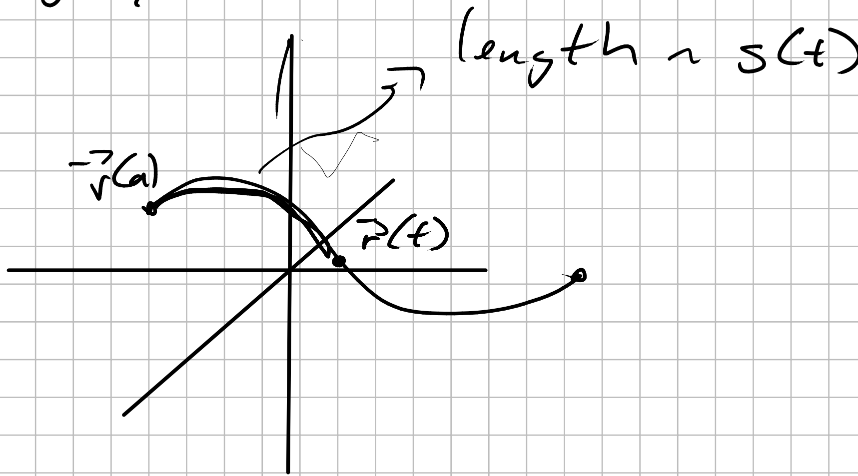
Curve parametrized by $\vec{r}(t)$ $a \le t \le b$

$C = \{(x,y,z) \in \mathbb{R}^3 \mid$ there is some $t \in [a,b]$ with $\vec{r}(t) = (x,y,z)\}$

"set of points ~~hit~~ by $\vec{r}(t)$



Arclength



$$s(t) = \int_a^t |\vec{r}\,'(t)|\, dt$$

Parameterization by arc length

$$s = s(t) \overset{\text{solve}}{\rightsquigarrow} t = t(s)$$

$$\vec{v}(s) = \vec{r}(t(s)) \qquad \underline{\text{param by arclength}}$$

showed in classwork

$$\frac{d\vec{r}}{ds} = \vec{T}(s)$$

Curvature

Defn: $\kappa(s) = \left| \dfrac{d\vec{T}}{ds} \right|$

can compute

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Lecture 7a

Curves from functions of two variables

Let $f(x,y)$ be a fun on a domain $D \subset \mathbb{R}^2$, and let

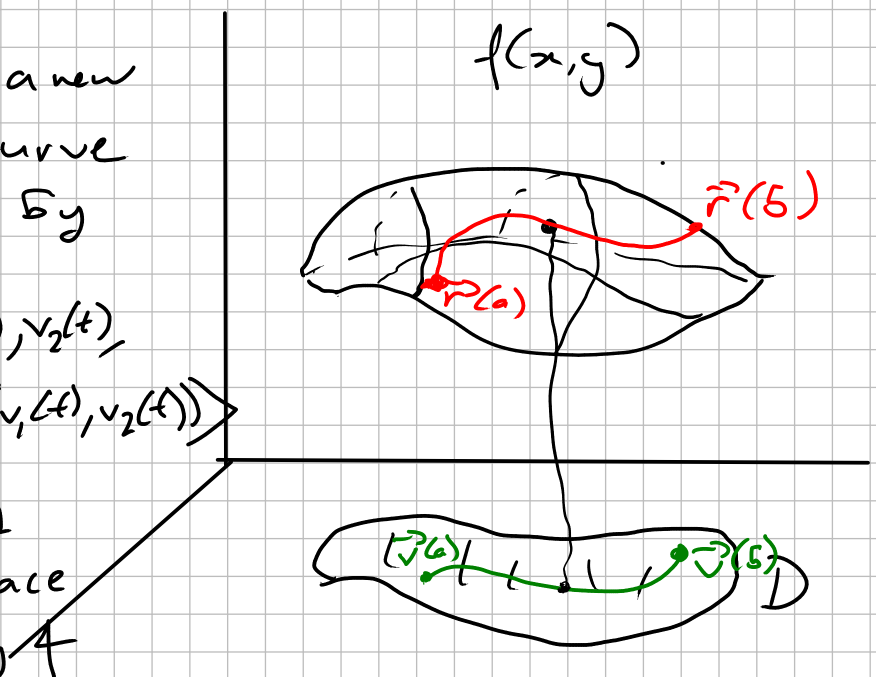
$$\vec{v} : [a,b] \longrightarrow D \subset \mathbb{R} \quad \text{be}$$
$$t \longmapsto \langle v_1(t), v_2(t) \rangle$$

a parameterized curve in x-y plane

<u>Define</u> a new param. curve in $\mathbb{R}^3$ by

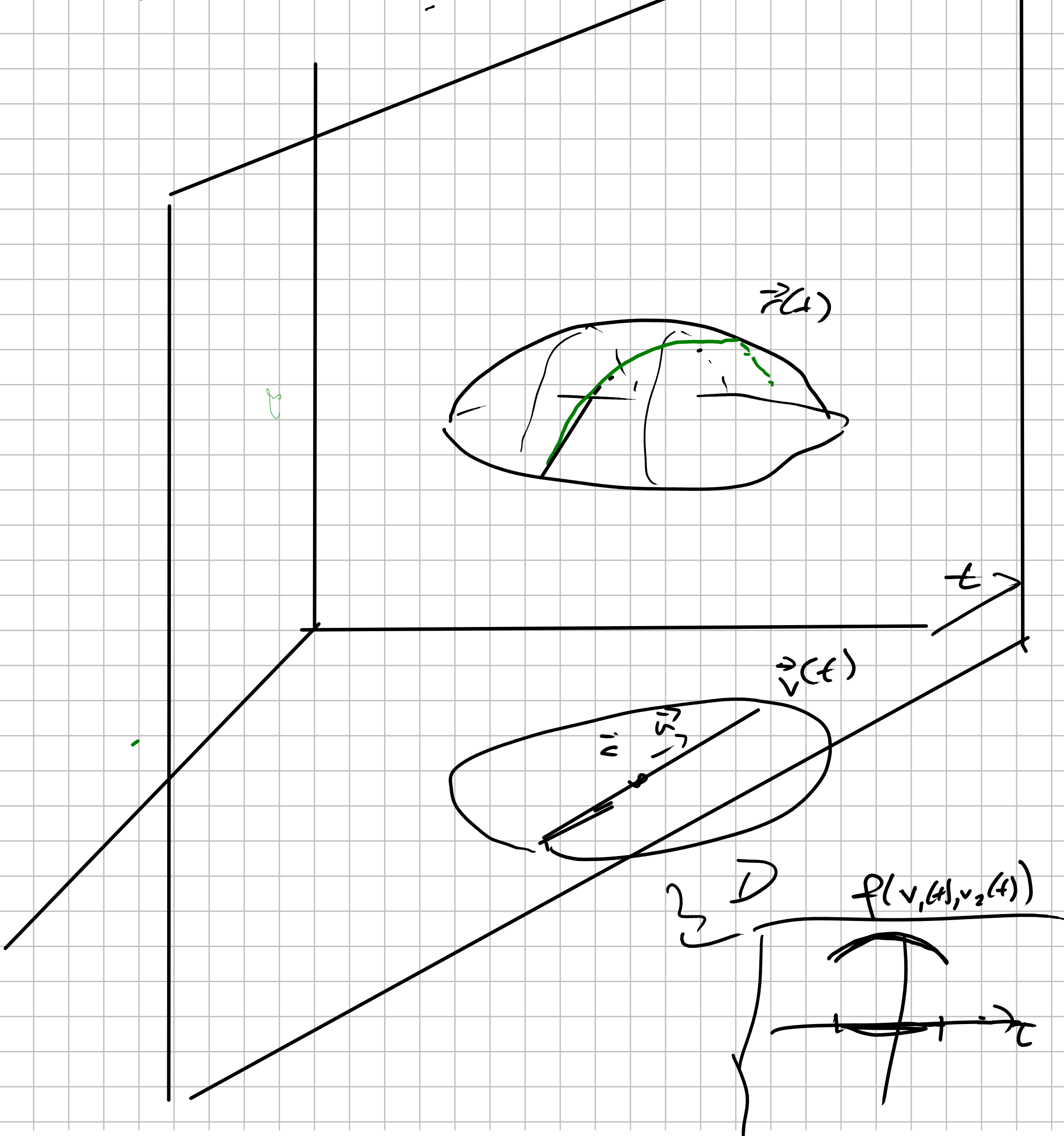
$$\vec{r}(t) = \langle v_1(t), v_2(t), f(v_1(t), v_2(t)) \rangle$$

curve <u>on</u> the surface defnd by f

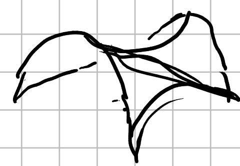


Special case: line //

$$\vec{v}(t) = \vec{c} + t\,\vec{u}$$



Eg, $f(x,y) = x^2 - y^2$



- $\vec{v}(t) = \langle 1, 1 \rangle + t \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

$$f(v_1(t), v_2(t)) = \left(\frac{1}{\sqrt{2}} t + 1\right)^2 - \left(-\frac{1}{\sqrt{2}} t + 1\right)^2$$

$$= \cancel{\frac{t^2}{2} - \frac{t^2}{2}} + \sqrt{2}\, t + \sqrt{2}\, t \cancel{+1 - 1}$$

$$= 2\sqrt{2}\, t$$

$\rightsquigarrow$ curve in the graph of $f(x,y)$

$$\vec{r}(t) = \langle 1, 1, 0 \rangle + t \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 2\sqrt{2} \rangle$$

$\rightarrow$ Line!

$$\vec{r}(t) = \langle 1, 1\rangle + t\langle 1, 0\rangle$$

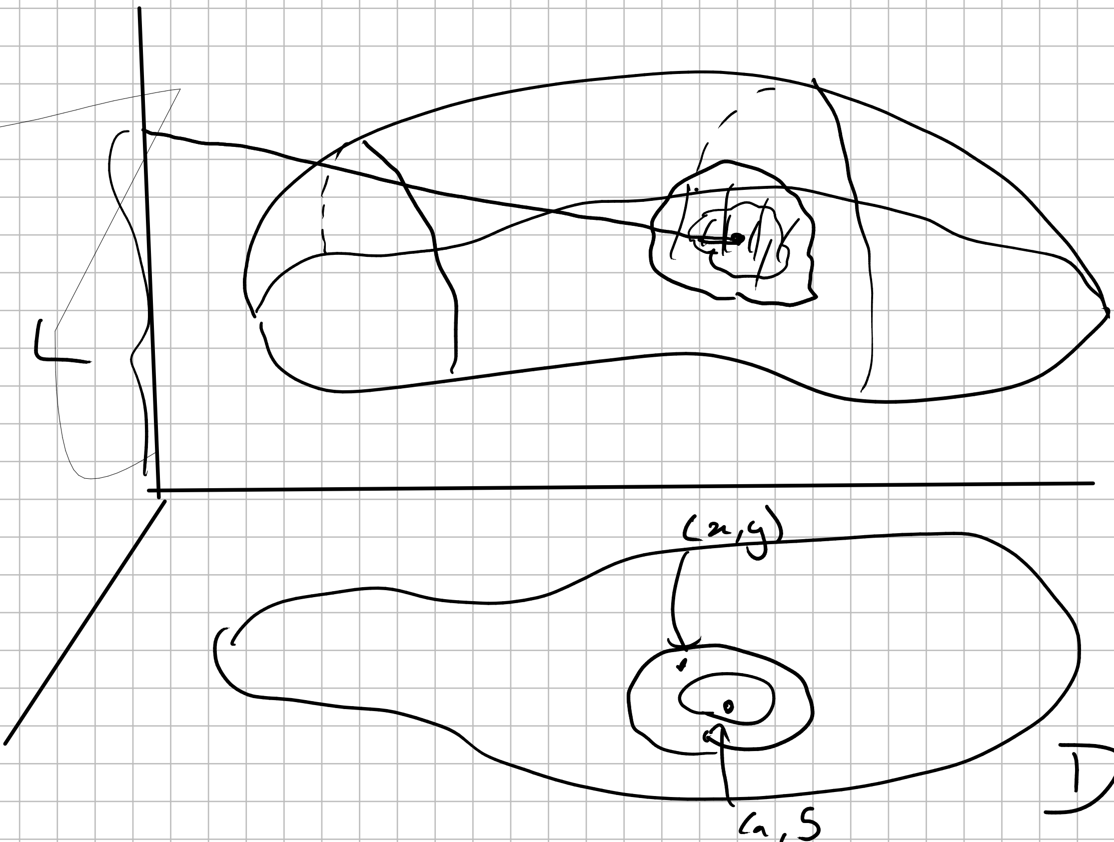
$$f(v_1(t), v_2(t)) = (t+1)^2 - 1^2$$

$$= t^2 + 2t \cancel{+1 - 1} = t^2 + 2t$$

$$\boxed{\vec{r}(t) = \langle 1+t, 1, t^2+2t\rangle}$$

Lecture 75: Limits & continuity

Let $f: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$



Idea (of limits) if (x,y) arbitrarily close to (a,b), $f(x,y)$ gets closer to $L \in \mathbb{R}$

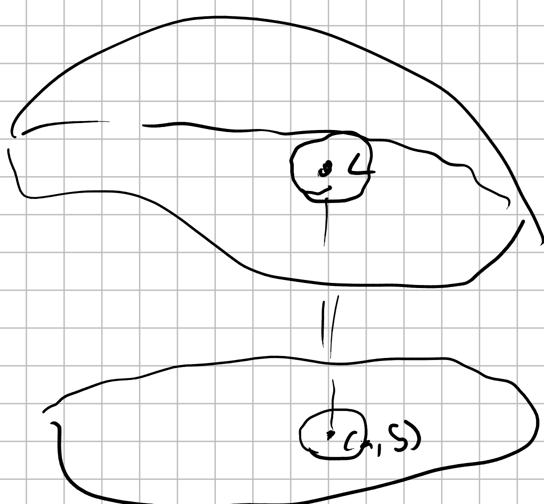
Formal Defn // The limit of f as (x,y) approaches (a,b) is $L \in \mathbb{R}$ if

For every $\varepsilon > 0$, $\exists\ \delta > 0$ such that

If $(x,y) \in D$ and $0 < d((x,y),(a,b)) < \delta$ then $|f(x,y) - L| < \varepsilon$

We write

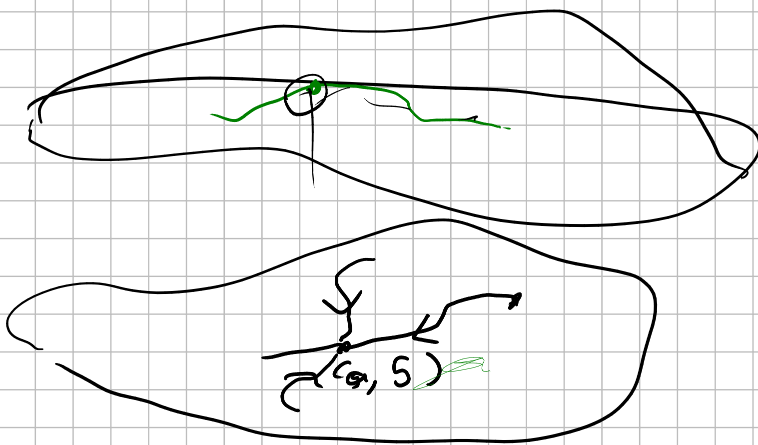
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$



Want $f(x,y)$ is within ε of L

choose small enough disk (radius δ) so that value of f on that disk are within ε of L

Slogan: "No matter which direction you approach (a,b) from, $f(x,y)$ always approaches L"



Let $\vec{r}: [p, q] \longrightarrow D$ continuous param. curve say $\vec{r}(c) = (a,b)$

If $\lim_{(x,y)\to(a,b)} f(x,y) = L$

then

$\lim_{t\to c} f(r_1(t), r_2(t))$ exists &

$\lim_{t\to c} f(r_1(t), r_2(t)) = L$

Every curve must yield the same limit.

Eg// $f(x,y) = \dfrac{y^3}{x^2+y^3}$

- $\vec{r}(t) = \langle t, 0 \rangle \rightsquigarrow \lim_{t\to 0} f(t,0) = 0$

- $\vec{r}(t) = \langle 0, t \rangle$

$$\lim_{t\to 0} f(0,t) = \lim_{t\to 0} \frac{t^3}{t^3} = 1$$

$\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{y^3}{x^2+y^3}$ does not exist

---

Defn// a function $f : D \longrightarrow \mathbb{R}$ of 2-vars is <u>continuous at $(a,b) \in D$</u> if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is <u>continuous on D</u> if it is continuous at every point in D

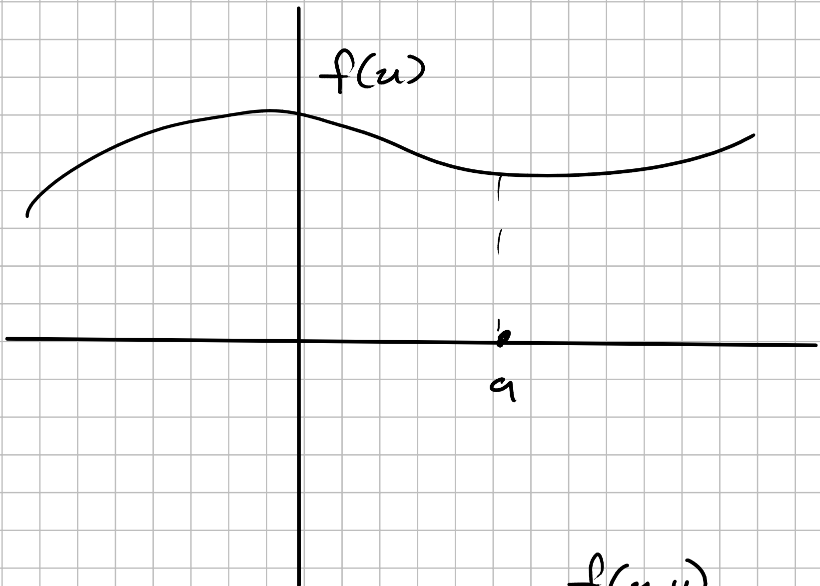
Rmk//
- every polynomial in x, y is cont. on $\mathbb{R}^2$
- Every rational fxn (ie ratio of polynomials

eg $\frac{y^3}{x^2+y^3}$) is continuous where the denominator is non-zero

[<u>Rmk</u>// For 3-var funs $g(x,y,z) \rightsquigarrow$ same basic defns]

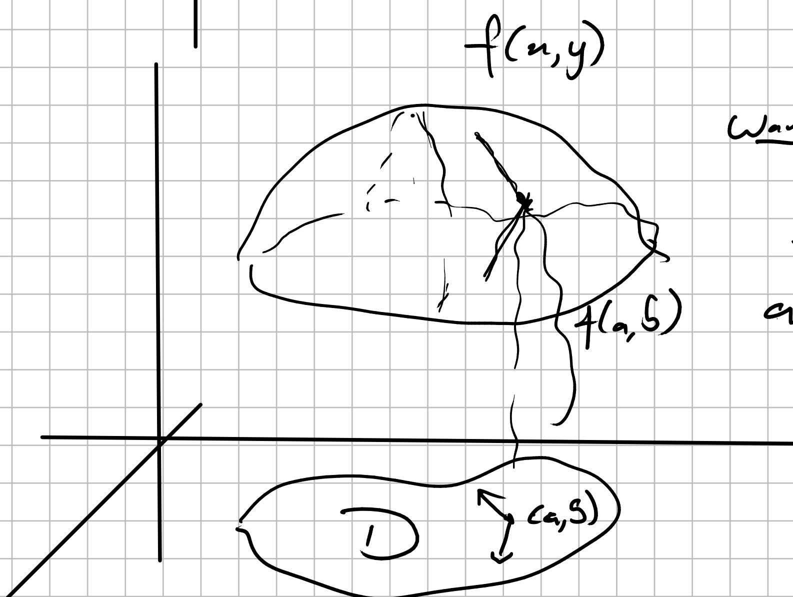
Lecture 8a

Want: Derivatives of 2-var. fns.



$\frac{df}{dx}(a)$ = rate of change of $f(x)$ at a.

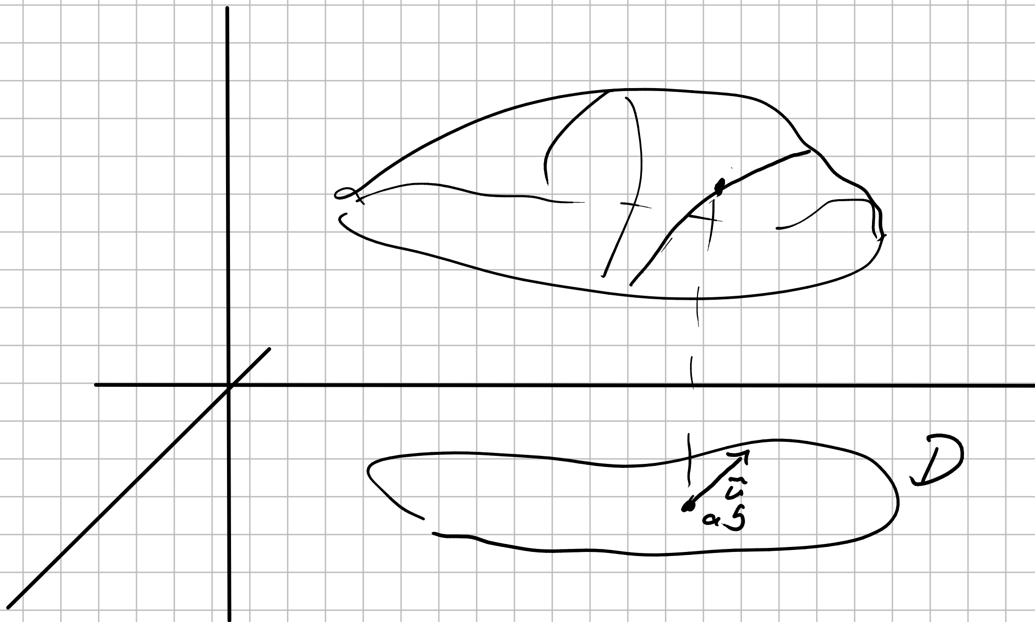
"rate of change of height at a"



Want: "rate of change of height at (a,b)"

Pick a direction

$\hat{u} = \langle u_1, u_2 \rangle$ unit vector $|\hat{u}| = 1$



$\rightsquigarrow$ line through (a, b) in the $\hat{u}$-dir.

$$\vec{l}(t) = \langle a, b \rangle + t\hat{u} = \langle a + tu_1, b + tu_2 \rangle$$

Height along $\vec{\ell}(t)$:

$f(\vec{\ell}(t))$ $\qquad t=0, \quad \vec{\ell}(t)=\langle a,b\rangle$

" the rate of change of $f(x,y)$ in the $\hat{u}$-direction at $\langle a,b\rangle$ is the rate of change of $f(\vec{\ell}(t))$ at $t=0$"

Defn// The <u>directional derivative</u> of f in the $\hat{u}$-direction at (a,b) in the domain of f is

$$(D_{\hat{u}} f)(a,b) := \left(\frac{d}{dt} f(\vec{\ell}(t))\right)(0)$$

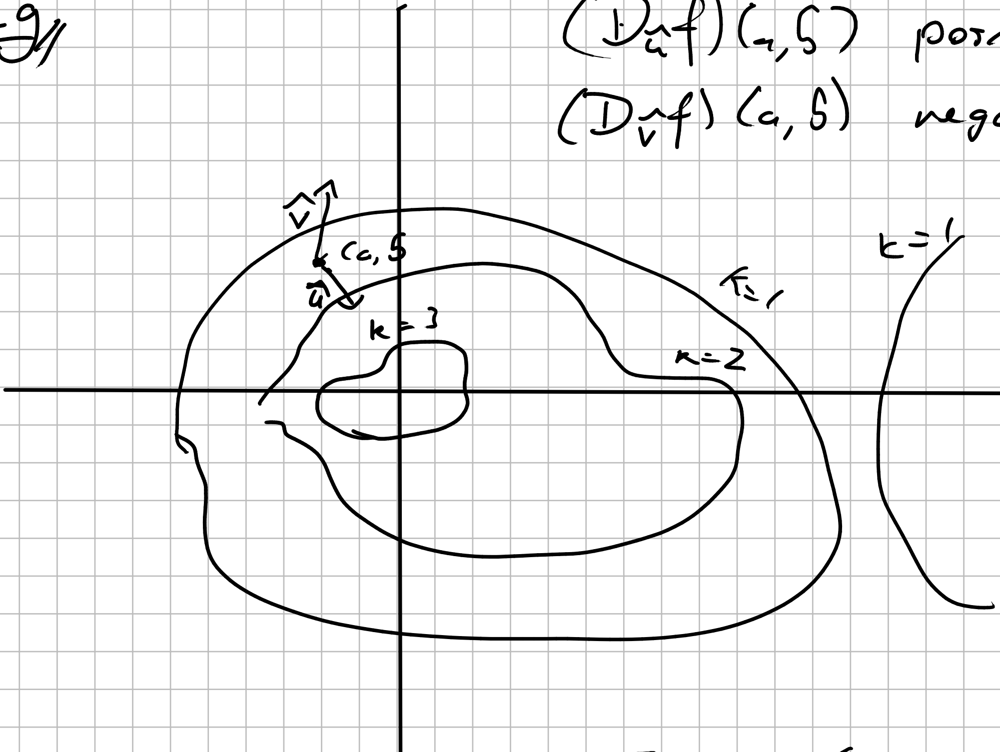
$$(D_{\hat{u}}(f))\langle a,b\rangle = \lim_{h\to 0} \frac{f(\vec{\ell}(0+h)) - f(\vec{\ell}(0))}{h}$$

$$\boxed{= \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}}$$

Eg//

$(D_{\hat{u}} f)(a,b)$ positive

$(D_{\hat{v}} f)(a,b)$ negative.



Eg// $f(x,y) = \sqrt{1-x^2-y^2}$ $\quad D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

at $(\frac{1}{2}, 0)$ in $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ - dire

$$\vec{\ell}(t) = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}t, \frac{1}{\sqrt{2}}t\right)$$

$$f(\vec{\ell}(t)) = \sqrt{1 - \left(\frac{1}{4} + \frac{1}{\sqrt{2}}t + \frac{1}{2}t^2\right) - \frac{1}{2}t^2}$$

$$f(\vec{\ell}(t)) = \sqrt{1 - \left(\tfrac{1}{4} + \tfrac{1}{\sqrt{2}}t + \tfrac{1}{2}t^2\right) - \tfrac{1}{2}t^2}$$

$$= \sqrt{\tfrac{3}{4} - \tfrac{1}{\sqrt{2}}t - t^2}$$

$$\frac{d}{dt}\left(f(\vec{\ell}(t))\right) = \frac{1}{2}\frac{1}{\sqrt{\tfrac{3}{4} - \tfrac{1}{\sqrt{2}}t - t^2}}\left(-2t - \tfrac{1}{\sqrt{2}}\right)$$

evaluate at $t=0$ $$= \frac{1}{2}\frac{1}{\sqrt{\tfrac{3}{4}}}\left(-\tfrac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\cancel{2}}\sqrt{\frac{\cancel{4}}{3}}\left(-\tfrac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{6}}$$

$$\left(D_{\hat{u}}(f)\right)\left(\tfrac{1}{2}, 0\right) = -\frac{1}{\sqrt{6}}.$$

Lecture 8b

Special case of directional derivatives

$u_1 = \langle 1, 0 \rangle \rightsquigarrow$ x-direction

$u_2 = \langle 0, 1 \rangle \rightsquigarrow$ y-direction

Defn. $f(x,y)$ be a 2-var fxn

the partial derivative in the x-direction at (a,b) is

$$f_x(a,b) = \left(\frac{\partial}{\partial x} f\right)(a,b) := \left(D_{\langle 1,0\rangle} f\right)(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

partial deriv. in the y-dir.

$$f_y(a,b) = \left(\frac{\partial}{\partial y} f\right)(a,b) = \left(D_{\langle 0,1\rangle} f\right)(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

Computation: to take $\frac{\partial}{\partial x}f$, take usual x-deriv, treating y as a constant.

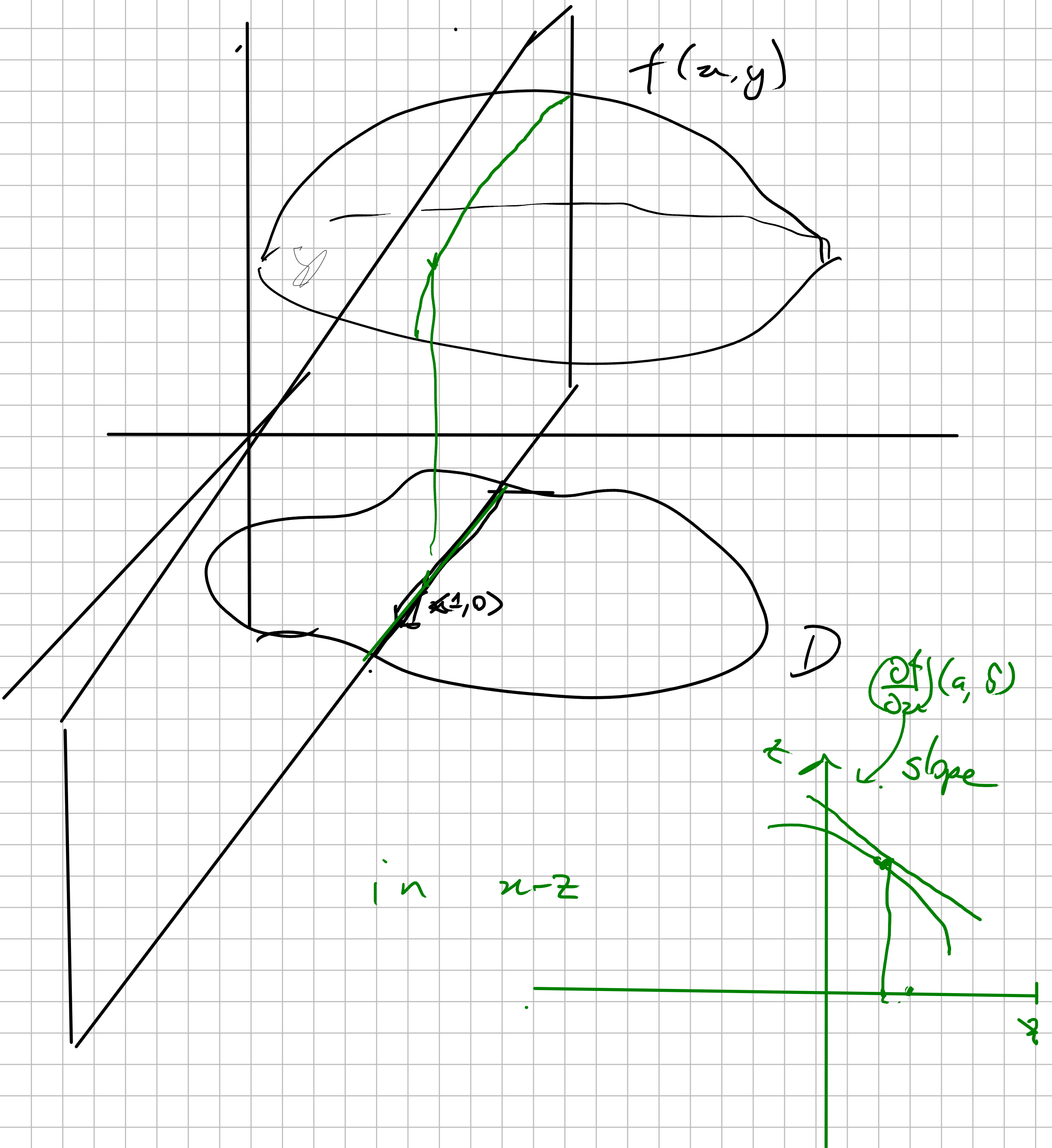
Eg// $f(x,y) = 3x^2 + y^4$

$f_x(x,y) := \frac{\partial f}{\partial x} = 6x \qquad f_y(x,y) = \frac{\partial f}{\partial y} = 4y^3$

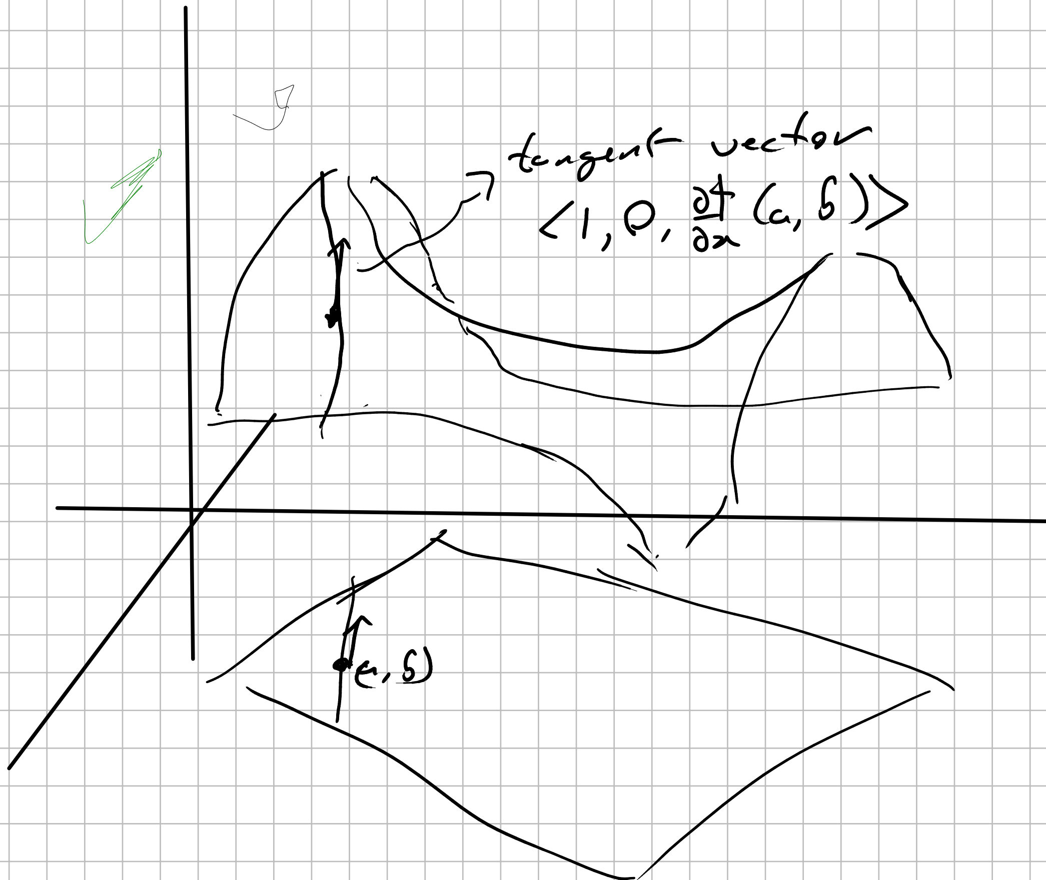
Eg// $f(x,y) = xy^3 - x^2y^2$

$f_x(x,y) = \frac{\partial f}{\partial x} = y^3 - 2xy^2$

$f_y(x,y) = \frac{\partial f}{\partial y} = 3xy^2 - 2x^2y$



f(x,y)
y
(1,0)
D
∂f/∂x (a,b)
slope
z
x
in x-z


tangent vector
$\langle 1, 0, \frac{\partial f}{\partial x}(a, b) \rangle$
(a, b)

Notations for higher derivs

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2 f}{\partial x^2}$$

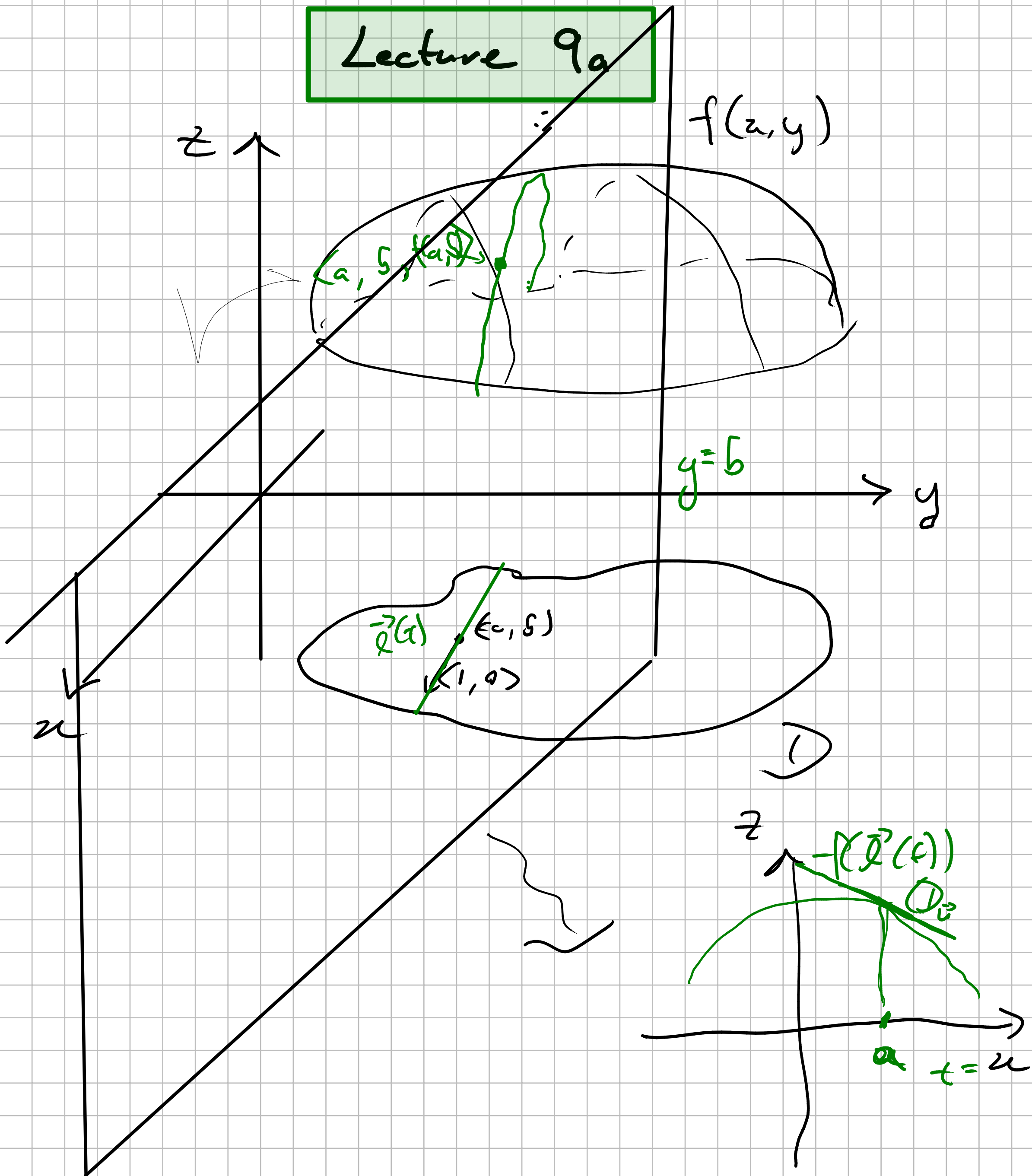
$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2 f}{\partial y \, \partial x}$$

Eg $f(x,y) = x^2 y^3$

$$f_y(x,y) = 3x^2y^2$$

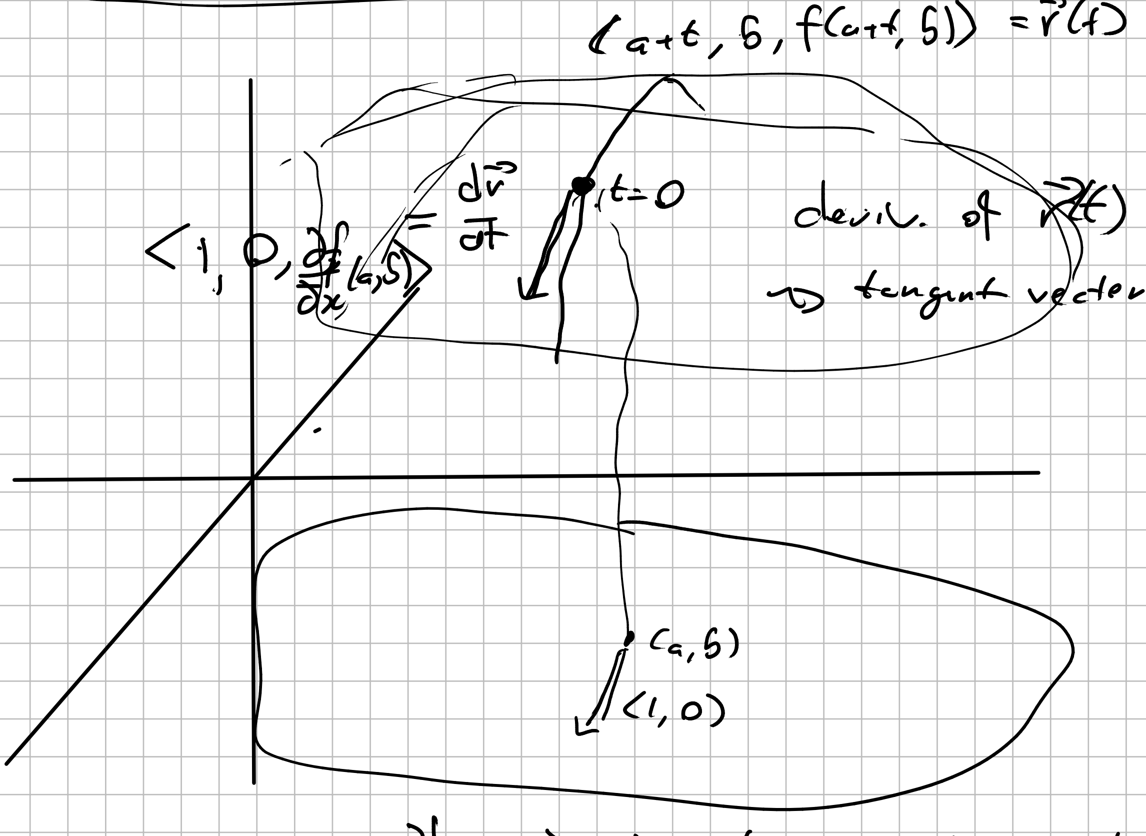
$$f_{yx}(x,y) = 6xy^2 \, .$$

Lecture 9a



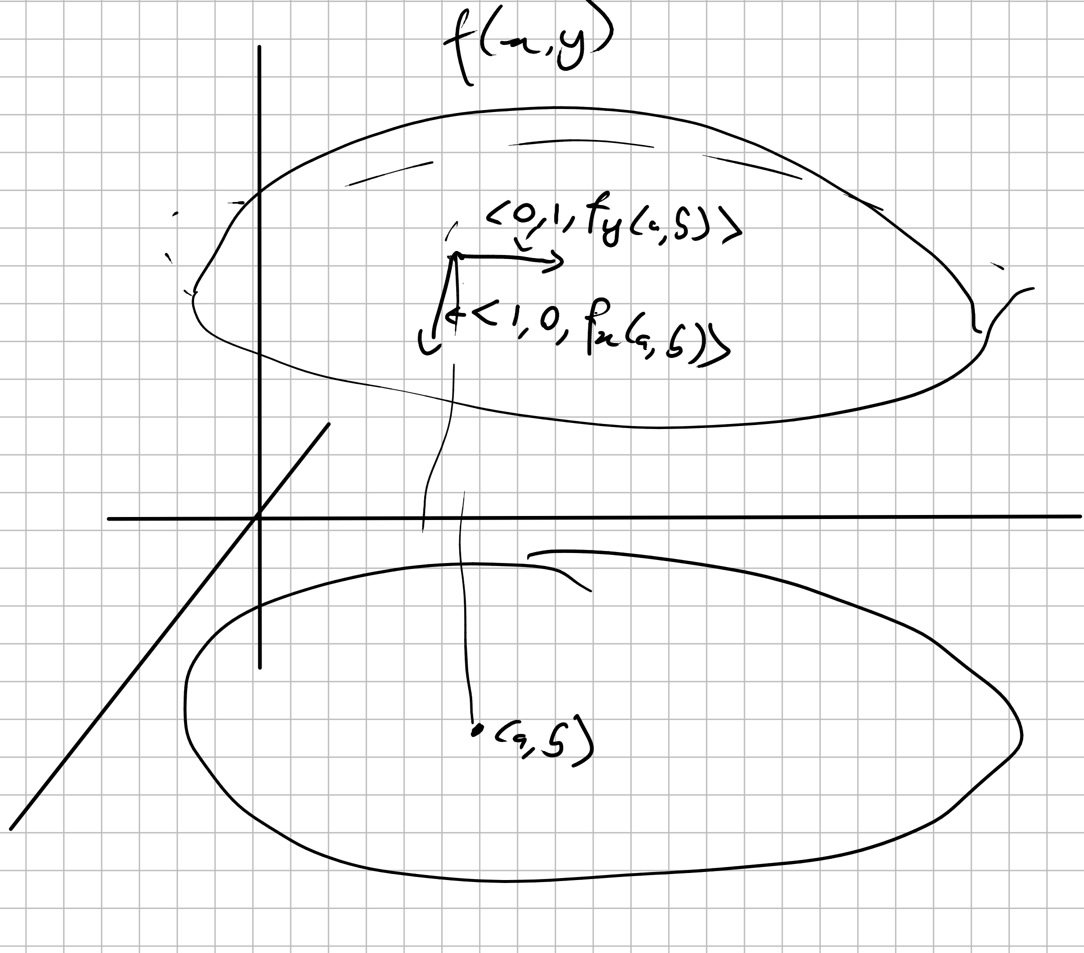
Slope of tangent line in $y=5$ plane to $f(\vec{\ell}(x))$ at $x=a$

is $(D_{\langle 1,0\rangle} f)(a,5) = \left(\frac{\partial}{\partial x} f\right)(a,5)$



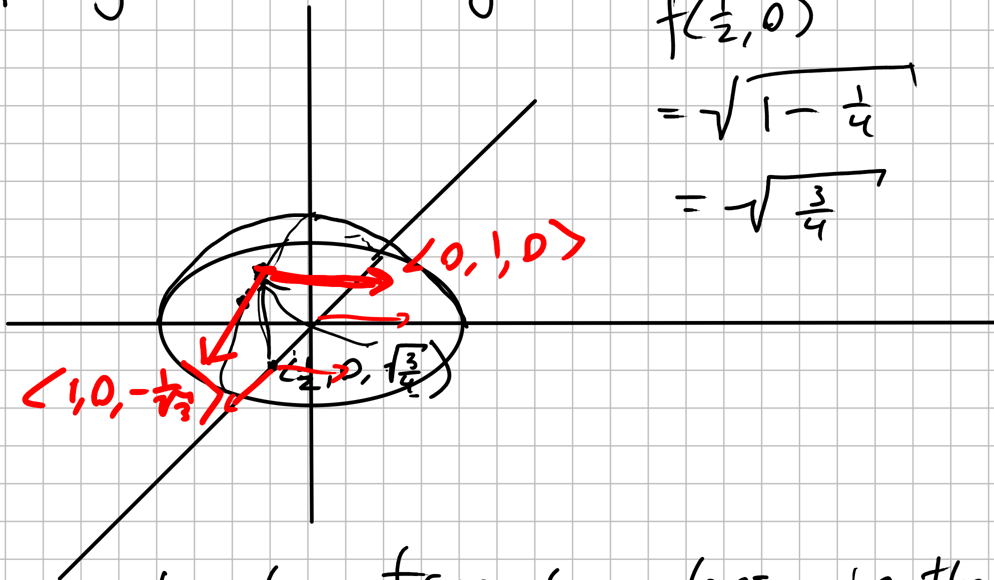
$\leadsto \langle 1, 0, \frac{\partial f}{\partial x}(a,5)\rangle$ is tangent to graph of $f(x,y)$ at $(a, 5, f(a,5))$

"partial derivatives are third component to a tangent vector to graph of $f(x,y)$ at a given pt.



$$f(x,y) = -\sqrt{1-x^2-y^2}$$

$$f(\tfrac{1}{2},0) = -\sqrt{1-\tfrac{1}{4}} = -\sqrt{\tfrac{3}{4}}$$



Want tangent vectors in the x & y-directions

$$\frac{df}{dx} = \frac{1}{2}\,\frac{1}{\sqrt{1-x^2-y^2}}\,(-2x)$$

$$f_x(\tfrac{1}{2},0) = \frac{1}{2}\,\frac{1}{\sqrt{\frac{3}{4}}}\,(-1) = -\frac{1}{\sqrt{3}}$$

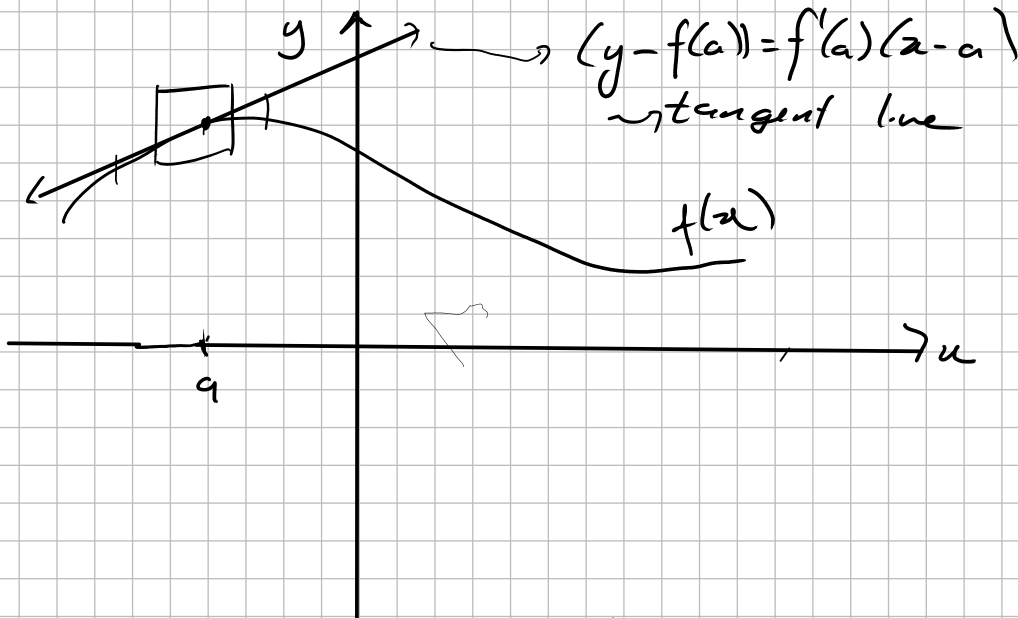
$$\frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{1}{\sqrt{1-x^2-y^2}}\,(-2y) = 0$$

x-dir tangent vector $\langle 1, 0, -\frac{1}{\sqrt{3}} \rangle$

y-dir —— " —— $\langle 0, 1, 0 \rangle$

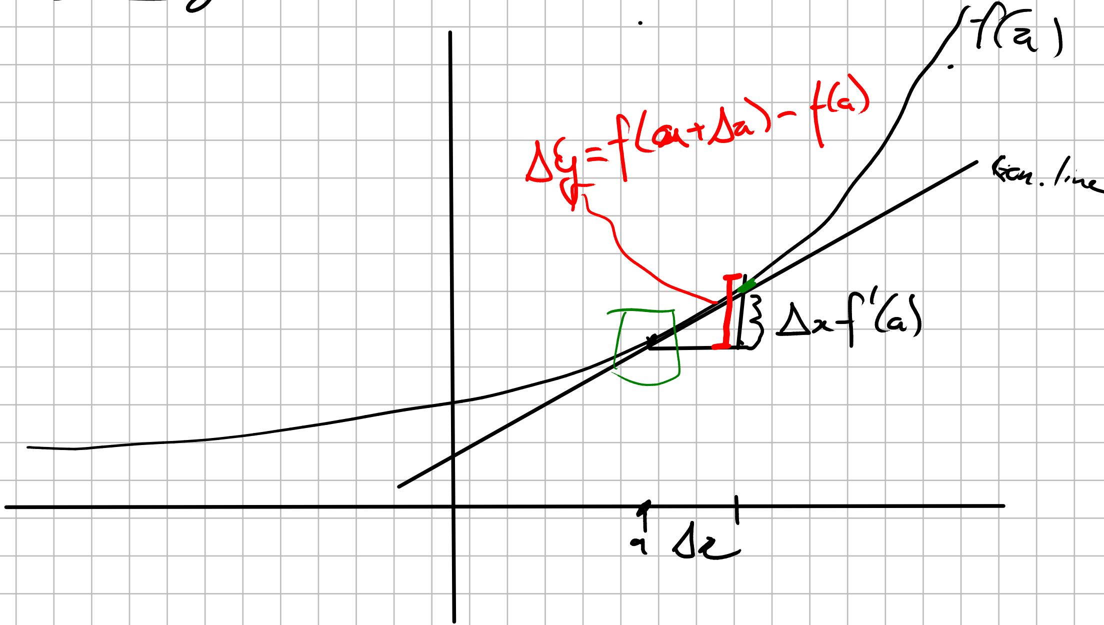
Lecture 9.5

- 1-var fun $f(x)$



We say this a <u>linear approximation</u> to $f(x)$ at a

Formally

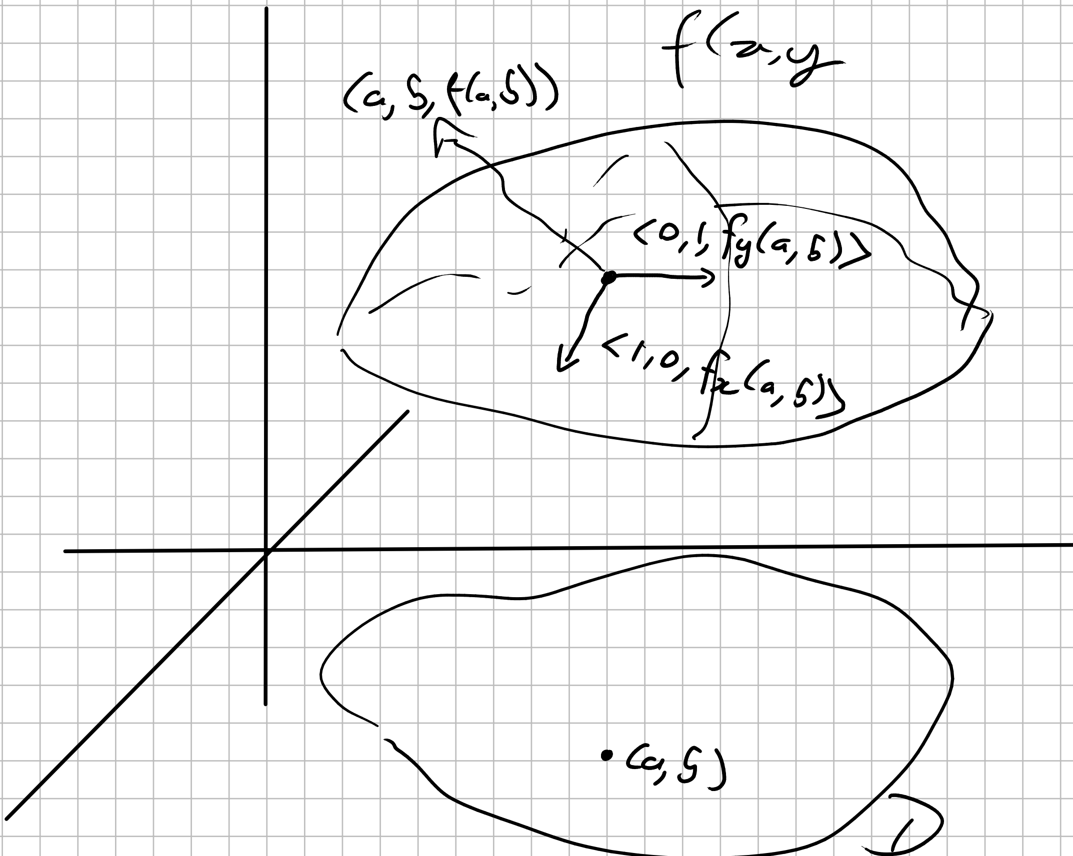


$$\Delta y = \Delta x f'(a) + \varepsilon \Delta x$$

$\rightsquigarrow$ ε as a fcn of Δx

$$\lim_{\Delta x \to 0} \varepsilon = 0$$

2-variables $f(x,y)$



two tangent vectors to graph at $(a, b, f(a,b))$

$\rightsquigarrow$ Write equation of the plane

Tangent plane

$$\langle 1, 0, f_x(a,b)\rangle \times \langle 0, 1, f_y(a,b)\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} = \underbrace{-f_x(a,b)\hat{i} - f_y(a,b)\hat{j} + \hat{k}}_{\text{normal vector}}$$

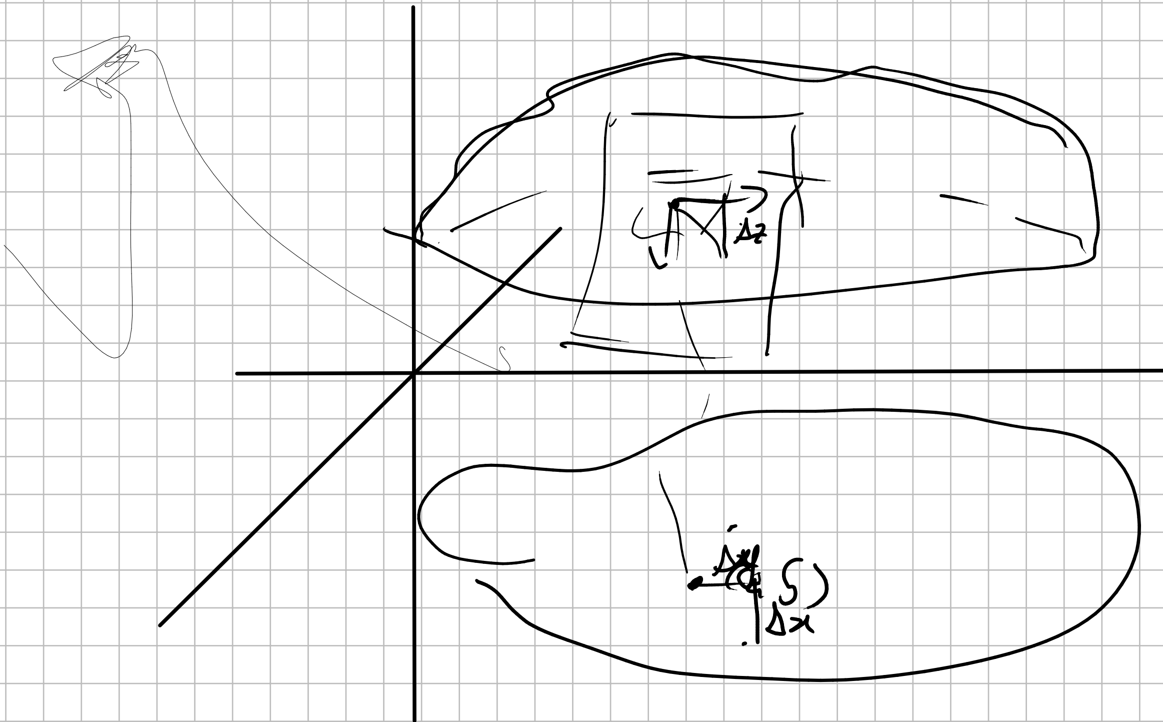
$$\left(\langle x,y,z\rangle - \langle a, b, f(a,b)\rangle\right)\cdot\langle -f_x, -f_y, 1\rangle = 0$$

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0$$

$$\Downarrow$$

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

Equation of the tangent plane to $f(x,y)$ at (a,b).



$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

Change in plane

$$f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

We say the tangent plane is a linear approx. to $f(x,y)$ at (a,b)

or

$f(x,y)$ is differentiable at (a,b)

if $(\Delta x, \Delta y) \longrightarrow (0,0) \Rightarrow \varepsilon_1 \to 0$ & $\varepsilon_2 \to 0$

Lecture 10a: Chain Rule

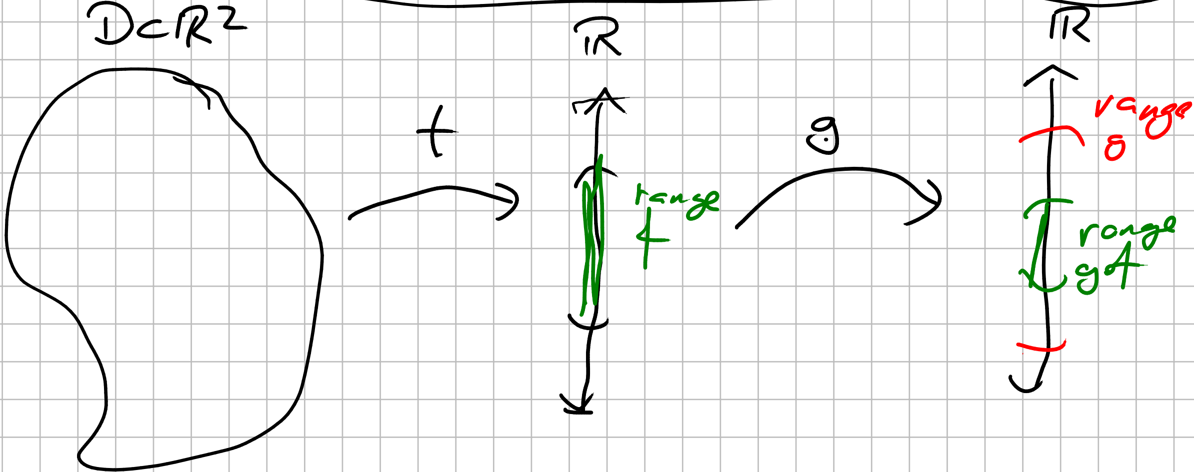
From Classwork + prev lecture //

If $f(x,y)$ is differentiable fxn. at (a,b) & $g(x)$ is differentiable at $c := f(a,b)$, the $g(f(x,y))$ is differentiable at (a,b) &

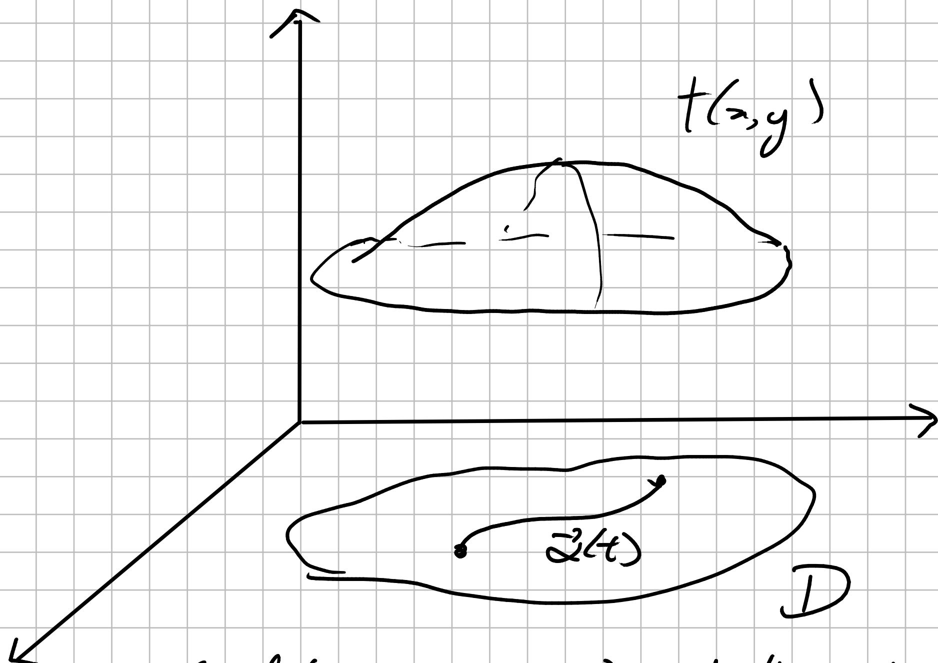
$$\frac{\partial}{\partial x}(g(f(x,y)))\Big|_{(a,b)} = g'(f(a,b))\, f_x(a,b)$$

$$\frac{\partial}{\partial x}(g\circ f) = (g'\circ f)\,\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial y}(g\circ f) = (g'\circ f)\,\frac{\partial f}{\partial y}$$



What about the other way?



Is $f(\alpha_1(t), \alpha_2(t))$ differentiable?

Thm [Chain rule #1]

If $f(x,y)$ is diff. & $\vec{\alpha}(t)$ is differentiable then $f(\alpha_1(t), \alpha_2(t))$ is differentiable &

$$\frac{d}{dt}(f(\vec{\alpha}(t)) = \frac{\partial f}{\partial x}\frac{d\alpha_1}{dt} + \frac{\partial f}{\partial y}\frac{d\alpha_2}{dt}$$

Pf// Show diff. at t_0. set

$(a,b) := \vec{\alpha}(t_0)$

Let $p(x,y)$ be the eqn of the tangent plane to $f(x,y)$ at (a,b).

$$p(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

$$\frac{d}{dt} f(\vec{\alpha}(t)) := \lim_{\Delta t \to 0} \frac{f(\alpha_1(t_0+\Delta t), \alpha_2(t_0+\Delta t)) - f(a,b)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(\alpha_1(t_0+\Delta t), \alpha_2(t_0+\Delta t)) - p(\alpha_1(t_0+\Delta t), \alpha_2(t_0+\Delta t))}{\Delta t} \quad 0$$

$$+ \lim_{\Delta t \to 0} \frac{f_x(a,b)(\alpha_1(t_0+\Delta t) - \alpha_1(t_0)) + f_y(a,b)(\alpha_2(t_0+\Delta t) - \alpha_2(t_0))}{\Delta t}$$

$$= \lim_{\Delta t \to 0} f_x(a,b) \frac{(\alpha_1(t_0+\Delta t) - \alpha_1(t_0))}{\Delta t}$$

$$+ \lim_{\Delta t \to 0} f_y(a,b) \frac{\alpha_2(t_0+\Delta t) - \alpha_2(t_0)}{\Delta t}$$

$$= f_x(a,b)\,\alpha_1'(t_0) + f_y(a,b)\,\alpha_2'(t_0)$$

$$= f_x(\alpha_1(t_0),\alpha_2(t_0))\,\alpha_1'(t_0) + f_y(\alpha_1(t_0),\alpha_2(t_0))\,\alpha_2'(t_0)$$

$\square$

Book $\rightsquigarrow$ $z = f(\vec{\alpha}(t))$ $\quad x = \alpha_1(t)$, $y = \alpha_2(t)$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Thm// [Chain rule #2]

If $f(x,y)$ is diff. and $g(s,t)$ & $h(s,t)$ are diff., then

$f(g(s,t),h(s,t))$ is differentiable

and

$$\frac{\partial}{\partial s}(f(g(s,t),h(s,t))) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial s}$$

$$\frac{\partial}{\partial t}(f(g(s,t),h(s,t))) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial t}$$

Lecture 105

$f(x,y) = \frac{1}{xy}$: diff if $xy \neq 0$

$g(s,t) = s^2+t^2+1$: diff. everywhere

$h(s,t) = \cos(st)+3$: diff everywhere

define

$$z(s,t) = f(g(s,t), h(s,t)) = \frac{1}{(s^2+t^2+1)(\cos(st)+3)}$$

By chain rule #2, since $g(s,t) > 0$
$h(s,t) > 0$

$\Rightarrow$ $z(s,t)$ diff. everywhere

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial s}$$

$$\frac{\partial f}{\partial x} = -\frac{1}{(xy)^2}\, y = -\frac{y}{(xy^2)} = -\frac{1}{x^2 y}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{y^2 x}$$

$$\frac{\partial f}{\partial x}(g(s,t), h(s,t)) = \frac{-1}{(s^2+t^2+1)^2(\cos(st)+3)}$$

$$\frac{\partial f}{\partial y}(g(s,t), h(s,t)) = \frac{-1}{(\cos(st)+3)^2(s^2+t^2+1)}$$

$$\frac{\partial g}{\partial s} = 2s \qquad \frac{\partial h}{\partial s} = -\sin(st)\, t$$

$$\boxed{\frac{\partial z}{\partial s} = \frac{-2s}{(s^2+t^2+1)^2(\cos(st)+3)} + \frac{\sin(st)\, t}{(\cos(st)+3)^2(s^2+t^2+1)}}$$

Comp. tools for Partial/Directional Derivs

- Compute $\frac{\partial f}{\partial x}$ by treating y as constant and taking usual x-deriv.

  Compute $\frac{\partial f}{\partial y}$ by treating x as constant and taking usual y-deriv.
- $f(x,y)$ is diff. at (a,b) if $f_x(x,y)$ & $f_y(x,y)$ exist near (a,b) and are continuous at (a,b).

- $D_{\hat{u}} f = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y}$

- Chain rules, differentiable $f(x,y)$, $\vec{\alpha}(t)$ $\vec{\alpha}(t) = \langle \alpha_1(t), \alpha_2(t) \rangle$, $g(s,t)$, $h(s,t)$, $k(x)$.

  (0) $\frac{\partial}{\partial x}\left[k(f(x,y))\right] = k'(f(x,y))\,\frac{\partial f}{\partial x}$

  $\frac{\partial}{\partial y}\left[k(f(x,y))\right] = k'(f(x,y))\,\frac{\partial f}{\partial y}$

(1) $\dfrac{d}{dt}\left[f(\vec{x}(t))\right] = \dfrac{\partial f}{\partial x}\dfrac{dx_1}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dx_2}{dt}$

(2) $\dfrac{\partial}{\partial s}\left[f(g(s,t), h(s,t))\right] = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial s}$

$\dfrac{\partial}{\partial t}\left[f(g(s,t), h(s,t))\right] = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial t}$

Lecture 11a: Gradient

Recall// In a prev. lect. we showed that if $f(x,y)$ is diff at (a,b), then

(1) $(D_{\hat{u}} f)(a,b)$ exists for any $\hat{u} \in \mathbb{R}^2$ unit vec.

(2)

$$(D_{\hat{u}} f)(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

$$= \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle$$

$$= \underbrace{\langle f_x(a,b), f_y(a,b)\rangle}_{\nabla f} \cdot \hat{u}$$

Q// What can can we say about the vector ∇f?

Defn// If $f(x,y)$ is a diff. fcn, we define the <u>gradient (vector) ∇f</u> to be

$$(\nabla f)(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

Lemma// Suppose $f(x,y)$ is a diff. fcn of 2 variables. Then the maximum value of $(D_{\hat{u}} f)(a,b)$ is $|(\nabla f)(a,b)|$ <u>and</u> this maximum occurs when $\hat{u}$ is in the same direction as $(\nabla f)(a,b)$.

Pf// Note

$$\begin{aligned} (D_{\hat{u}} f)(a,b) &= (\nabla f)(a,b) \cdot \hat{u} \\ &= |(\nabla f)(a,b)| \, \cancel{|\hat{u}|}^{1} \cos(\theta) \\ &= |(\nabla f)(a,b)| \cos(\theta) \end{aligned}$$

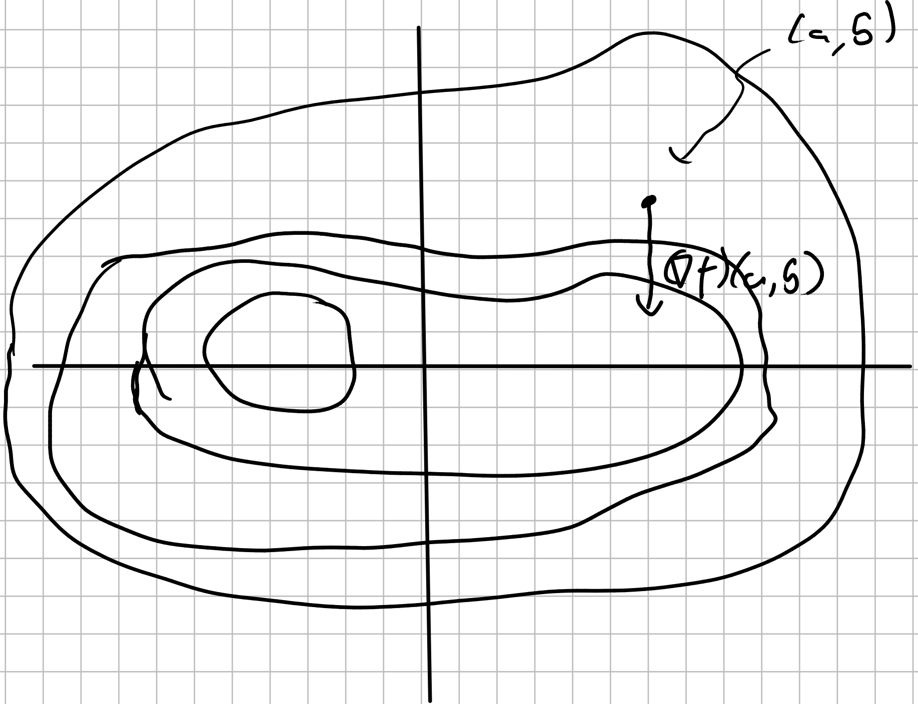
$\cos(\theta)$ is max. when $\theta = 0 \rightsquigarrow \cos(\theta) = 1$

$$= |(\nabla f)(a,b)|$$

$\square$

Visualization

Topo map (ie level curves)



Notice, It looks like $(\nabla f)(a,b)$ is perpendicular to level curves

Lemma Suppose $f(x,y)$ is diff. and let $\vec{r}(t) = \langle r_1(t), r_2(t) \rangle$ be a parameterization of level curve $f(x,y) = k$.
Then $(\nabla f)(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$, i.e. $(\nabla f)(r_1(t), r_2(t))$ is orthogonal to $\vec{r}\,'(t)$.

Pf Since $\vec{r}(t)$ param. $f(x,y) = k$

$$f(r_1(t), r_2(t)) = k$$

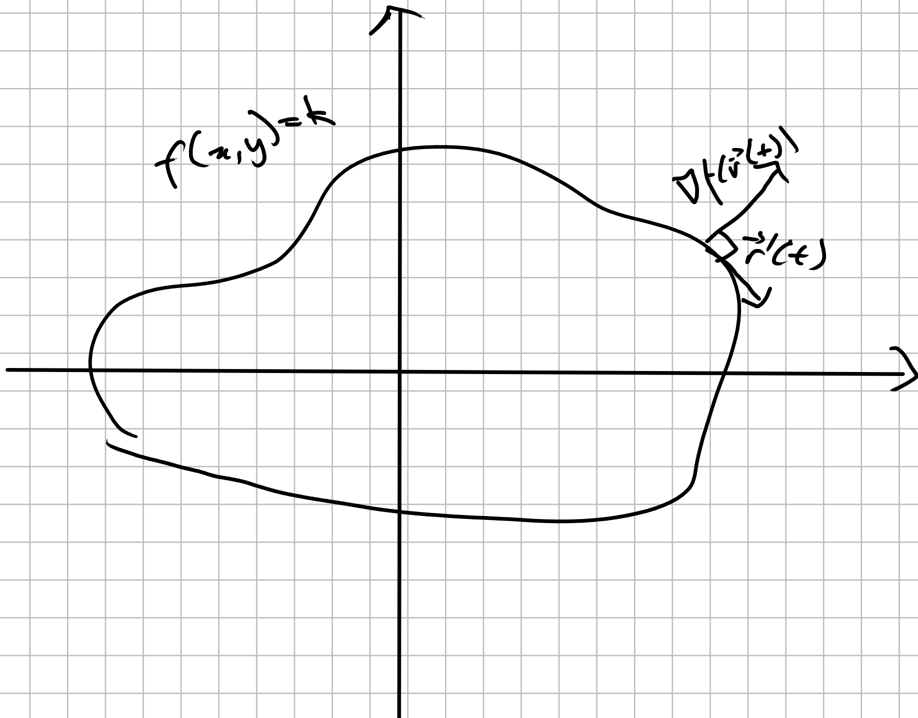
$$\Big\downarrow \frac{d}{dt}$$

$$\frac{d}{dt}\left[f(r_1(t), r_2(t)) \right] = 0$$

$$\frac{\partial f}{\partial x}\frac{dr_1}{dt} + \frac{\partial f}{\partial y}\frac{dr_2}{dt} = 0$$

$$\Downarrow$$

$$\left((\nabla f)(\vec{r}(t))\right) \cdot \vec{r}\,'(t) = 0 \qquad \square$$



Eg $f(x,y) = x^2+y^2$

level curve at k^2 is circle of radius k, param:

$\vec{r}(t) = \langle k\cos(t), k\sin(t)\rangle$

$\vec{r}'(t) = \langle -k\sin(t), k\cos(t)\rangle$

$$f_x(x,y) = 2x \rightsquigarrow f_x(\vec{r}(t)) = 2k\cos(t)$$

$$f_y(x,y) = 2y \rightsquigarrow f_y(\vec{r}(t)) = 2k\sin(t)$$

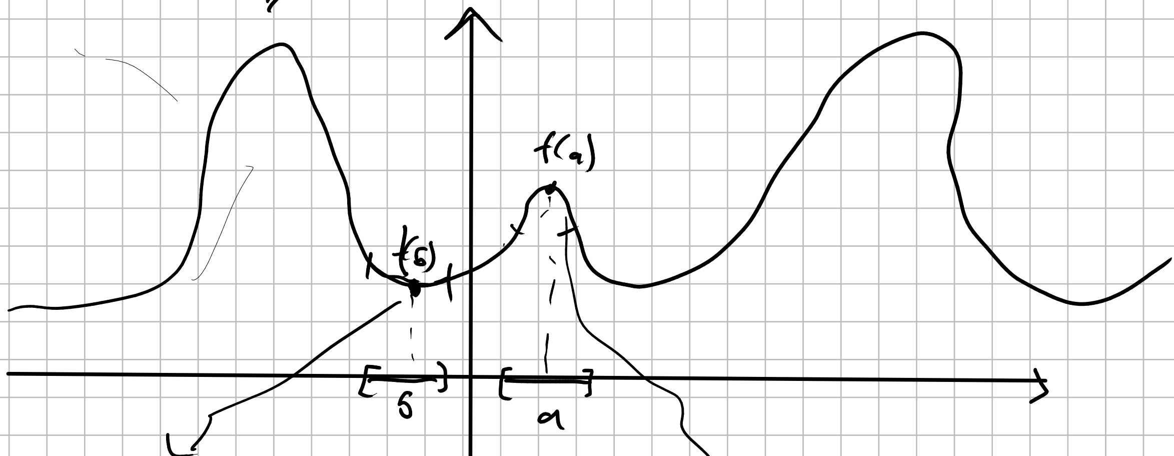
$$(\nabla f)(\vec{r}(t)) = \langle 2k\cos(t), 2k\sin(t) \rangle$$

$$(\nabla f(\vec{r}(t))) \cdot \vec{r}\,'(t) = -2k^2\cos(t)\sin(t) + 2k^2\sin(t)\cos(t)$$

$$= 0.$$

Lecture 115: Extrema

1-var Calc



Local minimum

Exists an interval $[e,f]$ around b s.t.

$f(b) \leq f(x) \quad \forall x \in [e,f]$

local maximum

Exists an interval $[c,d]$ around a s.t.

$f(a) \geq f(x) \quad \forall x \in [c,d]$

1st deriv. test (1-var)

Thm: If $f(x)$ has a local min or max at a, ~~th~~ and f is diff. at a, then $f'(a) = 0$.

Second deriv. test (1-var)

Thm // If $f''(a)$ exists and $f'(a)=0$ then

(1) If $f''(a)>0$, then $f(a)$ is a local min.

(2) If $f''(a)<0$, then $f(a)$ is a local maximum.

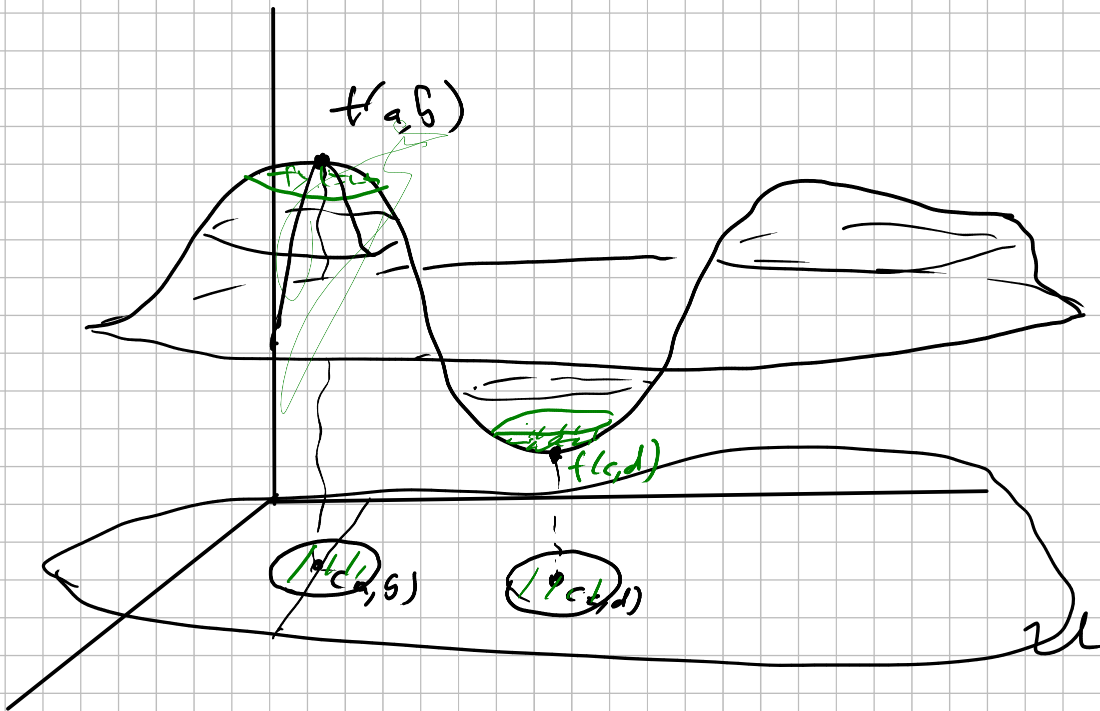
Want / 2-variable version

Defn // $f(x,y)$ is defined on a domain $U \subset \mathbb{R}^2$
We say f has a local maximum at $(a,b) \in U$ if there is a disk D around (a,b) s.t.

$$f(x,y) \le f(a,b) \quad \forall \ (x,y) \in D.$$

f has a local min. at $(a,b) \in U$ if there is a disk D around (a,b) s.t.

$$f(x,y) \ge f(a,b) \quad \forall \ (x,y) \text{ in } D.$$



Thm// [1st Deriv test for 2-variable fncs]
If $f(x,y)$ has a local max or min at (a,b), and $f_x(a,b)$ & $f_y(a,b)$ exist, then
$$f_x(a,b) = 0 \quad \& \quad f_y(a,b) = 0$$

Pf// Consider the path in $\mathbb{R}^2$
$\vec{r}(t) = \langle t, 5 \rangle$, then

$f(\vec{r}(t))$ is a 1-var fxn, which has a local max/min at $t = a$

so

$$f_x(a,5) = \frac{d}{dt} f(\vec{r}(t)) \Big|_{t=a} = 0 \qquad \square$$

Want// Second deriv. test

Problem// lots of second derivs
$f_{xx}(a,5)$, $f_{yy}(a,5)$, $f_{xy}(a,5)$, $f_{yx}(a,5)$.

Simplification

Thm [Clairaut's Thm]
Suppose $f(x,y)$ is defined on a disk D containing $(a,5)$, st $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are continuous on D then

$$f_{xy}(a,5) = f_{yx}(a,5)$$

Eg// $f(x,y) = x^2\cos(y)$

$f_x(x,y) = 2x\cos(y)$

$f_y(x,y) = -x^2\sin(y)$

$f_{xy}(x,y) = -2x\sin(y)$

$$\rotatebox{90}{=}$$

$f_{yx}(x,y) = -2x\sin(y)$

Defn// Suppose all second partial derivs of $f(x,y)$ are cont. on a disk

$$D = \{(x,y) \in \mathbb{R}^2 \mid d((x,y),(a,b)) < \varepsilon\}$$

centered at (a,b). The Hessian matrix of $f(x,y)$ at (a,b) is

$$H(a,b) = H = \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix}$$

The <u>Discriminant</u> of $f(x,y)$ at (a,b)

$$D(a,b) = D = \det(H) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

$$= f_{xx}(a,b) f_{yy}(a,b) - f_{yx}(a,b) f_{xy}(a,b)$$

$$= \boxed{f_{xx}(a,b) f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2}$$

Thm // [Second deriv. test, 2-var]

Suppose all second partials of $f(x,y)$ are continuous on a disk w/ center (a,b). Also suppose $f_x(a,b) = 0 = f_y(a,b)$. Then

(1) If $D = D(a,b) > 0$ & $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

(2) If $D = D(a,b) > 0$ & $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum.

(3) If $D < 0$, then $f(a,b)$ is neither a local max nor a local min "saddle pt."

Lecture 12a: Multivariable fns

Idea// Everything so far can be done for $f(x_1, x_2, \ldots, x_n)$ a function of n variables

Directional derivatives//

In n-dimension space $\mathbb{R}^n$ a unit vector is $\hat{v} := \langle v_1, v_2, \ldots, v_n \rangle$ s.t.

$$|\hat{v}| := \sqrt{v_1^2 + v_2^2 + \cdots v_n^2} = 1$$

Eqn of a line through $(b_1, b_2, \ldots, b_n)$ in $\hat{v}$-dir. is

$$\vec{r}(t) = \vec{b} + t\hat{v}$$

Define directional deriv of f. in the $\hat{v}$-dir. to be

$$(D_{\hat{v}} f)(b_1, \ldots, b_n) := \frac{d}{dt} f(\vec{r}(t)) \Big|_{t=0}$$

Partial derivatives

$\frac{\partial f}{\partial x_i}$ is the directional derivative in the $\langle 0, \dots, 0, \underbrace{1}_{i\text{th}}, 0, \dots, 0 \rangle$

Can be computed as usual 1-var. deriv. where everything except x_i is treated as a constant.

Eg. $f(x,y,z,w) = x^2 y + xyzw$

$$\frac{\partial f}{\partial x} = 2xy + yzw$$

$$\frac{\partial f}{\partial y} = x^2 + xzw$$

$$\frac{\partial f}{\partial z} = xyw$$

$$\frac{\partial f}{\partial w} = xyz$$

Chain rule

If $u(x_1, \ldots, x_n)$ is a dif. fun of n variables $x_1, \ldots, x_n$

& each x_i is a fun

$$x_i(t_1, \ldots, t_m)$$

of m variables, then

$$\frac{\partial}{\partial t_j}\left[u(x_1(\vec{t}), \ldots, x_n(\vec{t}))\right]$$

$$= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$

$$= \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial t_j}$$

Gradient Vector

Given $f(x_1, \ldots, x_n)$ diff. we define

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \leftarrow \text{the gradient of } f$$

Note that by chain rule

$(D_{\hat{v}} f) = (\nabla f) \cdot \hat{v}$

<u>Lemma</u>// Let $f(x_1, \dots, x_n)$ be a diff. fun of n variables

Let $\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^n$

$t \longmapsto \langle r_1(t), r_2(t) \dots, r_n(t) \rangle$

be a diff. curve in $\mathbb{R}^n$ s.t.

$f(\vec{r}(t)) = k \quad \forall \ t \in \mathbb{R}$. Then

$$\left[(\nabla f)(\vec{r}(t))\right] \cdot \vec{r}\,'(t) = 0$$

Pf// Chain rule $\square$

Lecture 12b: 3-variable fxns

Special case: functions of three variables

$$f(x,y,z)$$

Eg: $f(x,y,z) = x\sin(yz)$

$$\frac{\partial f}{\partial x} = \sin(yz)$$

$$\frac{\partial f}{\partial y} = x\cos(yz)z = xz\cos(yz)$$

$$\frac{\partial f}{\partial z} = xy\cos(yz)$$

Interpret these are rates of change of f in the $\langle 1,0,0\rangle$ $\langle 0,1,0\rangle$, & $\langle 0,0,1\rangle$ - directions respectively.

Gradient:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$\nabla f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

sends pt. (s_1, s_2, s_3) to

$\langle f_x(\vec{s}), f_y(\vec{s}), f_z(\vec{s})\rangle$

$$(D_{\hat{u}}f)(b_1, b_2, b_3) = (\nabla f)(b_1, b_2, b_3) \cdot \hat{u}$$

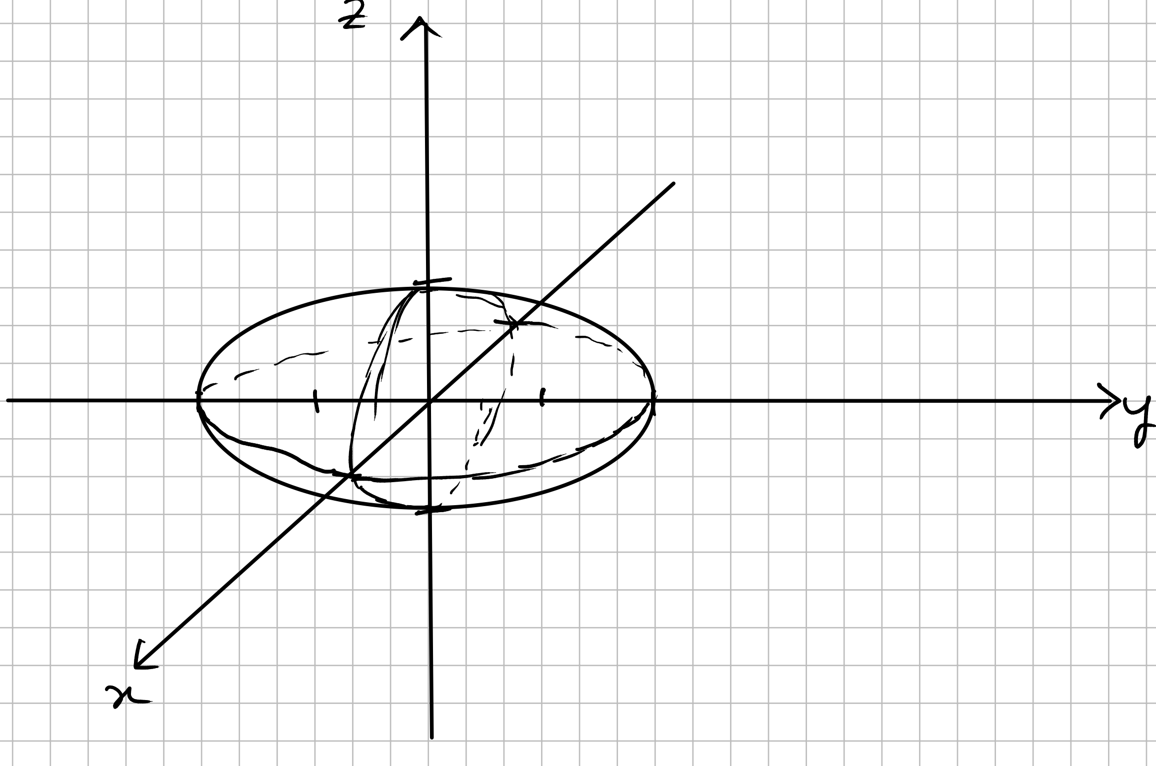
Level surfaces

Surface in $\mathbb{R}^3$ defined by

$$f(x,y,z) = k$$

Eg $f(x,y,z) = x^2 + \frac{1}{4}y^2 + z^2$

k = 1 $\quad x^2 + \frac{1}{4}y^2 + z^2 = 1 \rightsquigarrow$ ellipsoid



Even if we can't describe level surface as a fxn, we can write an eqn for its tangent plane at a point.

(b_1, b_2, b_3) s.t. $f(b_1, b_2, b_3) = k$

Lemma $\rightsquigarrow$ $(\nabla f)(b_1, b_2, b_3)$ is normal to the level surface given by $f(x,y,z) = k$.

Eg $f(x,y,z) = x^2 + \frac{1}{4}y^2 + z^2$

Find the eqn. of tangent plane to the level surface $f(x,y,z) = 1$ at the point $\langle \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$\nabla f = \langle 2x, \frac{y}{2}, 2z \rangle$, so at $(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ we get normal vector

$$\langle \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}} \rangle$$

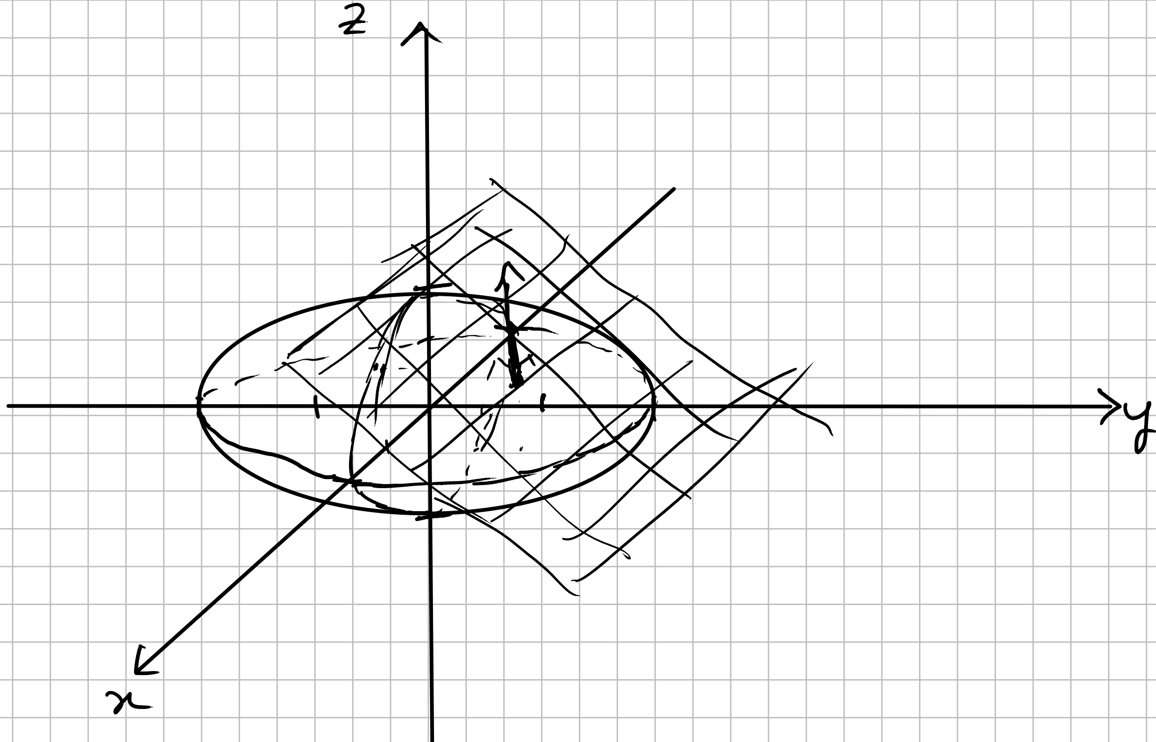
Plane goes through $\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

w/ normal $\left\langle \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right\rangle$

Eqn of the tangent plane is

$$\left(x-\frac{1}{\sqrt{3}}\right)\frac{2}{\sqrt{3}} + \left(y-\frac{2}{\sqrt{3}}\right)\frac{1}{\sqrt{3}} + \left(z-\frac{1}{\sqrt{3}}\right)\frac{2}{\sqrt{3}} = 0$$

Pictorially



Lecture 13a: Lagrange Multipliers

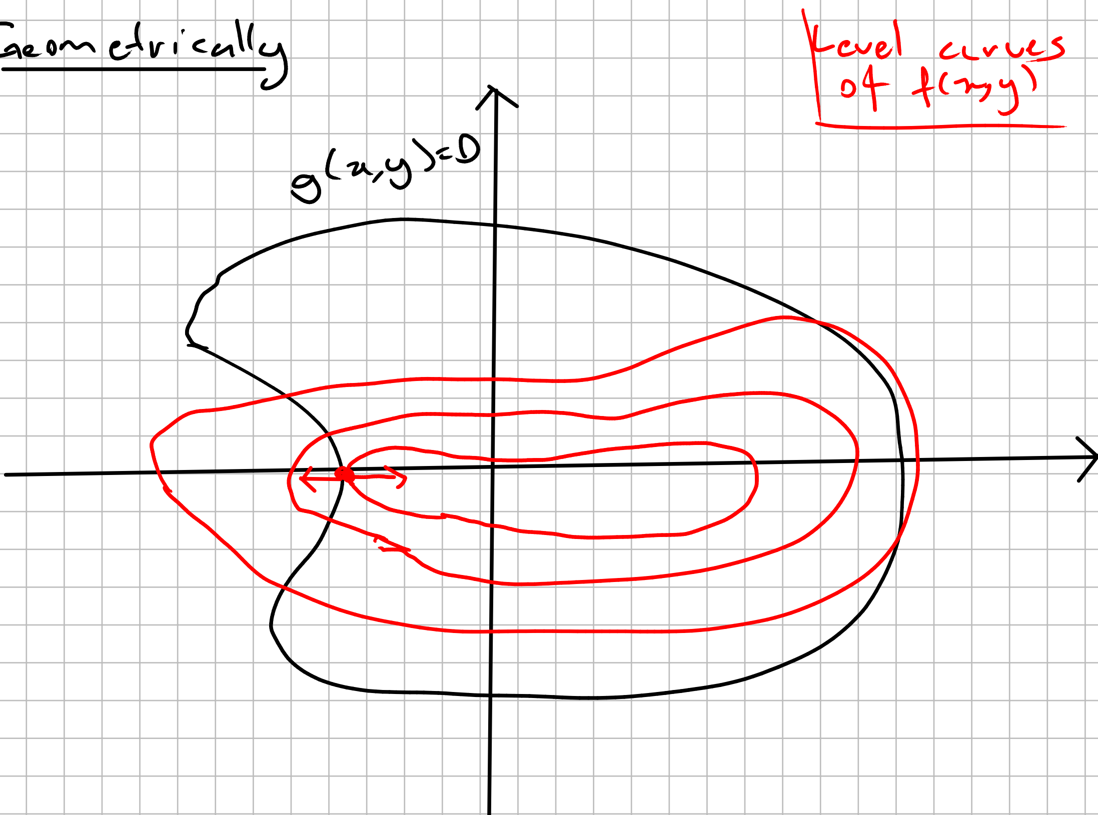
Classwork 8 we showed

Thm// Let $f(x,y)$ & $g(x,y)$ be differentiable 2-var. fns, and consider the curve defined by $g(x,y)=0$. Suppose (a,b) is a point on this curve such that f attains a local max/min at (a,b) on the curve.
Suppose ~~further that~~
- the curve has a diff. param. near (a,b)
- $\nabla g(a,b) \neq \vec{0}$

Then $\exists\ \lambda \in \mathbb{R}$ such that

$$\nabla f(a,b) = \lambda \nabla g(a,b)$$

Geometrically



We use this like "first deriv test"

Idea: Find values of x, y, and λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y) \Big\} \times 2 \text{ scalar eqns}$$

$$g(x,y) = 0$$

hold. Test these values to see which are max/min.

Eg. Find the points on the circle $x^2+y^2=320$ closest to & farthest from $(2,4)$

$$g(x,y) = x^2+y^2-320 = 0 \qquad \underline{\text{Constraint}}$$

<u>function</u> — want

$f(x,y) = d((x,y),(2,4))$

$$f(x,y) = d((x,y),(2,4))^2$$

$$= (x-2)^2+(y-4)^2$$

$$\nabla g = \langle 2x, 2y\rangle \qquad \nabla f = \langle 2(x-2), 2(y-4)\rangle$$

<u>$\lambda \nabla g = \nabla f$</u>

$$2(x-2) = 2\lambda x \rightsquigarrow x = \frac{-4}{2\lambda-2}$$

$$2(y-4) = 2\lambda y \qquad y = \frac{-8}{2\lambda-2}$$

<u>$g(x,y)=0$</u>

$$\frac{16}{(2\lambda-2)^2} + \frac{64}{(2\lambda-2)^2} = 320$$

$$\frac{80}{(2\lambda-2)^2} = 320 \rightsquigarrow \frac{1}{(2\lambda-2)^2} = 4$$

$$1 = (2\lambda-2)^2\, 4$$

$$1 = 4(4\lambda^2 - 8\lambda + 4)$$

$$1 = 16\lambda^2 - 32\lambda + 16$$

$$0 = 16\lambda^2 - 32\lambda + 15$$

$$\lambda = \frac{32 \pm \sqrt{32^2 - 4(16)15}}{2(16)} = \frac{32 \pm \sqrt{32\cdot 32 - 30(32)}}{32}$$

$$= \frac{32 \pm \sqrt{2\cdot 32}}{32} \qquad 2\cdot 32 = 4\cdot 16$$

$$= \frac{32 \pm 2\cdot 4}{32} = 1 \pm \frac{1}{4} = \frac{5}{4} \text{ or } \frac{3}{4}$$

$\lambda = \frac{5}{4}$

$$x = \frac{-4}{\frac{5}{2} - \frac{4}{2}} = \frac{-4}{\frac{1}{2}} = -8$$

$$y = \frac{-8}{\frac{5}{2} - \frac{4}{2}} = \frac{-8}{\frac{1}{2}} = -16$$

$\lambda = \frac{3}{4}$

$$x = 8, \quad y = 16$$

$f(-8,-16) = (-8-2)^2 + (-16-4)^2 = 100 + 400 = 500$

$f(8,16) = 6^2 + 12^2 = 180$

Max is $\sqrt{500}$ & occurs at $(-8,-16)$

Min is $\sqrt{180}$ & occurs at $(8,16)$

<u>Eg</u>// Find the extrema of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \le 4$

(1) find crit. pts

$$\frac{\partial f}{\partial x} = 8x = 0 \quad \Updownarrow \quad x = 0 \qquad\qquad \frac{\partial f}{\partial y} = 20y = 0 \quad \Updownarrow \quad y = 0$$

$f(0,0) = 0$

(2) $g(x,y) = x^2 + y^2 - 4$ constraint

$\nabla g = \langle 2x, 2y \rangle \qquad \nabla f = \langle 8x, 20y \rangle$

$$\left.\begin{array}{l} 8x = 2\lambda x \\ 20y = 2\lambda y \end{array}\right\}$$ two poss. solns

$$\begin{cases} x = 0, & \lambda = 10 \\ y = 0, & \lambda = 4 \end{cases}$$

<u>$x = 0$</u>

$g(0, y) = 0 \Rightarrow y = \pm 2$

<u>$y = 0$</u>

$g(x, 0) = 0 \Rightarrow x = \pm 2$

$f(0,2) = f(0,-2) = 10(2)^2 = 40$

$f(2,0) = f(-2,0) = 4(2)^2 = 16$

max is 40 and occurs at $(0,2)$
$(0,-2)$

min is 0 and occurs at $(0,0)$.

Lecture 13b

Thm// Suppose $f(x,y,z)$ & $g(x,y,z)$ are diff. 3-var fxns, define a surface by $g(x,y,z)=0$. Suppose (a,b,c) is on this surface and f attains a local max/min on the surface at (a,b,c). Assume

- the surface is "smooth" near (a,b,c)
- $\nabla g(a,b,c) \neq 0$

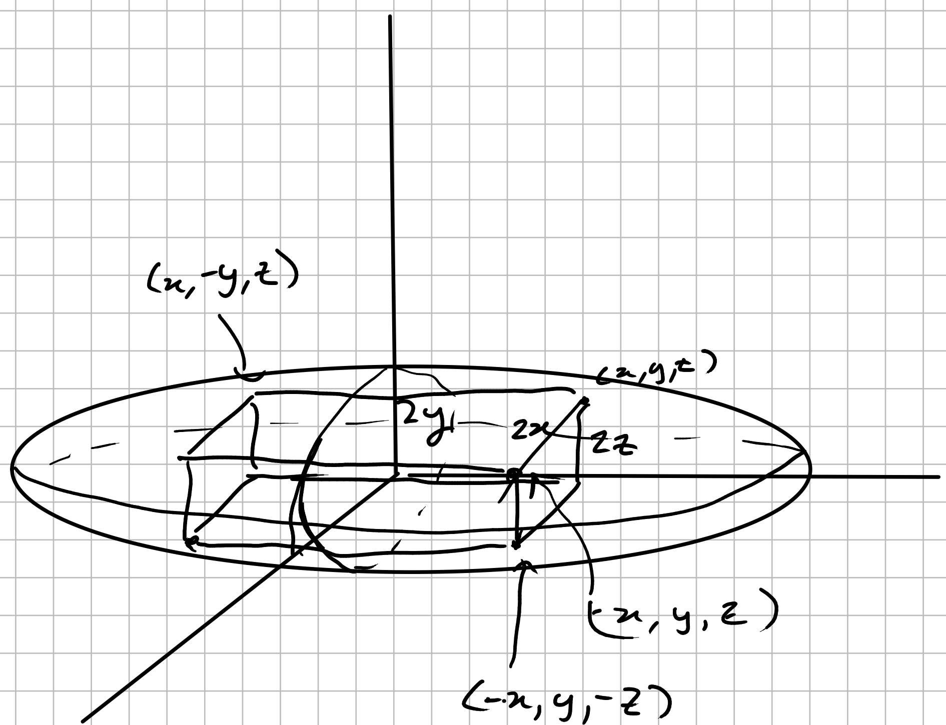
Then $\exists\ \lambda \in \mathbb{R}$ such that

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c)$$

Eg// A rectangular prism w/ sides parallel to the coordinate planes has vertices lying on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Find maximum volume.



volume $f(x,y,z) = (2x)(2y)(2z) = 8xyz$

Constraint //

$$g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

Fxn //

$$f(x,y,z) = 8xyz$$

$$\nabla f = \langle 8yz, 8xz, 8xy \rangle$$

$$\nabla g = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

$$\nabla f = \lambda \nabla g$$

$$\left.\begin{aligned} 8yz &= \frac{2\lambda x}{a^2} \quad \times x \\ 8xz &= \frac{2\lambda y}{b^2} \quad \times y \\ 8xy &= \frac{2\lambda z}{c^2} \quad \times z \end{aligned}\right\} \rightsquigarrow \begin{aligned} 8xyz &= \frac{2\lambda x^2}{a^2} \\ 8xyz &= \frac{2\lambda y^2}{a^2} \\ 8xyz &= \frac{2\lambda z^2}{c^2} \end{aligned}$$

$$\frac{2\lambda x^2}{a^2} = \frac{2\lambda y^2}{b^2} = \frac{2\lambda z^2}{c^2}$$

Two options

(1) $\lambda = 0$

(2) $y^2 = \frac{b^2}{a^2} z^2 \qquad z^2 = \frac{c^2}{a^2} x^2$

In case (1)

At least 1 of $x, y, \&\ z$ are zero

$\rightsquigarrow f(x,y,z) = 0$

In case (2)

$g(x,y,z) = 0 \rightsquigarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$

$$\frac{3}{a^2} x^2 = 1$$

$$x = \pm \frac{a}{\sqrt{3}}$$

$\rightsquigarrow y = \pm \frac{b}{\sqrt{3}} \qquad z = \pm \frac{c}{\sqrt{3}}$

pts $\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{b}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}\right)$

$$f\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{b}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}\right) = \pm\frac{8abc}{3\sqrt{3}}$$

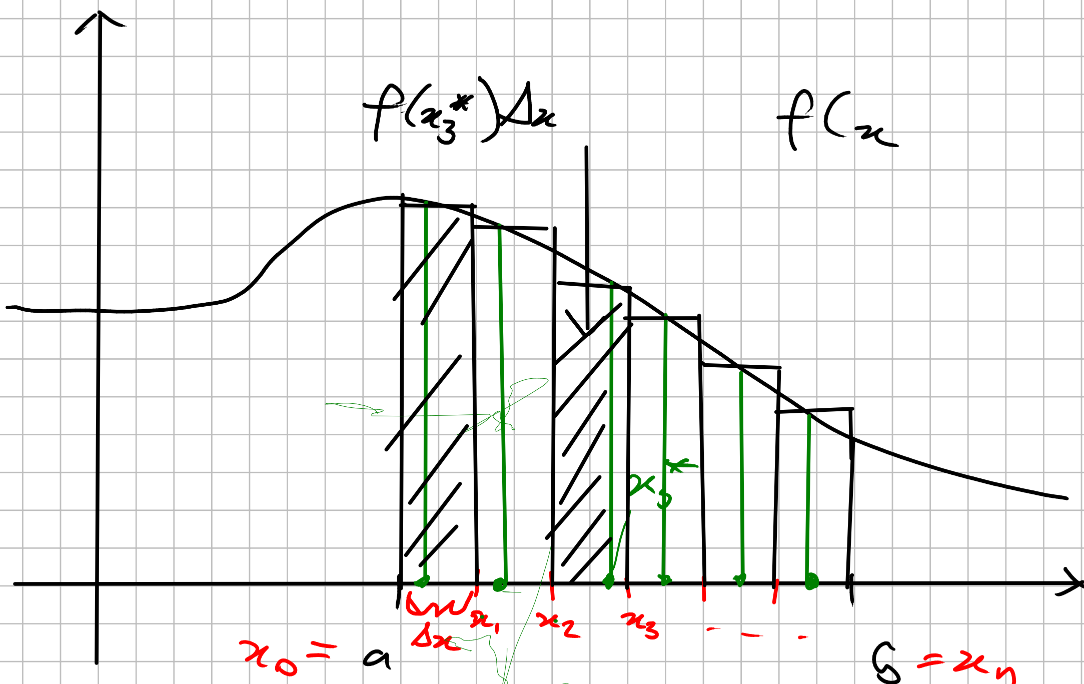
Max volume, $\frac{8abc}{3\sqrt{3}}$, occurs

For the prism w/ vertices

$$\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{b}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}\right)$$

Lecture 14 // Integration

Recall // Defn of 1-var integral



(1) Divide $[a, b]$ into n intervals of length $\Delta x = \frac{b-a}{n}$

(2) Choose a "test pt" in each interval $x_i^* \in [x_{i-1}, x_i]$

(3) Approximate area by <u>Riemann sum</u>

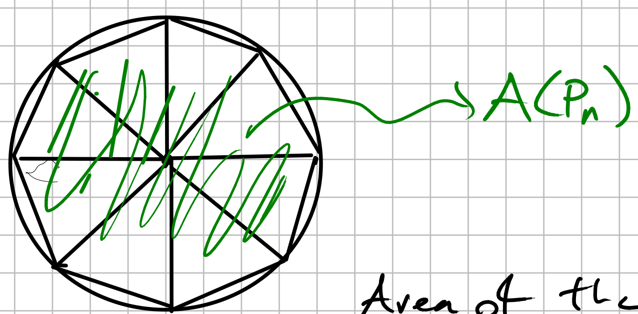
$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

(4) Define

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

<u>Many similar techniques</u>

Eg: Approximat area of a circle by an inscribed regular n-gon P_n

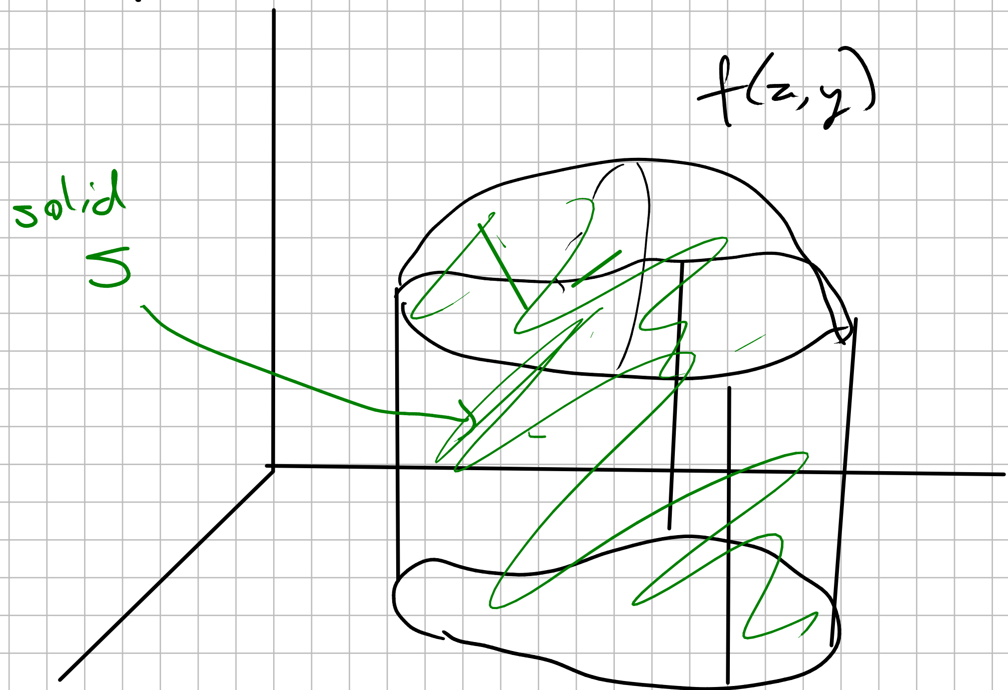


Area of the circle

$$\lim_{n\to\infty} A(P_n)$$

Want: use same ideas to approximate Volumes

Idea: start w/ a 2-variable fxn $f(x,y)$ on a domain D



Want: Volume of S

Problem Looks tricky!

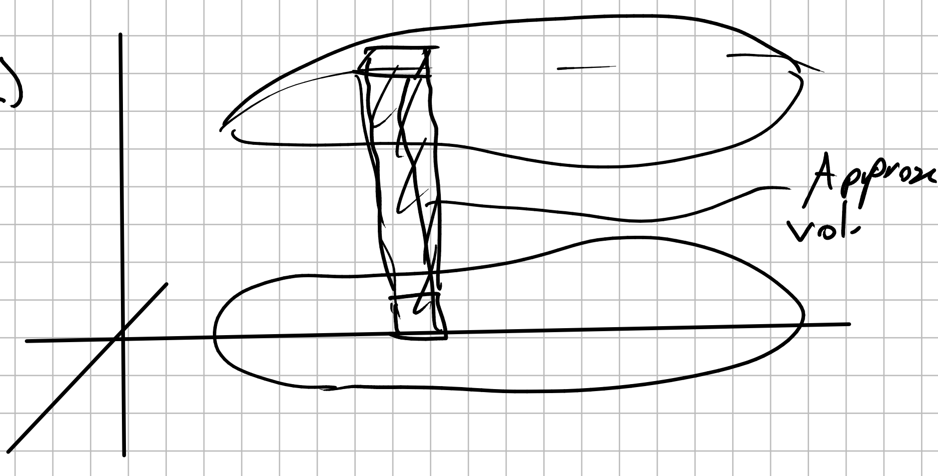
Need to approximate 2 things

- Shape of D
- volume of pieces of D

Might think

(1) approx "shape"

(2)

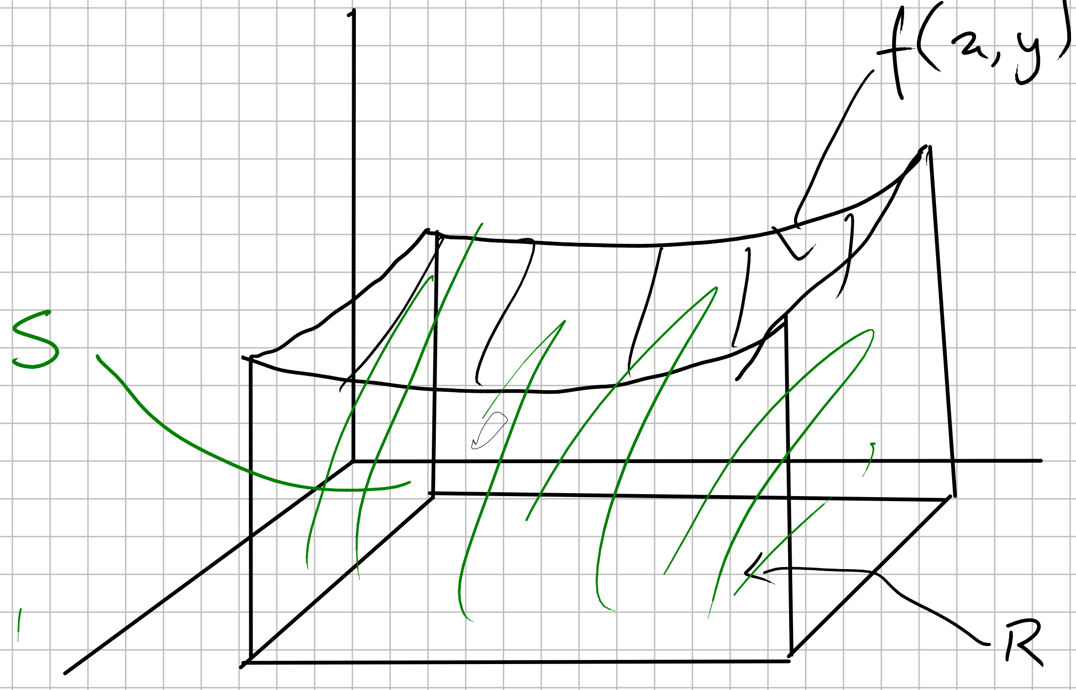


We take an <u>easier</u> route

- First just define integral on Rectangular domains.

Lecture 14: Integration of 2-var fxns

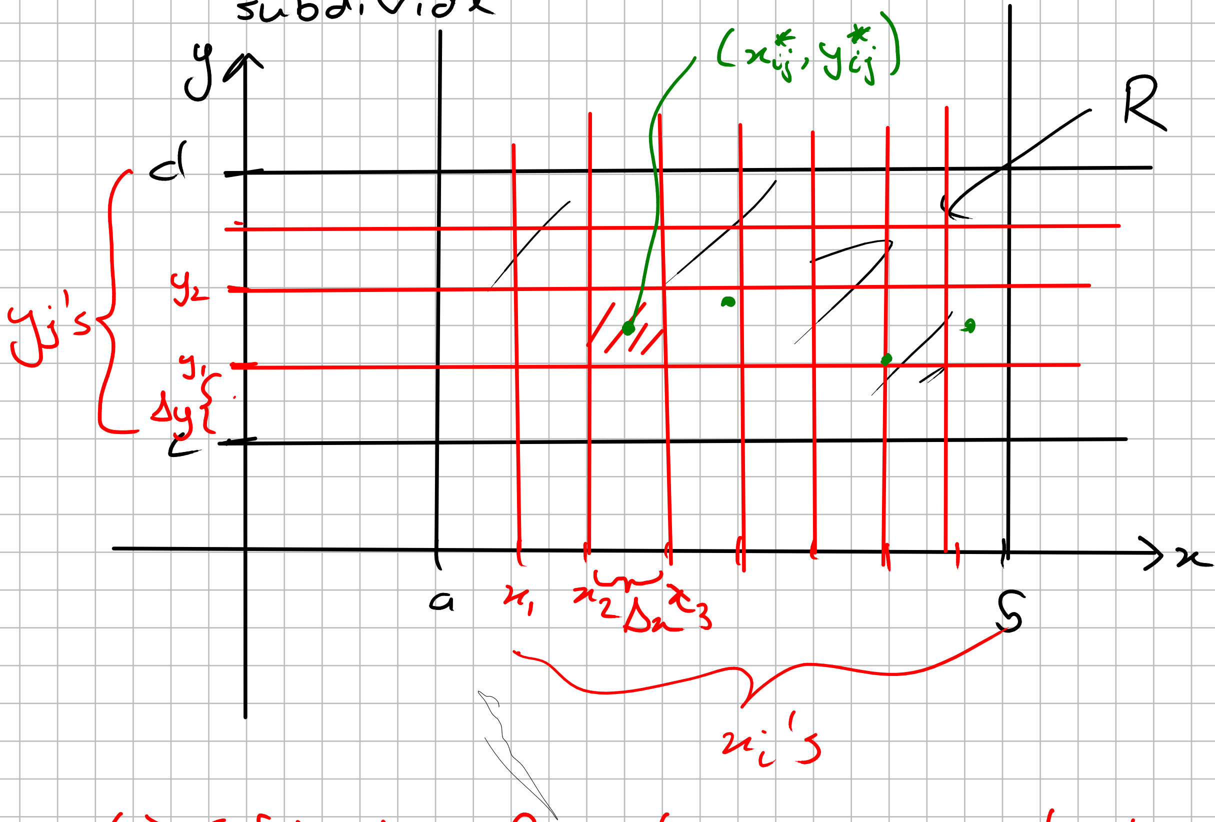
<u>Setup</u>// $f(x,y)$ is a 2-var fxn defined on $R=[a,b]\times[c,d]$



We <u>want</u> the volume of

$$S=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; a\le x\le b,\ c\le y\le d,\ 0\le z\le f(x,y)\right\}$$

Good news: Rectangles are easy to subdivide

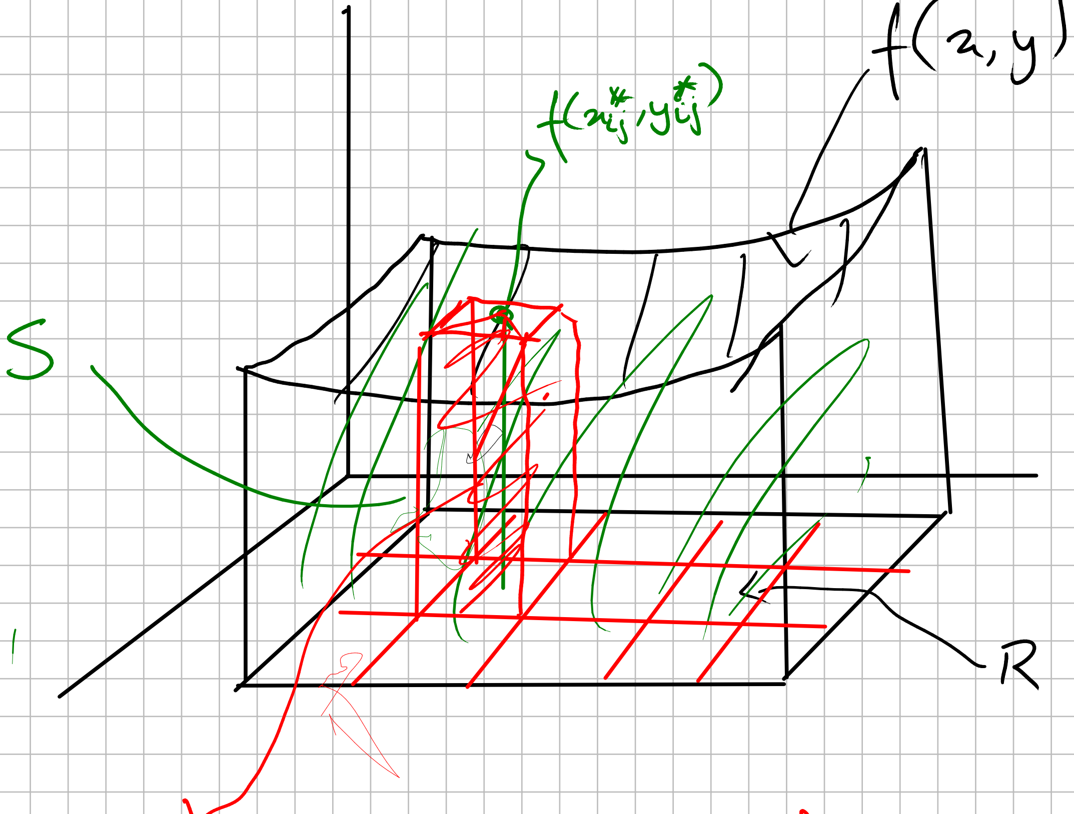


(1) Subdivide R into $n \times m$ rectangles of area $\Delta A = \Delta x \Delta y$

$$\Delta x = \frac{b-a}{n}$$
$$\Delta y = \frac{d-c}{m}$$

(2) Choose "test pts" (x_{ij}^*, y_{ij}^*) in each rectangle

From this, we approximate Vol(S)



$$Vol = f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y = f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

So we can approx. Vol(S) by

$$\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

Defn// The <u>double integral</u> of $f(x,y)$ on R is

$$\iint_R f(x,y)\,dA := \lim_{m,n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

If this lim. exists, we say f is <u>integrable</u> on R

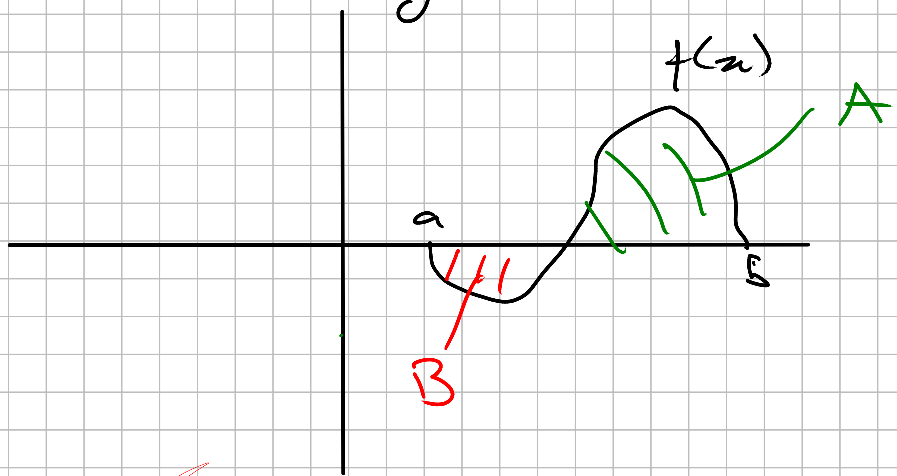
<u>Integrability</u>// $f(x,y)$ is integrable on R

(1) If $f(x,y)$ is continuous on R

(2) $\exists$ $M \in \mathbb{R}$ s.t. $|f(x,y)| \leq M$ on R (ie $f(x,y)$ <u>bounded</u> on R) & f is discontinuous at <u>only</u> a finite set of smooth curves in R

(3) Other case — ~

Rmks (1) If $f(x,y)$ is not ≥ 0 on R, the $\iint_R f(x,y)\,dA$ isn't really the volume of S, since we're subtracting volume for the negative part. Analogy



$$\int_a^b f(x)\,dx = A - B$$

(2) If $f(x,y)$ is integrable, we can use any test pts in our approximation.

- Midpoints
- corner pts

⋮

How does this work for $f(x,y)$ on some general domain D?

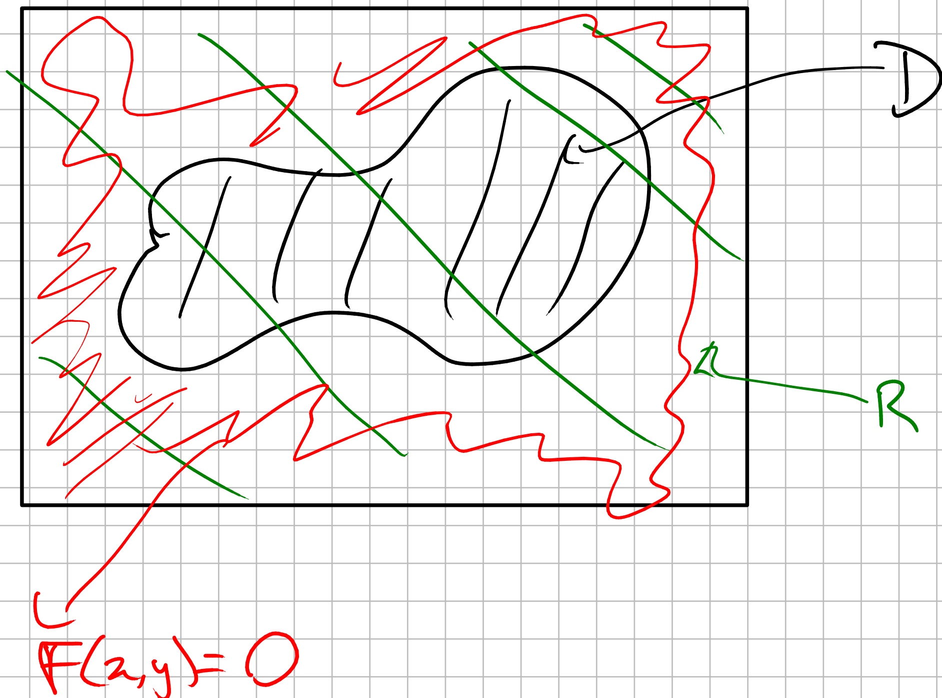
Defn// Let $f(x,y)$ be defined on a bounded region D. Let R be a rectangle containing D. Define a new fxn $F(x,y)$ on R by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{else} \end{cases}$$

Then define

$$\iint_D f(x,y)\, dA := \iint_R F(x,y)\, dA$$

if $F(x,y)$ is integrable.



Next time:

- Computing Double integrals
- Fubini thms

Lecture 16a

In class:

Thm: (change of coords for polar rectangles)

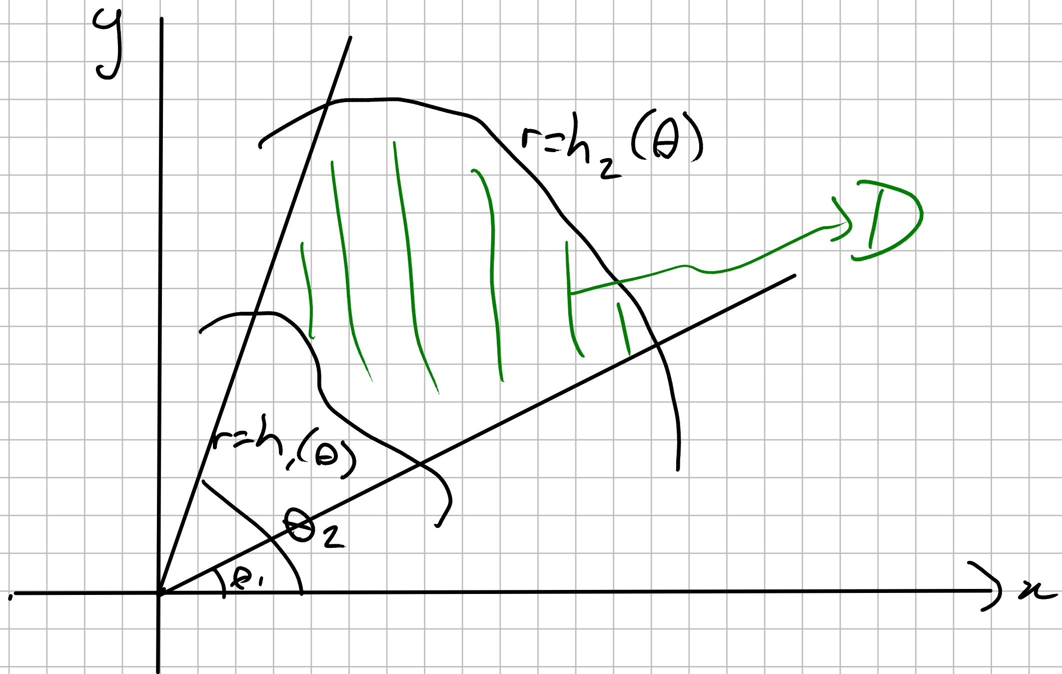
If $f(x,y)$ is continuous on a polar rectangle

$$P = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x = r\cos(\theta) & r_1 \le r \le r_2 \\ y = r\sin(\theta) & \theta_1 \le \theta \le \theta_2 \end{matrix} \right\}$$

then

$$\iint_P f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos(\theta), r\sin(\theta))\, r \, dr\, d\theta$$

We can also do this more generally



Thm// Suppose f is continuous on

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{ll} x = r\cos\theta & \theta_1 \le \theta \le \theta_2 \\ y = r\sin\theta & h_1(\theta) \le r \le h_2(\theta) \end{array} \right\}$$

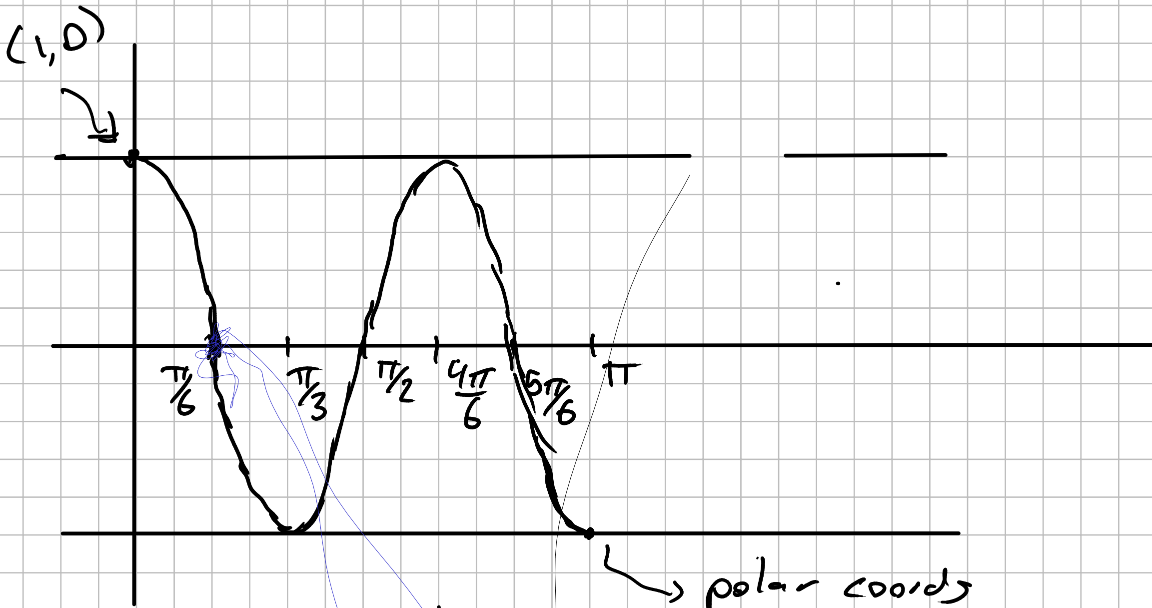
where $h_1(\theta)$ & $h_2(\theta)$ are cont.

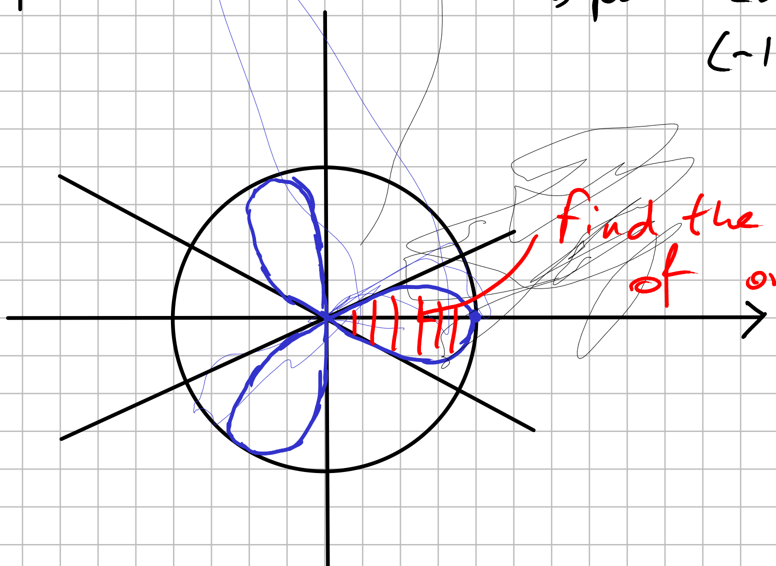
Then

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta$$

Eg// Consider the curve

$$r = \cos(3\theta) \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$





find the area of one "petal"

Region in question is

$$-\pi/6 \le \theta \le \frac{\pi}{6} \qquad 0 \le r \le \cos(3\theta)$$

so area of the petal D is

$$\iint_D 1\, dA = \int_{-\pi/6}^{\pi/6} \int_0^{\cos(3\theta)} r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\pi/6} \frac{1}{2}\cos^2(3\theta)\, d\theta$$

half-angle formula

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{4}(1+\cos(6\theta))\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin(6\theta)}{6}\right]_{\theta=-\pi/6}^{\pi/6}$$

$$= \boxed{\frac{\pi}{12}}$$

Lecture 16b

Recap//

$f(x,y)$ continuous on $D \subseteq \mathbb{R}^2$ Bounded

(1) We defined double integrals for D a rectangle in $\mathbb{R}^2$ ie $[a,b]\times[c,d]$

$$\iint_D f(x,y)\,dA := \lim_{m,n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y$$

similarly for $D \subset R$, where $R=[a,b]\times[c,d]$. D bounded domain we defined

$$\iint_D f(x,y)\,dA := \iint_R F(x,y)\,dA$$

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{else} \end{cases}$$

(2) Fubini's Thms

(a) If $D = [a,b] \times [c,d]$ is a rectangle

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

(b) If D is a Type I region

$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

w/ g_1 & g_2 cont. Then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(c) If D is a Type II region

$$D = \{(x,y) \in \mathbb{R}^2 \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

h_1 & h_2 cont. Then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

(3) General properties

(a) $\displaystyle\iint_D f(x,y) + g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

(b) $c \in \mathbb{R}$

$$\iint_D c f(x,y)\,dA = c \iint_D f(x,y)\,dA$$

(c) $f(x,y) \ge g(x,y)$ for all $(x,y) \in D$

$$\iint_D f(x,y)\,dA \ge \iint_D g(x,y)\,dA$$

(d) If $D = D_1 \cup D_1$ intersecting only in their boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

---

(4) Change of coords. for polar coords.

(a) $D = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \begin{matrix} x = r\cos\theta & r_1 \le r \le r_2 \\ y = r\sin\theta & \theta_1 \le \theta \le \theta_2 \end{matrix}\right\}$

then

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

(b) $D = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \begin{matrix} x = r\cos\theta & \theta_1 \le \theta \le \theta_2 \\ y = r\sin\theta & h_1(\theta) \le r \le h_2(\theta) \end{matrix}\right\}$

then

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$